Debt Contracts and Repayment Schedules

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1 Basics

We can discuss Debt Contracts (DC) as an application of the Second Best contracts.[1] Suppose There is a risk-neutral bank B, and a risk-neutral entrepreneur, E. The entrepreneur has project/ideas but no money; Bank has money but no ideas. They sign a contract; B lends an amount I to E. The contract can be a DC or a 'non-debt' contract (NDC).

Let q denotes the output/revenue/profit from the project; $q \in [0, \infty)$, and $q = q(e, \theta)$; where e denoted the effort put in by E, $e \in \mathcal{E} \subseteq \mathcal{R}_+$ and $\theta \in \Theta$. Θ is the set of states of nature and captures randomness of the project outcome. The cost of effort function is $\psi(e)$; $\psi^{'}(e) > 0$ and $\psi^{''}(e) > 0$. Let $F(q|e)$ be the conditional cumulative distribution of q; and $f(q|e)$ be the associated conditional density function.

Note: $\frac{\partial q(e,\theta)}{\partial e} \geq 0 \Rightarrow F_e(q|e) \leq 0$ and $\frac{\partial q(e,\theta)}{\partial e} > 0 \Rightarrow F_e(q|e) < 0$. Assume, $(\forall \theta \in \Theta) \left[\frac{\partial q(e,\theta)}{\partial e} \geq 0\right]$, and $(\exists \theta \in \Theta) \left[\frac{\partial q(e,\theta)}{\partial e} > 0\right]$.

Let $E(q|e) = \int_0^\infty qf(q|e)dq$. Assume for all e, $\int_0^\infty qf(q|e)dq \geq 0$, $\int_0^\infty qf(q|0)dq = 0$ and that the Monotone Likelihood Ratio Property (MLRP) Holds. That is, $\frac{d}{dq}\left[\frac{f_e(q|e)}{f(q|e)} \geq 0\right], i.e., \quad \frac{d}{dq}\left[\frac{\partial \ln f(q|e)}{\partial e} > 0\right]$. Payoff functions are defined as follows: The risk-neutral Bank's payoff function is $V(.)$, $\quad V' > 0, \quad V'' = 0$, which gives us $V(x) = x$. The risk-neutral Entrepreneur's payoff function is $u(w, e) = u(w) - \psi(e), \quad u' > 0, \quad u'' = 0$, where $\psi(e)$ is the (money) cost of effort e, $\psi' > 0$ and $\psi'' \geq 0$. This gives us $u(w, e) = w - \psi(e)$. Let

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[†]This write up has not been proof-read. Please watch out for typos.

[1]This note is based on Innes, Robert (1990). *Journal of Economic Theory.* Also see book BM Section 4.6.2.

$r(q)$ be the contract, or the repayment schedule signed by B and E.

A general repayment schedule is a function: $r(q) : \mathfrak{R}_+ \mapsto \mathfrak{R}$. However,

Definition 1 *Limited Liability Contract (LLC) is a repayment schedule function $r(q) : \mathfrak{R}_+ \mapsto \mathfrak{R}_+$, such that $0 \leq r(q) \leq q$.*

Definition 2 *Debt Contract (DC) is a repayment schedule function $r(q) : \mathfrak{R}_+ \mapsto \mathfrak{R}_+$, such that $0 \leq r(q) \leq q$, i.e., two sided limited lability holds, and $0 \leq r'(q)$, i.e., monotonicity holds.*

2 Optimum Repayment Schedules

Let the entrepreneur make the contractual offer to the bank. Under the FB, the entrepreneur will solve:

$$\max_{r(q),e}\left\{\int_0^\infty w(q)f(q|e)dq - \psi(e)\right\} \equiv \max_{r(q),e}\left\{\int_0^\infty [q - r(q)]f(q|e)dq - \psi(e)\right\}$$

s.t. IR for the bank, i.e., $\int_0^\infty r(q)f(q|e)dq \geq (1+\rho)I$, where ρ is the return in the outside option for the bank.[2] We assume $\rho = 0$. Clearly, E will choose $r(q)$ such that IR binds. We assume that $r(q)$ is integrable. So, the entrepreneur's problem can be written as:

$$\max_{e}\left\{\int_0^\infty [q - I]f(q|e)dq - \psi(e)\right\} \equiv \max_{e}\left\{\int_0^\infty qf(q|e)dq - \psi(e)\right\}.$$

Assuming that this programme is strictly concave, the FB effort, e^*, solves the following FOC

$$\int_0^\infty qf_e(q|e)dq - \psi'(e) = 0 \tag{1}$$

Under a SB contract, the entrepreneur will solve:

$$\max_{r(q),e}\int_0^\infty [q - (r(q))]f(q|e)dq - \psi(e) \tag{2}$$

s.t. IR for the bank and IC for the entrepreneur. Notice, $r(q) : \mathfrak{R}_+ \mapsto \mathfrak{R}$. However that even for a given $r(q)$, neither a solution of (2) nor its uniqueness

[2]E.g., assuming that the lending market is competitive, ρ can be taken as returns from the safe investment, say in govt. securities.

is guaranteed. One approach can be to assume that for any given $r(q)$, (2) has unique solution. Alternatively, to ensure a solution assume there exists an effort level e_{max}, such that

$$\lim_{e \to e_{max}} \left\{ \int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e) \right\} < \int_0^\infty [q - (r(q))] f(q|0) dq - \psi(0).$$

that is, there is dis-utility of effort which grows 'large' as $e \to e_{max}$. So, the entrepreneur's effort level can be restricted to $[0, e_{max}]$ with any loss of generality. Moreover, notice the following: For all e

$$\left[\int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e) \le \int_0^\infty q f(q|e) dq - \psi(e) \right]$$

Also, for any given $0 \le r(q) \le q$, the following holds: $(\forall e \in [0, e_{max}])$

$$\begin{aligned}
\int_0^\infty 0.f(q|e) dq - \psi(e) &\le \int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e) \\
\int_0^\infty [q - k^*] f(q|e) dq - \psi(e) &\le \int_0^\infty 0.f(q|e) dq - \psi(e),
\end{aligned}$$

where $k^* = \int_0^\infty q f(q|e_{max}) dq$. Now, since $\int_0^\infty q f(q|e) dq - \psi(e)$ and $\int_0^\infty [q - k^*] f(q|e) dq - \psi(e)$ are continuous, they are bounded on the compact set $[0, e_{max}]$. Therefore, $\int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e)$ is bounded on the compact set $[0, e_{max}]$, we have already assumed it to be continuous. Now, in view of the Weierstrass Theorem, for given $0 \le r(q) \le q$, there is exists at least one solution to (2). [*Of course, you could simply assume that (2) has a unique solution*].

From our results on SB contracts, we know that if very high penalties are possible, the FB outcome can be approached arbitrarily closely.

2.1 Optimum Limited Liability Schedule

Under a limited liability (LL) contract, the entrepreneur will solve:

$$\max_{r(q), e} \int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e) \tag{3}$$

$$\begin{aligned}
\int_0^\infty [q - (r(q))] f_e(q|e) dq &= \psi'(e) && (4) \\
\int_0^\infty r(q) f(q|e) dq &= I && (5) \\
0 \le r(q) &\le q && (6)
\end{aligned}$$

where (4) is the IC for the E, (5) is the IR for the B, and (6) is the two-way limited liability (LL) constraint. We will assume that (3) is strictly concave. Specifically, assume that the above programme is strictly concave and therefore has a unique solution. For a quick moment ignore the LL constraint. The Lagrangian associated with IR and IC with (3) is:

$$\begin{aligned}\mathcal{L} &= \int_0^\infty [q-(r(q))]\, f(q|e)dq - \psi(e) + \mu\left[\int_0^\infty [q-(r(q))] f_e(q|e)dq - \psi'(e)\right] \\ &+ \lambda\left[\int_0^\infty r(q)f(q|e)dq - I\right]\end{aligned}$$

or, re-writing

$$\begin{aligned}\mathcal{L} &= \int_0^\infty r(q)\left[\lambda - \mu\frac{f_e(q|e)}{f(q|e)} - 1\right] f(q|e)dq \\ &+ \int_0^\infty q\left[1+\mu\frac{f_e(q|e)}{f(q|e)}\right] f(q|e)dq - \psi(e) - \mu\psi'(e) - \lambda I. \end{aligned} \tag{7}$$

It can be shown that IC will bind, i.e, $\mu > 0$. So, in view of $0 \leq r(q) \leq q$, the optimum repayment schedule is: $r^*(q) = \begin{cases} q & \forall q \left(1+\mu\frac{f_e(q|e)}{f(q|e)} < \lambda\right) \\ 0 & \forall q \left(1+\mu\frac{f_e(q|e)}{f(q|e)} > \lambda\right)\end{cases}$. That is,

$$r^*(q) = \begin{cases} q & \forall q \left(\frac{f_e(q|e)}{f(q|e)} > \frac{\lambda-1}{\mu}\right) \\ 0 & \forall q \left(\frac{f_e(q|e)}{f(q|e)} < \frac{\lambda-1}{\mu}\right)\end{cases}. \tag{8}$$

MLRP means that the ratio $\frac{f_e(q|e)}{f(q|e)}$ is an increasing function of q. Suppose there exists a value of revenue, say $q = Z$ such that for all $q > Z$, $\frac{f_e(q|e)}{f(q|e)} > \frac{\lambda-1}{\mu}$. So, the optimum contract can be written as:

$$r^*(q) = \begin{cases} 0 & \text{if } q > Z \\ q & \text{if } q < Z\end{cases} \tag{9}$$

That is, E commits to a high effort by offering to B all of the output below the threshold Z. Note that under the optimum LL contract, the effort $e(r^*(q))$ will solve the following FOC:

$$\int_Z^\infty q f_e(q|e)dq - \psi'(e) = 0. \tag{10}$$

A comparison of (1) and (10) shows that $e(r^*(q)) < e^*$.

2.2 Optimum Debt Contract

By definition, a debt contract is monotonic. Since a DC specifies $r(a) : \Re_+ \mapsto \Re_+$ such that

$$r^D(q) = \begin{cases} D & \text{if } q > D \\ q & \text{if } q \leq D \end{cases} \tag{11}$$

that is, $r^D(q) = \min\{q, D\}$. Clearly, a debt contract satisfies LL constraint. Moreover, $r'(a) \geq 0$ holds, i.e., DC is monotonic. Under a DC, the entrepreneur's payoff is

$$\int_0^\infty [q - r^D(q)] f(q|e) dq - \psi(e) = \int_0^D [q - q] f(q|e) dq + \int_D^\infty [q - D] f(q|e) dq - \psi(e).$$

Under a debt contract, the entrepreneur will solve:

$$\max_{r(q),e} \int_0^\infty [q - (r(q))] f(q|e) dq - \psi(e) \tag{12}$$

s.t.

$$\begin{aligned} \int_0^\infty [q - (r(q))] f_e(q|e) dq &= \psi'(e) \\ \int_0^\infty r(q) f(q|e) dq &= I \\ 0 \leq r(q) &\leq q \\ 0 &\leq r'(q) \end{aligned} \tag{13}$$

where (13) is the monotonicity constraint. In view of the above, assume that (12) is strictly concave. Clearly, E would want to choose minimum value of D, say D_0, that satisfies IR, i.e.,

$$\int_0^{D_0} q f(q|e^D) dq + [1 - F(D_0|e^D)] D_0 = I,$$

where $e^D(D_0)$ solves (IC)

$$\int_{D_0}^\infty (q - D_0) f_e(q|e) dq - \psi'(e) = 0. \tag{14}$$

Since $\int_0^\infty [q - r^D(q)] f(q|e) dq - \psi(e)$ is continuous in e and D, in view of Maximum theorem it follows that e^D is a continuous function of D; in fact, $\frac{\partial e^D(D)}{\partial D} < 0$. Now, the question is whether a Debt contract can induce the FB effort.

A comparison of (1) and (14) shows that $e^D(D_0) < e^*$ i.e., debt contract cannot induce the FB effort. Moreover, a comparison of (10) and (14) shows that in general, $e(r^*(q)) \neq e^D$. Innes (1990) formally shows that $e^D(D_0) < e(r^*(q))$. However, the debt contract is the best the most efficient among the monotonic contracts.

You should think of an intuitive argument to explain $e^D(D_0) < e(r^*(q))$. ***Hint:*** Think in terms of constraints facing a Debt contract as opposed to a LL contract, to guess the payoff of E under the two contracts. Now, use strict concavity of the payoff function for E and the facts that $e^D(D_0) < e^*$ and $e(r^*(q)) < e^*$.