Moral Hazard: Characterization of SB

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Characterization of Second Best Contracts I

General Model: Suppose

- $q = q(e, \theta)$, where
- Θ is the set of states of nature and captures randomness.
- $e \in \mathcal{E} \subseteq \mathcal{R}$ and $\theta \in \Theta$
- $\frac{\partial q(e,\theta)}{\partial e} \geq 0$.

Payoff functions:

- Principal is risk neutral or risk-neutral. So, let

$$V(q, w) = q - w, \quad V' > 0, \quad V'' \leq 0$$

- Agent is (weakly) risk-averse. So, let

$$u(w, e) = u(w) - \psi(e), \quad u' > 0, \quad u'' \leq 0,$$

where $\psi(e)$ is the dis-utility of effort e, $\psi' > 0$ and $\psi'' \geq 0$.

Characterization of Second Best Contracts II

Contract:

- The set of contracts is

$$\mathcal{A} = \{(q, w) : q \in \mathcal{R}_+, w(q) \in \mathcal{R}\}.$$

- $\bar{w}$ = Certainty equivalent of the reservation (outside) wage
- $\bar{u} = u(\bar{w})$, the reservation utility

When $u'' = 0$ holds, i.e., when the agent is risk-neutral, the FB can be achieved by selling the output to the agent.

So assume that the agent is risk-averse, i.e., $u'' < 0$.

Characterization of Second Best Contracts III

The P will solve:

$$\max_{w(q)} E\{V(q - w(q))\}$$

s.t. IR

$$E\{u(w(q)) - \psi(e)\} \geq \bar{u} \qquad (IR)$$

and

$$e = \arg\max_{\hat{e}}\{E\{u(w(q) - \psi(\hat{e})\}\} \qquad (IC)$$

However,

- For given level of e, output q can be treated as a random variable.
- Assume $q \in [\underline{q}, \overline{q}]$. Let,
- $F(q|e)$ is a conditional cumulative distribution of q
- $f(q|e)$ is the associated conditional density function

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- Note: $F(q|e)$ is a distribution induced by the distribution of θ on Θ.
- $F(q|e)$ is induced through the production technology function $q = q(e, \theta)$
- Note: $\frac{\partial q(e,\theta)}{\partial e} \geq 0 \Rightarrow F_e(q|e) \leq 0$ and $\frac{\partial q(e,\theta)}{\partial e} > 0 \Rightarrow F_e(q|e) < 0$
- We will assume that $F(q|e)$ satisfies First Order Stochastic Dominance.

Definition

Distribution $F(q|e)$ satisfies First Order Stochastic Dominance w.r.t effort if

$$(\forall q)[F_e(q|e) \leq 0] \;\&\; (\exists q)[F_e(q|e) < 0]$$

Clearly $\frac{\partial q(e,\theta)}{\partial e} > 0 \Rightarrow F_e(q|e) < 0$ is sufficient for $F(q|e)$ to satisfy the First Order Stochastic Dominance.

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Therefore, in the above setup, the P will solve:

$$\max_{w(q)}\{\int_{\underline{q}}^{\overline{q}} V(q - w(q))f(q|e)dq\}$$

s.t. IR

$$\int_{\underline{q}}^{\overline{q}} u(w(q))f(q|e)dq - \psi(e) \geq \bar{u}. \tag{1}$$

and

$$e = \arg\max_{\hat{e}}\{\int_{\underline{q}}^{\overline{q}} u(w(q))f(q|\hat{e})dq - \psi(\hat{e})\} \tag{IC}$$

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For the time being assume that the agent's payoff function is concave. So, replacing IC with the foc and the relevant soc, we get

$$\int_{\underline{q}}^{\overline{q}} u(w(q))f_e(q|e)dq - \psi'(e) \quad = \quad 0 \tag{2}$$

$$\int_{\underline{q}}^{\overline{q}} u(w(q))f_{ee}(q|e)dq - \psi''(e) \quad < \quad 0 \tag{3}$$

Form the Lagrangian using (1) and (2)

$$\begin{aligned} \mathcal{L}() \quad = \quad & \int_{\underline{q}}^{\overline{q}} V(q - w(q))f(q|e)dq \\ + \quad & \lambda[\int_{\underline{q}}^{\overline{q}} u(w(q))f(q|e)dq - \psi(e) - \bar{u}] \\ + \quad & \mu[\int_{\underline{q}}^{\overline{q}} u(w(q))f_e(q|e)dq - \psi'(e)] \end{aligned} \tag{4}$$

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the foc w.r.t. $w(q)$ is

$$(\forall q)[\frac{V'(q-w(q))}{u'(w(q))}=\lambda+\mu\frac{f_e(q|e)}{f(q|e)}] \tag{5}$$

Moreover, when the P is risk-neutral, the foc is

$$(\forall q)[\frac{1}{u'(w(q))}=\lambda+\mu\frac{f_e(q|e)}{f(q|e)}] \tag{6}$$

Note that risk-sharing is FB *only if*

$$(\forall q)[\frac{V'(q-w(q))}{u'(w(q))}=\text{constant}], i.e.,$$

$$(\forall q)[\frac{1}{u'(w(q))}=\text{constant}] \tag{7}$$

(7) follows from the *Borch Rule*.

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Some Conclusions:

- (7) requires that both $w(q)$ and $q - w(q)$ are (weakly) increasing functions of q;
- Assuming P is risk-neutral, the FB risk sharing is independent of the distribution function $F(q|e)$ for the random variable q;
- From (7), it can be (by differentiating) shown that

$$0 \leq w'(q) < 1; \tag{8}$$

- Risk-sharing will be as required by (7) *only if* $\mu = 0$, or
- if

$$(\forall q)[\frac{f_e(q|e)}{f(q|e)} = k] \tag{9}$$

for some real number k.

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Since, for all e, $\int_{\underline{q}}^{\overline{q}} f(q|e)dq = 1$ holds, therefore, $\int_{\underline{q}}^{\overline{q}} f(q|e)kdq = k$ and $\int_{\underline{q}}^{\overline{q}} f_e(q|e)dq = 0$. That is, (9) implies (can hold only if)

$$0 = \int_{\underline{q}}^{\overline{q}} f_e(q|e)dq = \int_{\underline{q}}^{\overline{q}} f(q|e)kdq = k,$$

that is, $k = 0$.
That is, (9) holds only if

$$(\forall q)[f_e(q|e) = 0]$$

But, we are not interested in such a scenario.
Moreover, as we prove below, $\mu > 0$. Therefore, risk sharing in not FB.

Characterization of Second Best Contracts X

Let $w_\lambda(q)$ solve

$$\frac{V'(q - w(q))}{u'(w(q))} = \lambda, \tag{10}$$

where λ is the same as in (5). Recall, $w(q)$ solves (5), i.e.,

$$(\forall q)[\frac{V'(q - w(q))}{u'(w(q))} = \lambda + \mu \frac{f_e(q|e)}{f(q|e)}]$$

Therefore, $\mu > 0$ implies that the SB contract is such that:

$$\begin{cases} w(q) \geq w_\lambda(q), & \text{on } Q_+ = \{q|f_e(q|e) \geq 0\}; \\ w(q) < w_\lambda(q), & \text{on } Q_- = \{q|f_e(q|e) < 0\}. \end{cases} \tag{11}$$

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Remark

Note that

$$\frac{\partial \ln f(q|e)}{\partial e} = \frac{f_e(q|e)}{f(q|e)}, i.e.,$$

$\frac{f_e(q|e)}{f(q|e)}$ *is the derivative of the likelihood function* $\ln f(q|e)$*;*
$\ln f(q|e) = \ln Prob\{e|q\}$.

Definition

Continuous Output Case: Monotone Likelihood Ratio Property (MLRP):
Distribution $F(q|e)$ satisfies MLRP if

$$\frac{d}{dq}[\frac{f_e(q|e)}{f(q|e)} \geq 0], i.e., \quad \frac{d}{dq}[\frac{\partial \ln f(q|e)}{\partial e} \geq 0].$$

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Definition

Discrete Output Case: Monotone Likelihood Ratio Property (MLRP): Assuming two output levels; q_1 and q_2. Distribution $F(q|e)$ satisfies MLRP if $(\forall e > \bar{e})$,

$$\frac{\pi(q_i|\bar{e})}{\pi(q_i|e)} = \frac{f(q_i|\bar{e})}{f(q_i|e)}$$

is (weakly) decreasing in i., i.e., if $(\forall e > \bar{e})$, $\frac{f(q_i|e)-f(q_i|\bar{e})}{f(q_i|e)}$ is (weakly) increasing in i.

Proposition

The contract satisfies monotonicity iff $F(q|e)$ satisfies Monotone Likelihood Ration Property*, i.e.,*

$$\frac{dw}{dq} \geq 0 \Leftrightarrow \frac{d}{dq}[\frac{f_e(q|e)}{f(q|e)} \geq 0].$$

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Special Case: Let $q \in \{q_L, q_H\}$ and $e \in \{e_L, e_H\}$. Now, assuming that the P is risk-neutral and wants to induce e_H, the foc can be written as

$$\frac{1}{u'(w(q_L))} = \lambda + \mu[1 - \frac{f(q_L|e_L)}{f(q_L|e_H)}]$$
$$\frac{1}{u'(w(q_H))} = \lambda + \mu[1 - \frac{f(q_H|e_L)}{f(q_H|e_H)}]$$

Now, if q_H is more likely when $e = e_H$, and the q_L is more likely when $e = e_L$, we get $w_H > w_L$, i.e.,

$$[\frac{f(q_H|e_H)}{f(q_H|e_L)} > 1 \text{ and } \frac{f(q_L|e_L)}{f(q_L|e_H)} > 1] \Rightarrow w_H > w_L.$$

That is, the contract is monotonic in output.

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Proposition

Now when $F(q|e)$ satisfies First Order Stochastic Dominance w.r.t effort, $\mu > 0$, i.e., IC will bind.

Proof: Suppose $\mu \leq 0$ holds. Differentiating (4), w.r.t. e gives

$$\begin{aligned}\int_{\underline{q}}^{\overline{q}} V(q - w(q)) f_e(q|e) dq &\quad + \\ \lambda[\int_{\underline{q}}^{\overline{q}} u(w(q)) f_e(q|e) dq - \psi(e) - \bar{u}] &\quad + \\ \mu[\int_{\underline{q}}^{\overline{q}} u(w(q)) f_{ee}(q|e) dq - \psi^{''}(e)] &\quad = \quad 0 \qquad (12)\end{aligned}$$

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In view of (2) and (3), $\mu \leq 0$ implies:

$$\int_{\underline{q}}^{\overline{q}} V(q - w(q)) f_e(q|e) dq \leq 0. \tag{13}$$

Let $w_\lambda(q)$ solve (10), i.e.,

$$\frac{V'(q - w(q))}{u'(w(q))} = \lambda$$

Recall, $w(q)$ solves (5), i.e.,

$$(\forall q)[\frac{V'(q - w(q))}{u'(w(q))} = \lambda + \mu \frac{f_e(q|e)}{f(q|e)}]$$

Therefore, $\mu \leq 0$ implies:

$$\begin{cases} w(q) \leq w_\lambda(q), & \text{on } Q_+ = \{q|f_e(q|e) \geq 0\}; \\ w(q) > w_\lambda(q), & \text{on } Q_- = \{q|f_e(q|e) < 0\}. \end{cases} \tag{14}$$

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Therefore, we get

$$\int_{\underline{q}}^{\overline{q}} V(q - w(q)) f_e(q|e) dq \geq \int_{\underline{q}}^{\overline{q}} V(q - w_\lambda(q)) f_e(q|e) dq. \qquad (15)$$

In view of $F_e(\underline{q}, e) = F_e(\overline{q}, e) = 0$, integration by parts gives us

$$\int_{\underline{q}}^{\overline{q}} V(q - w_\lambda(q)) f_e(q|e) dq = - \int_{\underline{q}}^{\overline{q}} V'(q - w_\lambda(q))(1 - w'_\lambda(q)) F_e(q|e) dq. \quad (16)$$

Hold RHS to be fixed (assume $\mu = 0$) and differentiate (5) w.r.t. q to get

$$w'_\lambda(q) = \frac{V''(q - w_\lambda(q))}{\lambda u''(w_\lambda(q)) + V''(q - w_\lambda(q))}$$

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In view of $\lambda > 0$, this gives $1 > w'_\lambda(q) \geq 0$. Also, $V' > 0$ and $F(q|e)$ satisfies FOSD. Therefore,

$$-\int_{\underline{q}}^{\overline{q}} V'(q - w_\lambda(q))(1 - w'_\lambda(q))F_e(q|e)dq > 0.$$

That is, we get

$$\int_{\underline{q}}^{\overline{q}} V(q - w_\lambda(q))f_e(q|e)dq > 0. \tag{17}$$

But (13) and (17) imply a contradictions. Therefore, $\mu > 0$. $Q.E.D.$

Non-monotonic Contracts I

Example

Consider the following probability density function:

	$f(q_L \mid e)$	$f(q_M \mid e)$	$f(q_H \mid e)$
e_L	0.5	0.5	0
e_H	0.4	0.1	0.5

where $q_H > q_M > q_L$. Note here MLRP is violated.

Exercise Show that the SB contact is such that $w_H > w_L > w_M$, i.e., the contract is non-monotonic.

Limitation of Non-monotonic Contracts?