Deterministic Dynamic Programming

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1 Introduction

A representative household has a unit endowment of labor time every period, of which it can choose n_t labor. It values only consumption every period, and wishes to choose $(C_t)_0^\infty$ to attain

$\sup \sum_{t=0}^\infty \beta^t U(C_t)$

subject to

$C_t + i_t \leq F(k_t, n_t)$ (1)

$k_{t+1} = (1-\delta)k_t + i_t$, k_0 given. (2)

$U' > 0$, so replace $\leq$ with $=$ in Eq. (1).

Since leisure gives no utility, $n_t = 1$ for every t.

Let $f(k_t) = F(k_t, 1) + (1-\delta)k_t$. Then (1) + (2) yields a single constraint for each period t. Also, $C_t = f(k_t) - k_{t+1}$. And we can rewrite the problem as:

Choose $(k_{t+1})_0^\infty$ to attain

$\sup \sum_{t=0}^\infty \beta^t U(f(k_t) - k_{t+1})$ (3)

subject to $0 \leq k_{t+1} \leq f(k_t)$, $t = 0, 1, 2, \ldots$, k_0 given (4)

This is a *sequences problem* or **SP**. It's a special case of the SP Problem below: Let the states $x_t \in X$, and let $\Gamma(x)$ be the set of feasible choices of state, from state x.

$$(\mathrm{SP}) \sup_{(x_{t+1})_0^\infty} \sum_{t=0}^\infty \beta^t F(x_t, x_{t+1})$$

$$\text{subject to } x_{t+1} \in \Gamma(x_t),\ t = 0, 1, 2, \ldots, x_0 \in X \text{ given}$$

2 Bellman's Equation

Let $v^*(x_0)$ be the value function for SP. We want to talk about the relationship between SP, and the functional equation FE below.

$$\text{FE} \quad v^*(x) = \sup_{y \in \Gamma(x)} \left[F(x,y) + \beta v^*(y) \right], \ \forall x \in X$$

The value function of SP satisfies FE under much more general assumptions that in the Lemma below. But we then show that under these assumptions, a solution to FE exists and is unique, so if we solve FE, we will have derived the value function for the original SP problem. Moreover, it can be shown that the optimal policy function can be obtained as the argmax of FE.

Lemma 1 *Suppose* $0 < \beta < 1$*,* F *is bounded, and* $\Gamma(x)$ *is nonempty for all* $x \in X$*. Let* $v^* : X \to \Re$ *be the value function for SP. The* v^* *satisfies FE (the Bellman Equation).*

Proof. Note that since F is bounded and $0 < \beta < 1$, $v^*(x)$ is a real number for every $x \in X$.

(i) For all $x \in X$, $v^*(x) \geq F(x,y) + \beta v^*(y)$, for all $y \in \Gamma(x)$.

Indeed pick any $\epsilon > 0$ and any $y \in \Gamma(x)$. By the sup definition of $v^*(y)$, there must exist a feasible sequence $(y, x_2, x_3, \ldots)$ that gives payoff arbitrarily close to $v^*(y)$, i.e.

$$u(y, x_2, \ldots) \equiv \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+2}) \geq v^*(y) - \epsilon$$

where $x_{t+1} = y$ for $t = 0$. So, the feasible sequence $(x, y, x_2, x_3, \ldots)$ gives payoff

$$\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = F(x,y) + \beta u(y, x_2, \ldots) \geq F(x,y) + \beta v^*(y) - \beta\epsilon$$

where $x_0 = x, x_1 = y$. So we have

$$v^*(x) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \geq F(x,y) + \beta v^*(y) - \beta\epsilon$$

and (i) follows since ϵ is arbitrary.

(ii) For all $\epsilon > 0$, there exists $y \in \Gamma(x)$ s.t.

$v^*(x) - \epsilon \leq F(x,y) + \beta v^*(y)$.

Indeed, since $v^*(x)$ is a supremum, there exists a feasible sequence $(x, y, x_2, \ldots) \equiv (x_0, x_1, x_2, \ldots)$ satisfying the first inequality below.

$v^*(x) - \epsilon \leq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = F(x,y) + \beta \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+2}) \leq F(x,y) + \beta v^*(y)$.

The result follows from (i) and (ii). ∎

To show that there is a unique v that solves FE, the Bellman Equation (so that by the above lemma, it must be that this $v = v^*$ above), we use the following, heavily used, intermediate result due to Blackwell.

Theorem 1 *(Blackwell). Let $M \subset bU$ s.t. $u \in M$ and $a \in [0,\infty) \Rightarrow u + a \in M$ (where $u+a$ is the function defined by $u(x) + a, \forall x \in U$). Let $T : M \to M$ be an operator satisfying (i) Monotonicity: $u \leq v \Rightarrow Tu \leq Tv$, and (ii) Discounting: There exists $\lambda \in [0,1)$ s.t. $\forall u \in M$ and $a \in [0,\infty)$, $T(u+a) \leq Tu + \lambda a$. Then T is a uniformly strict contraction with modulus λ on the metric space (M, d_∞).*

Proof of Blackwell's theorem. Interpret the inequalities as holding pointwise for all $x \in U$.

$u(x) = v(x) + u(x) - v(x) \leq v(x) + |u(x) - v(x)| \leq v(x) + ||u - v||_\infty$.

Since T is monotone, applying it to both sides we have

$Tu \leq T(v + ||u-v||_\infty) \leq Tv + \lambda ||u - v||_\infty$.

The last inequality is due to the discounting property of T.

Thus $Tu - Tv \leq \lambda ||u-v||_\infty$.

Reversing the roles of u and v, we get

$Tv - Tu \leq \lambda ||u - v||_\infty$.

Combining the two, $|Tu - Tv| \leq \lambda||u-v||_\infty$.

Since this inequality holds when the left hand is evaluated at arbitrary $x \in U$, it holds for the sup. Thus

$||Tu - Tv||_\infty \leq \lambda||u-v||_\infty$.

That is, T is a uniformly strict contraction. ■

Now for the uniqueness result.

Theorem 2 *Let $X \subset \Re^l$ be convex, and $\Gamma : X \to X$ be nonempty, compact-valued and continuous. Let $F : gr(\Gamma) \to \Re$ be bounded and continuous, and $0 < \beta < 1$. Then the operator T on bcX, the space of bounded continuous functions on X, given below*

$$(Tf)(x) = \max_{y \in \Gamma(x)} [F(x,y) + \beta f(y)]$$

maps into bcX, has a unique fixed point $v \in bcX$, and for all $v_0 \in bcX$, $||T^n v_0 - v||_1 \to 0$ as $n \to \infty$.

Proof. For each $x \in X$, the RHS is a maximization problem over the compact set $\Gamma(x)$; the max exists by continuity. Since F and f are bounded, so is Tf. And since F and f are continuous and Γ is a compact-valued, continuous correspondence, Tf is continuous by the Theorem of the Maximum. SO, $T : bcX \to bcX$.

Note that T also satisfies monotonicity and discounting and so is a uniformly strict contraction with modulus β. Moreover, bcX is a complete metric space. So the result follows from Banach's contraction mapping theorem. ■

Remarks.

(1). For the representative agent growth model that we started with, the Bellman Equation is:

$$v^*(k) = \max_{0 \leq y \leq f(k)} [U(f(k) - y) + \beta v^*(y)]$$

Suppose the optimizing value of y is $\sigma(k)$. Clearly, it can only depend on the initial capital stock k at the beginning of the time period. And not otherwise on the entire history preceding k. Thus it is in fact optimal to choose today's action as a function only of the current state.

(2) For ease of notation, call v^* simply v. Suppose v is differentiable. And suppose $\sigma(k)$ is an interior max. Then the first order condition for the Bellman problem, obtained by differentiating the RHS wrt y, and then setting $y = \sigma(k)$, is:

$$U'[f(k) - \sigma(k)] = \beta v'(\sigma(k))$$

This says that at the optimum, the marginal utility of consuming current output equals the marginal value of allocating it to capital and enjoying higher consumption next period on.

An envelope theorem due to Benveniste and Scheinkman also holds, which says that to obtain $v'(k)$, we can differentiate both sides of Bellman's equation wrt k, ignoring the second-order changes on the optimal choice of y. And then replace y with $\sigma(k)$. So,

$$v'(k) = f'(k)U'[f(k) - \sigma(k)]$$

In applications, we may use this condition in the FOC to get an Euler equation.