Debt Contracts

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Debt Contracts I

Innes (1990, JET)
Suppose

- There is a risk-neutral Bank B, and A risk-neutral Entrepreneur E
- The Entrepreneur has project/ideas but no money; Bank has money but no ideas
- They sign a contract; B lends I amount to E.
- Contract can be a 'debt' contract or some other contract.

Let

- $q \in [0, \infty$ denotes the output/revenue/profit from the project
- $q = q(e, \theta)$, where
- Θ is the set of states of nature and captures randomness.
- $e \in \mathcal{E} \subseteq \mathcal{R}$ and $\theta \in \Theta$

Debt Contracts II

- $\frac{\partial q(e,\theta)}{\partial e} \geq 0$.

Let

- $F(q|e)$ is a conditional cumulative distribution of q
- $f(q|e)$ is the associated conditional density function
- Note: $\frac{\partial q(e,\theta)}{\partial e} \geq 0 \Rightarrow F_e(q|e) \leq 0$ and $\frac{\partial q(e,\theta)}{\partial e} > 0 \Rightarrow F_e(q|e) < 0$.
- Assume: $E(q|0) = 0$ and the Monotone Likelihood Ratio Property (MLRP) holds. That is,

$$\frac{d}{dq}[\frac{f_e(q|e)}{f(q|e)} \geq 0], i.e., \quad \frac{d}{dq}[\frac{\partial \ln f(q|e)}{\partial e} \geq 0].$$

Payoff functions:

- Risk-neutral Bank's payoff function is

$$V(x) = x, \quad V' > 0, \quad V'' = 0$$

Debt Contracts III

- Risk-neutral Entrepreneur's payoff function is $u(w,e) = u(w) - \psi(e)$, $u' > 0$, $u'' = 0$, i.e., $u(w,e) = w - \psi(e)$, where $\psi(e)$ is the (money) cost of effort e, $\psi' > 0$ and $\psi'' \geq 0$.

Let

$r(q)$ be the contract, repayment schedule.

Definition

Limited Liability Contract is a repayment schedule $r(q)$, such that: $r(q) \leq q$

Definition

Debt Contract is a repayment schedule $r(q)$, such that:

- $0 \leq r(q) \leq q$, i.e., two sided limited lability
- $0 \leq r'(q)$, i.e., monotonicity

FB Repayment Schedule I

Let the entrepreneur make the offer to B. Under the FB, the entrepreneur will solve:

$$\max_{r(q),e} \int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e)$$

s.t. IR, i.e., $\int_0^\infty r(q) f(q|e) dq \geq I$. Clearly, IR will bind. So, the entrepreneur solves:

$$\max_{e} \int_0^\infty [q - I] f(q|e) dq - \psi(e)$$

Assuming that this programme is strictly concave, the FB effort, e^*, solves the following FOC is given by

$$\int_0^\infty q f_e(q|e) dq - \psi'(e) = 0 \tag{0.1}$$

SB Repayment Schedule I

Under a two-way limited liability SB contract, the entrepreneur will solve

$$\max_{r(q),e} \int_0^\infty [q-(r(q))]f(q|e)dq - \psi(e) \tag{0.2}$$

s.t. e solves (0.3)

$$\begin{aligned} \int_0^\infty [q-(r(q))]f_e(q|e)dq &= \psi'(e) && (0.3) \\ \int_0^\infty r(q)f(q|e)dq &= I && (0.4) \\ 0 \leq r(q) &\leq q && (0.5) \end{aligned}$$

where $0 \leq r(q) \leq q$ is the two-way limited liability constraint.

- Does the above programme have a solution?
- If there is a solution, is it unique?

SB Repayment Schedule II

To ensure a solution, assume there exists an effort level e_{max}, such that

$$\lim_{e \to e_{max}} \int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e) < \int_0^\infty [q - (r(q))] f(q|0) dq - \psi(0)$$

So, the entrepreneur's effort level can be restricted to $[0, e_{max}]$. Note: we have the following: For all e,

$$\left[\int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e) \leq \int_0^\infty q f(q|e) dq - \psi(e) \right]$$

Further, for all $e \in [0, e_{max}]$,

$$\int_0^\infty [0 - (r(0))] f(q|e) dq - \psi(e) \leq \int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e).$$

SB Repayment Schedule III

Let,

$$k^* = \int_0^\infty q f(q|e_{max}) dq.$$

Further, we have for all $e \in [0, e_{max}]$,

$$\int_0^\infty [q - k^*] f(q|e) dq - \psi(e) \leq \int_0^\infty [0 - (r(0))] f(q|e) dq - \psi(e)$$

Note $\left[\int_0^\infty q f(q|e) dq - \psi(e)\right]$ and $\left[\int_0^\infty [q - k^*] f(q|e) dq - \psi(e)\right]$

- are continuous functions of e
- so, they are bounded on the compact set $[0, e_{max}]$.
- Therefore, for given $r(q)$, there is exists at least one solution to the following:

$$\max_e \left\{ \int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e) \right\}.$$

SB Repayment Schedule IV

We will assume that (0.2) has unique solution. The Lagrangian associated with IR and IC with (0.2) is:

$$\begin{aligned}\mathcal{L} &= \int_0^\infty [q-(r(q))]\, f(q|e)dq - \psi(e) + \mu\left[\int_0^\infty [q-(r(q))] f_e(q|e)dq - \psi(e)\right] \\ &+ \lambda\left[\int_0^\infty r(q)f(q|e)dq - I\right]\end{aligned}$$

or, re-writing

$$\begin{aligned}\mathcal{L} &= \int_0^\infty r(q)\left[\lambda - \mu\frac{f_e(q|e)}{f(q|e)} - 1\right] f(q|e)dq \\ &+ \int_0^\infty q\left[1+\mu\frac{f_e(q|e)}{f(q|e)}\right] f(q|e)dq - \psi(e) - \mu - \psi'(e) - \lambda I \qquad (0.6)\end{aligned}$$

SB Repayment Schedule V

It can be shown that IC will bind, i.e, $\mu > 0$. So, the optimum repayment schedule is:

$$r^*(q) = \begin{cases} q & \forall q \left[1 + \mu \frac{f_e(q|e)}{f(q|e)} < \lambda \right] \\ 0 & \forall q \left[1 + \mu \frac{f_e(q|e)}{f(q|e)} > \lambda \right] \end{cases} \tag{0.7}$$

$$r^*(q) = \begin{cases} q & \forall q \left[\frac{f_e(q|e)}{f(q|e)} > \frac{\lambda - 1}{\mu} \right] \\ 0 & \forall q \left[\frac{f_e(q|e)}{f(q|e)} < \frac{\lambda - 1}{\mu} \right] \end{cases} \tag{0.8}$$

We know that there exists a value of revenue, say $q = Z$ such that for all $q > Z$, $\frac{f_e(q|e)}{f(q|e)} > \frac{\lambda - 1}{\mu}$. (Why?)
So, the optimum contract is

$$r^*(q) = \begin{cases} 0 & \text{if } q > Z \\ q & \text{if } q < Z \end{cases} \tag{0.9}$$

SB Repayment Schedule VI

Under the optimum LL contract, $e(r^*(q))$ solves the following FOC:

$$\int_Z^\infty q f_e(q|e)dq - \psi'(e) = 0. \tag{0.10}$$

A comparison of (0.1) and (0.10) shows that

$$e(r^*(q)) < e^*.$$

Optimum Debt Contract I

A debt contract is monotonic. Let,

$$r^D(q) = \begin{cases} D & \text{if } q > D \\ q & \text{if } q \leq D \end{cases} \tag{0.11}$$

that is, $r^D(q) = \min\{q, D\}$. Under a debt contract, the entrepreneur's payoff is

$$\int_0^\infty [q - r^D(q)] f(q|e) dq - \psi(e) = \int_0^D [q - q] f(q|e) dq + \int_D^\infty [q - D] f(q|e) dq - \psi(e).$$

So, he wants to choose minimum value of D such that

$$\int_0^D q f(q|e^D) dq + [1 - F(D|e^D)] D = I,$$

where e^D solves the following FOC:

$$\int_D^\infty (q - D) f_e(q|e) dq = \psi'(e).$$

Optimum Debt Contract II

Note:

- $\int_0^{\infty}[q - r^D(q)]f(q|e)dq - \psi(e)$ is continuous in e and D,
- in view of Maximum theorem, e^D is a continuous function of D.

Question

Can a Debt contract can induce the FB effort?

Comparison of

- (0.1) and (0.12) shows that $r^D(q)$ cannot induce the FB effort; $r^D(q) < r^{FB}$.
- (0.10) and (0.12) shows that $r^D(q)$ will be different from the SB effort.

Optimum Debt Contract III

Question

Is a Debt better than the optimum payment schedule discussed above? (Assume the above, problem is strictly concave)

Innes (1990) showed that,

- DC is the most efficient among the class of monotonic contracts.