Fundamental Theorems of Welfare Economics

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Lecture 6

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The First Fundamental Theorem of Welfare Economics: Consider an exchange economy \((u^i, e^i)_{i \in N}\).

**Theorem**

If \(u^i\) is strictly increasing for all \(i = 1, \ldots, N\), then 
\[ \text{W}(((u^i, e^i)_{i \in N}) \subseteq \text{C}((u^i, e^i)_{i \in N}). \]

That is,

- Every WE/Competitive equilibrium is Pareto optimum;
- Every WE/Competitive equilibrium is in the Core.

**Question**

- What if the Core allocations are highly unequal
- Can markets lead to equitable outcomes?
Suppose:

- \( y = (y^1, \ldots, y^N) \) is any feasible Pareto optimum allocation.
- \( y = (y^1, \ldots, y^N) \) may or may not be equitable across individuals.

Question

- If desired, can \( y = (y^1, \ldots, y^N) \) be achieved as a competitive equilibrium?
- If yes, what are the condition for \( y = (y^1, \ldots, y^N) \) be achieved as a competitive equilibrium?
An Example I

Consider a $2 \times 2$ economy:

- $u^1(.) = x_1^1 + 2x_2^1$, and $u^2(.) = x_1^2 x_2^2$
- Therefore, $MRS_1 = \frac{1}{2}$ and $MRS_2 = \frac{x_2^2}{x_1^2}$
- Let $e^1(.) = (1, \frac{1}{2})$, and $e^2(.) = (0, \frac{1}{2})$
- Assume that individuals act as price-takers

Given any price vectors, in equilib. person 2 will consume $(x_1^2, x_2^2)$ such that:

$MRS_2 = \frac{p_1}{p_2}$ and all income is spent.

That is, the demanded bundle $(x_1^2, x_2^2)$ will be such that:

\[ \frac{x_2^2}{x_1^2} = \frac{p_1}{p_2}, \text{ i.e.,} \]

\[ p_2 x_2^2 = p_1 x_1^2 \text{ and} \]

\[ p_1 x_1^2 + p_2 x_2^2 = p_1 \cdot 0 + \frac{p_2}{2} \]
An Example II

This gives us:

\[ 2p_1 x_1^2 = \frac{p_2}{2} \text{ i.e.,} \]

\[ x_1^2 = \frac{p_2}{4p_1} \text{ Moreover,} \]

\[ x_2^2 = \frac{1}{4} \]

For the 1st person, the following holds:

\[ \frac{p_1}{p_2} > \frac{1}{2} \Rightarrow \text{only 2nd good is demanded} \]

\[ \frac{p_1}{p_2} < \frac{1}{2} \Rightarrow \text{only 1st good is demanded} \]

\[ \frac{p_1}{p_2} = \frac{1}{2} \Rightarrow \text{any } (x_1^1, x_2^1) \text{ on the budget line can be demanded.} \]
An Example III

That is, the demanded bundle \((x_1^1, x_2^1)\) will be such that: if \((x_1^1, x_2^1) \gg (0, 0)\).

\[
MRS_1 = \frac{p_1}{p_2}, \text{ i.e. } \frac{1}{2} = \frac{p_1}{p_2}
\]

\[
p_1 x_1^1 + p_2 x_2^1 = p_1 + \frac{p_2}{2}.
\]

Otherwise, only one good is demanded.

So, the plausible equilibrium price vector will have: \(\frac{p_1}{p_2} = \frac{1}{2}\). Why?

Let \((p_1, p_2) = (1, 2)\). At this price:

- For 2nd person, the demanded bundle \(x^2 = (x_1^2, x_2^2) = (1/2, 1/4)\)
- For 1st person, the bundle \(x^1 = (x_1^1, x_2^1) = (1/2, 3/4)\) lies on the budget line.
An Example IV

Therefore,
\((x^1, x^2)\), where \(x^1 = ((1/2, 1/4) \text{ and } x^2 = (1/2, 3/4))\), along with 
\((p_1, p_2) = (1, 2)\) is a competitive equilibrium.

**Remark**

WE exists even though preferences are not strictly quasi-concave.

**Question**

*For the above economy, suppose we are told that a WE exists. How can we find the WE?*

**Note**

- We know that WE is PO and is a Core allocation
- So, we can start with the set of PO points.
An Example V

The locus of tangencies of ICs, i.e, where

\[ MRS_1 = MRS_2, \text{ i.e. } \frac{1}{2} = \frac{x_2^2}{x_1^2} \]

\[ x_1^2 = 2x_2^2. \]

The only consistent point is

\[ \mathbf{x}^1 = (1/2, 3/4) \text{ and } \mathbf{x}^2 = (1/2, 1/4). \]

Now, question is:

- \( \mathbf{x}^1 = (1/2, 3/4) \) and \( \mathbf{x}^2 = (1/2, 1/4) \) is a WE?

- For what price vector, the utility maximizers persons 1 an 2 will choose \( \mathbf{x}^1 = (1/2, 3/4) \) and \( \mathbf{x}^2 = (1/2, 1/4) \), respectively?
An Example VI

Given the nature of the preferences: Try any \( p = (p_1, p_2) \) such that \( \frac{p_1}{p_2} = \frac{1}{2} \).

**Question**

*Consider, another PO allocation such as \((y^1, y^2)\) where \( y^1 = (1/4, 5/8) \) and \( y^2 = (3/4, 3/8) \). Can we induce it as WE?*

Yes, try this by keeping \( p = (p_1, p_2) \) such that \( \frac{p_1}{p_2} = \frac{1}{2} \), but by choosing

\[
T_1 = -\frac{1}{2} \quad \text{and} \quad T_2 = \frac{1}{2}
\]

Now, in equi. person 2 will consume \((x^2_1, x^2_2)\) such that: \( \frac{x^2_2}{x^2_1} = \frac{p_1}{p_2} \), i.e,

\[
p_2.x^2_2 = p_1.x^2_1 \quad \text{and} \quad p_1 x^2_1 + p_2 x^2_2 = p_1.0 + p_2.\frac{1}{2} + T_2
\]
You can check that $T_2 = \frac{1}{2}$ induces the 2nd person to buy $3/8$. When $T_1 = -\frac{1}{2}$ and $T_2 = \frac{1}{2}$, the only solution to 2’s problem is

$$(y_1^2, y_2^2) = (3/4, 3/8).$$

Also, person 1 demands $y^1 = (1/4, 5/8)$. 
The Second Fundamental Theorem of Welfare Economics:

**Theorem**

If $u^i$ is continuous, strictly increasing, and strictly quasi-concave for all $i = 1, \ldots, N$, then any Pareto optimum allocation, $y = (y^1, \ldots, y^N)$, such that $y^i >> 0$,

- can be achieved as competitive equilibrium with suitable transfers.
- That is, $y = (y^1, \ldots, y^N)$ is a WE with suitable transfer.
- With suitable transfers, market can achieve any of the socially desirable allocation as competitive equilibrium.
2nd Theorem: Transfer of Goods

Suppose, \( \mathbf{y} = (y^1, ..., y^N) \) is a feasible PO allocation, and we want to achieve allocation as \( \mathbf{y} = (y^1, ..., y^N) \) a competitive equilibrium. There are two solutions.

Choose, \( \mathbf{\dot{e}} = (\dot{e}^1, ..., \dot{e}^N) \) such that: For all \( i = 1, ..., N \)

\[
y^i = e^i + \dot{e}^i.
\]

It can be easily seen that there exists a price vector such that \( \mathbf{y} = (y^1, ..., y^N) \) a competitive equilibrium. Let \( \mathbf{\dot{p}} = (p'_1, ..., p'_M) \) be such a price vector.

Remark

Choose any \( \mathbf{\dot{e}} = (\dot{e}^1, ..., \dot{e}^N) \) such that the new endowment vectors \( (e^1 + \dot{e}^1, ..., e^N + \dot{e}^N) \) lies on the budget line generated by the price vector \( \mathbf{\dot{p}} = (p'_1, ..., p'_M) \).
2nd Theorem: Cash Transfer I

Consider an exchange economy \((u^i, e^i)_{i \in N}\). Let, 
\[ x = (x^1, \ldots, x^N) \] be an the equilibrium without transfers. Clearly, for all 
\[ i = 1, \ldots, N \]
\[ p \cdot x^i = p \cdot e^i \]

Now, consider ‘cash’ transfers; individual \(i\) gets \(T_i\). 
Let, \( y = (y^1, \ldots, y^N) \) be the equilibrium after ‘cash’ transfers. Now: For all 
\[ i = 1, \ldots, N \]
\[ p \cdot y^i = p \cdot e^i + T^i, \text{ i.e.,} \]

for all \( i = 1, \ldots, N \)
\[ \sum_{j=1}^{M} p_j y^i_j = \sum_{j=1}^{M} p_j e^i_j + T^i \] \hspace{1cm} (1)
That is,

\[
\sum_{i=1}^{N} \left( \sum_{j=1}^{M} p_j y^i_j \right) = \sum_{i=1}^{N} \left( \sum_{j=1}^{M} p_j e^i_j \right) + \sum_{i=1}^{N} T^i, \ i.e.,
\]

\[
\sum_{i=1}^{N} T^i = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} p_j y^i_j \right) - \sum_{j=1}^{M} \left( \sum_{i=1}^{N} p_j e^i_j \right)
\]

\[
= \sum_{j=1}^{M} p_j \left( \sum_{i=1}^{N} y^i_j - \sum_{i=1}^{N} e^i_j \right)
\]

\[
= 0.
\]