Macroeconomics: A Dynamic General Equilibrium Approach

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Lecture Notes, DSE

February 2-12, 2016
As we have stated before, modern macroeconomics is based on a dynamic general equilibrium approach which postulates that:

- Economic agents are continuously optimizing/re-optimizing subject to their constraints and subject to their information set up. They optimize not only over their current choice variables but also the choices that would be realized in future.
- All agents have rational expectations: thus their ex ante optimal future choices would ex post turn out to be less than optimal if and only if their information set was incomplete and/or there are some random elements in the economy which cannot be anticipated perfectly.
- The optimal choice of all agents are then mediated through the markets to produce an outcome for the macroeconomy.
This approach is ‘**dynamic**’ because agents are making choices over variables that relate to both present and future.

This approach is ‘**equilibrium**’ because the outcome for the macro-economy is the aggregation of individuals’ ‘equilibrium’ behaviour.

This approach is ‘**general equilibrium**’ because it simultaneously takes into account the optimal behaviour of different types of agents in different markets and ensures that all markets clear.
DSG Approach vis-a-vis Traditional Macroeconomics

- The need to build macro-models based on internally-consistent, dynamic optimization exercises of rational agents arose once it was realized that ad-hoc micro foundations for the aggregative system may not be consistent with one another.
- This begs the following question: Why do we need such optimization based micro-founded framework at all?
- Why cannot we just take the aggregative equations as a representation of the macro-economy and try to estimate various parameters, using aggregative data?
- After all, if we are ultimately interested in knowing how the macroeconomy would respond to various kinds of policy shocks, all that we need to do is to econometrically estimate the parameters of the aggregative system.
- Then from the estimated parameter values or coefficients, we can predict the implications of various policy changes.
Indeed, this is exactly how macroeconomic analysis was conducted traditionally!

As we have already seen, traditional macroeconomics was based on some aggregative behavioural relationship (e.g., Keynesian Savings Function - which postulates a relationship between aggregate income and aggregate savings; Phillips Curve - which posits a relationship between unemployment rate and inflation rate).

Often one would construct detailed behavioural equations for the macroeconomy and would try to estimate the parameters of these equations using time series data.

To be sure some of these equations would be dynamic in nature.

But **optimization** over time was not considered to be important or even relevant. Indeed, the concept of optimization itself - either by households or firms or even government - was rather alien in the field Macroeconomics.
The need to build macro models based explicitly on agents’ optimization exercises came from the so-called Lucas Critique.

Lucas (1976) argued that aggregative macro models which are estimated to predict outcomes of economic policy changes are useless simply because the estimated parameters themselves may depend on the existing policies.

As the policy changes, these coefficients themselves would change, thereby generating wrong predictions!

His solution was to build macroeconomic models with clear and specific microeconomic foundations - models that are explicitly based on agents’ optimization exercises.
Such models will enable us to differentiate between true parameters - primitives like tastes, technology etc - which are independent of the government policies, and variables that treated as exogenous by the agents but are actually endogenous and are influenced by government policies.

Moreover such models would take into account agents’ expectations about government policies.

Predictions based on such microfounded models would be more accurate than the aggregative models which club all the true parameters as well as other policy-related parameters together.
Let us see exactly what Lucas critique means in the context of a simple example.

Consider the Keynesian savings function, specified as an aggregative relationship:

$$S_t = \alpha_1 + \alpha_2 Y_t + \epsilon_t$$

An aggregative macro model would take the above behavioural relationship as given and would estimate the coefficients $\alpha_1$ and $\alpha_2$ from data.

We have already provided a micro-foundation for this kind of Keynesian Consumption/Savings function.

Let us re-visit that exercise.
Micro-foundation of Keynesian Savings Function:

- We assume that the economy consists of a finite number \( (H) \) of identical households. We can then talk in terms of a ‘representative’ household.

- Let us define a 2-period utility maximization problem of the representative household as:

\[
\text{Max. } \{c_t, c_{t+1}\} \quad \log(c_t) + \beta \log(c_{t+1})
\]

subject to,

\[
\begin{align*}
(i) \quad P_t c_t + s_t &= y_t; \\
(ii) \quad P_{t+1}^e c_{t+1} &= (1 + r_{t+1}^e) s_t + y_{t+1}^e.
\end{align*}
\]

- From (i) and (ii) we can eliminate \( S_t \) to derive the life-time budget constraint of the household as:

\[
P_t c_t + \frac{P_{t+1}^e c_{t+1}}{(1 + r_{t+1}^e)} = y_t + \frac{y_{t+1}^e}{(1 + r_{t+1}^e)}
\]
From the FONCs:

\[
\frac{c_{t+1}}{\beta c_t} = (1 + r^e_{t+1}) \left( \frac{P_t}{P^e_{t+1}} \right).
\]

Solving we get:

\[
P_t c_t = \frac{1}{(1 + \beta)} \left[ y_t + \frac{y^e_{t+1}}{(1 + r^e_{t+1})} \right]
\]

Thus

\[
s_t = \frac{\beta}{(1 + \beta)} y_t - \frac{1}{(1 + \beta)} \left[ \frac{y^e_{t+1}}{(1 + r^e_{t+1})} \right]
\]

Aggregating over all households:

\[
S_t = \frac{\beta}{(1 + \beta)} Y_t - \frac{1}{(1 + \beta)} \left[ \frac{Y^e_{t+1}}{(1 + r^e_{t+1})} \right]
\]

Notice that an aggregative model would equate \( \frac{\beta}{(1 + \beta)} \) to \( \alpha_2 \) and

\[\frac{1}{(1 + \beta)} \left[ \frac{Y^e_{t+1}}{(1 + r^e_{t+1})} \right]\]

\( \alpha_1 \).
While the coefficient $\alpha_2$ is indeed based on true parameters (primitives) and would therefore be unaffected by policy changes, coefficient $\alpha_1$ is not.

Any policy that changes the household’s expectation about its future income or future rate of interest rate would affect $\alpha_1$.

Thus predicting outcomes of such a policy based on the estimated values of the aggregative equations would be wrong.
The Lucas critique and the consequent logical need to develop a unified micro-founded macroeconomic framework which would allow us to accurately predict the macroeconomic outcomes in response to any external shock (policy-driven or otherwise) led to emergence of the modern dynamic general equilibrium approach.

As before, there are two variants of modern DGE-based approach:

- One is based on the assumption of perfect markets (the Neoclassical/RBC school). As is expected, this school is critical of any policy intervention, in particular, monetary policy interventions.
- The other one allows for some market imperfections (the New-Keynesian school). Again, true to their ideological underpinning, this school argues for active policy intervention.
However, both frameworks are similar in two fundamental aspects:

- Agents optimize over infinite horizon; and
- Agents are forward looking, i.e., when they optimize over future variable they base their expectations on all available information – including information about (future) government policies. In other words, agents have rational expectations.

We now develop the choice-theoretic frameworks for households and firms under the DGE approach.

As before, we shall assume that the economy is populated by $H$ identical households so that we can talk in terms of a representative household.
Household’s Choice Problem under Perfect Markets: Infinite Horizon

- Let us examine the consumption-savings choices of the representative household over infinite horizon when markets are perfect.
- To simplify the analysis, we shall only focus on the consumption choice of the household and ignore the labour-leisure choice (for the time being).
- At any point of time the household is endowed with one unit of labour - which it supplies inelastically to the market.
- We shall also ignore prices and the concomitant role of money and focus only on the ‘real’ variables.
- Let $a_t$ denote the asset stock of the household at the beginning of period $t$.
- Then Income of the household at time $t$: $y_t = w_t + r_t a_t$.
- We shall assume that savings of an household in any period are invested in various forms of assets (all assets have the same return), which augments the household’s asset stock in the next period.
If we do not allow intra-household borrowing, then the representative household $h$'s problem would given by:

$$
\begin{align*}
\max \sum_{t=0}^{\infty} \beta^t u(c_t^h) \\
\text{s.t.} \\
&c_t^h \leq w_t + r_t a_t^h \quad \text{for all } t \geq 0; \\
&a_{t+1}^h = w_t + r_t a_t^h + (1 - \delta) a_t^h - c_t^h; \quad a_t^h \geq 0 \text{ for all } t \geq 0; \quad a_0^h \text{ given.}
\end{align*}
$$
Notice that the household is solving this problem at time 0. Therefore, in order to solve this problem the households would have to have some expectation about the entire time paths of $w_t$ and $r_t$ from $t = 0$ to $t \to \infty$.

We shall however assume that households’ have rational expectations. In this model with complete information and no uncertainty, rational expectation is equivalent to perfect foresight. We shall use these two terms here interchangeably.

By virtue of the assumption of rational expectations/perfect foresight, the agents can correctly guess all the future values of the market wage rate and rental rate, but they still treat them as exogenous.

As atomistic agents, they believe that their action cannot influence the values of these ‘market’ variables.
Household’s Choice Problem: Infinite Horizon (Contd.)

- Notice that once we choose our consumption time path \( \{ c_t^h \}_{t=0}^\infty \), the corresponding time path of the asset level \( \{ a_{t+1}^h \}_{t=0}^\infty \) would automatically get determined from the constraint functions (and vice versa).

- So in effect in this constrained optimization problem, we only have to choose one set of variables directly. We call them the control variables. Let our control variable for this problem be \( \{ c_t^h \}_{t=0}^\infty \).

- We can always treat \( c_0, c_1, c_2, \ldots \) as independent variables and solve the problem using the standard Lagrangean method.

- The only problem is that there are now infinite number of such choice variables \( (c_0, c_1, c_2, \ldots, c_\infty) \) as well as infinite number of constraints (one for each time period from \( t = 0, 1, 2, \ldots, \infty \)) and things can get quite intractable.

- Instead, we shall employ a different method - called Dynamic Programming - which simplifies the solution process and reduces it to a univariate problem.
Consider the following canonical discrete time dynamic optimization problem:

\[
\text{Max. } \sum_{t=0}^{\infty} \beta^t \tilde{U}(t, x_t, y_t)
\]

subject to

(i) \( y_t \in \tilde{G}(t, x_t) \) for all \( t \geq 0 \);

(ii) \( x_{t+1} = \tilde{f}(t, x_t, y_t); x_t \in X \) for all \( t \geq 0 \); \( x_0 \) given.

Here \( y_t \) is the control variable; \( x_t \) is the state variable; \( \tilde{U} \) represents the instantaneous payoff function.

(i) specifies what values the control variable \( y_t \) is allowed to take (the feasible set), given the value of \( x_t \) at time \( t \);

(ii) specifies evolution of the state variable as a function of previous period’s state and control variables (state transition equation).
It is often convenient to use the state transition equation given by (ii) to eliminate the control variable and write the dynamic programming problem in terms of the state variable alone:

\[
\begin{align*}
\text{Max} & \quad \sum_{t=0}^{\infty} \beta^t U(t, x_t, x_{t+1}) \\
\text{subject to} & \quad (i) \quad x_{t+1} \in G(t, x_t) \text{ for all } t \geq 0; \quad x_0 \text{ given.}
\end{align*}
\]

We are going to focus on \textit{stationary} dynamic programming problems, where time \( t \) does not appear as an independent argument either in the objective function or in the constraint function:

\[
\begin{align*}
\text{Max} & \quad \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \\
\text{subject to} & \quad (i) \quad x_{t+1} \in G(x_t) \text{ for all } t \geq 0; \quad x_0 \text{ given.}
\end{align*}
\]
Stationary Dynamic Programming: Value Function & Policy Function

- Ideally we should be able to solve the above stationary dynamic programming problem by employing the Lagrange method. Let \( \{x_{t+1}\}_{t=0}^{\infty} \) denote such a solution.
- We can then write the maximised value of the objective function as a function of the parameters alone, in particular as a function of \( x_0 \):

\[
V(x_0) \equiv \text{Max}_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) ; \quad x_{t+1} \in G(x_t) \text{ for all } t \geq 0;
\]

\[
= \sum_{t=0}^{\infty} \beta^t U(x_t^*, x_{t+1}^*).
\]

- The maximized value of the objective function is called the value function.
- The function \( V(x_0) \) represents the value function of the dynamic programming problem at time 0.
Suppose we were to repeat this exercise again the next period i.e. at \( t = 1 \).

Now of course the time period \( t = 1 \) will be counted as the initial point and the corresponding initial value of the state variable will be \( x_1^* \).

Let \( \tau \) denote the new time subscript which counts time from \( t = 1 \) to \( \infty \). By construction then, \( \tau \equiv t - 1 \).

When we set the new optimization exercise (relevant for \( t = 1, 2, \ldots, \infty \)) in terms of \( \tau \) it looks exactly similar. In particular, the new value function will be given by:

\[
V(x_1^*) = \max_{\{x_{\tau+1}\}_{\tau=0}^{\infty}} \sum_{\tau=0}^{\infty} \beta^\tau U(x_\tau, x_{\tau+1}); \quad x_{\tau+1} \in G(x_\tau) \text{ for all } \tau \geq 0;
\]

\[
= \sum_{\tau=0}^{\infty} \beta^\tau U(x_\tau^*, x_{\tau+1}^*).
\]
Noting the relationship between $t$ and $\tau$, we can immediately see that the two value functions are related in the following way:

\[
V(x_0) = \sum_{t=0}^{\infty} \beta^t U(x_t^*, x_{t+1}^*)
\]

\[
= U(x_0, x_1^*) + \beta \sum_{t=1}^{\infty} \beta^{t-1} U(x_t^*, x_{t+1}^*)
\]

\[
= U(x_0, x_1^*) + \beta \sum_{\tau=0}^{\infty} \beta^\tau U(x_\tau^*, x_{\tau+1}^*)
\]

\[
= U(x_0, x_1^*) + \beta V(x_1^*).
\]

The above relationship is the basic functional equation in dynamic programming which relates two successive value functions recursively. It is called the **Bellman Equation**. It breaks down the infinite horizon dynamic optimization problem into a two-stage problem:

- what is optimal today ($x_1^*$);
- what is the optimal continuation path ($V(x_1^*)$).
Since the above functional relationship holds for any two successive values of the state variable, we can write the Bellman Equation more generally as:

\[ V(x) = \max_{\tilde{x} \in G(x)} \left[ U(x, \tilde{x}) + \beta V(\tilde{x}) \right] \text{ for all } x \in X. \tag{1} \]

The maximizer of the right hand side of equation (2) is called a policy function:

\[ \tilde{x} = \pi(x), \]

which solves the RHS of the Bellman Equation above.

If we knew the value function \( V(.) \) and were it differentiable, we could have easily found the policy function by solving the following FONC (called the Euler Equation):

\[ \tilde{x} : \frac{\partial U(x, \tilde{x})}{\partial \tilde{x}} + \beta V'(\tilde{x}) = 0. \tag{2} \]
Unfortunately, the value function is not known.

In fact we do not even know whether it exists; if yes then whether it is unique, whether it is continuous, whether it is differentiable etc.

A lot of theorems in Dynamic Programming go into establishing conditions under which a value exists; is unique and has all the nice properties (continuity, differentiability and others).

For now, without going into further details, we shall simply assume that all these conditions are satisfied for our problem.

In other words, we shall assume that for our problem the value function exists and is well-behaved (even though we do not know its precise form).

Once the existence of the value function is established, we can then solve the FONC (3) (the Euler Equation) to get the policy function.

But there is still one hurdle: what is the value $V'(\tilde{x})$?

Here the Envelope Theorem comes to our rescue.
Recall that $V(\tilde{x})$ is nothing but the value function for the next period where $\tilde{x}$ is next period’s initial value of the state variable (which is given - from next period’s perspective).

Since the Bellman equation is defined for all $x \in X$, we therefore get a similar relationship between $\tilde{x}$ and its subsequent state value ($\hat{x}$):

$$V(\tilde{x}) = \max_{\hat{x} \in \mathcal{G}(\tilde{x})} \left[ U(\tilde{x}, \hat{x}) + \beta V(\hat{x}) \right].$$

Then applying Envelope Theorem:

$$V'(\tilde{x}) = \frac{\partial U(\tilde{x}, \hat{x})}{\partial \tilde{x}}.$$

(3)

Combining the Euler Equation (3) and the Envelope Condition (4), we get the following equation:

$$\frac{\partial U(x, \tilde{x})}{\partial \tilde{x}} + \beta \frac{\partial U(\tilde{x}, \hat{x})}{\partial \tilde{x}} = 0 \text{ for all } x \in X.$$
Replacing $x, \tilde{x}, \hat{x}$ by their suitable time subscripts:

$$\frac{\partial U(x_t, x_{t+1})}{\partial x_{t+1}} + \beta \frac{\partial U(x_{t+1}, x_{t+2})}{\partial x_{t+1}} = 0; \ x_t \text{ given.} \quad (4)$$

Equation (5) is a difference equation which we should be able to solve to derive the time path of the state variable $x_t$ (and consequently that of the control variable $y_t$).

Since it is a difference equation of order 2, apart from the initial condition, we need another boundary condition.

Typically such a boundary condition is provided by the following transversality condition:

$$\lim_{t \to \infty} \beta^t \frac{\partial U(x_t, x_{t+1})}{\partial x_t} x_t = 0. \quad (5)$$
We now provide some sufficient conditions for the Value function of the above stationary dynamic programming problem to exist, to be twice continuously differentiable, to be concave etc.

We just state the theorems here without proof. Proofs can be found in Acemoglu (2009).

1. Let $G(x)$ be non-empty-valued, compact and continuous in all $x \in X$ where $X$ is a compact subset of $\mathbb{R}$. Also let $U : X_G \rightarrow \mathbb{R}$ is continuous, where $X_G = \{(x_t, x_{t+1}) \in X \times X : x_{t+1} \in G(x_t)\}$. Then there exits a unique and continuous function $V : X \rightarrow \mathbb{R}$ that solves the stationary dynamic programming problem specified earlier.

2. Let us further assume that $U : X_G \rightarrow \mathbb{R}$ is concave and is continuously differentiable on the interior of its domain $X_G$. Then the unique value function defined above is strictly concave and is differentiable.
Even when the dynamic programming problem is non-stationary, we can find analogous sufficient conditions that will ensure the existence, uniqueness, concavity and differentiability of the corresponding value function.

Then we can proceed exactly as above to write down the Bellman equation that relates the value functions of two successive time periods and then solve for the optimal policy function from the corresponding Euler Equation and the Envelope condition.

All the economic problems that we would be looking at in this course will satisfy these sufficiency properties.

So we shall stop bothering about this sufficiency condition from now on and focus on applying the dynamic programming technique to the economic problems at hand.

Interested students can look up Acemoglu (2009): Introduction to Modern Economic Growth, Chapter 6, for the theorems and proofs.
Recall that we had specified the representative household’s optimization problem under infinite horizon as:

\[
\text{Max. } \left\{ \{c^h_t\}_{t=0}^\infty, \{a^h_{t+1}\}_{t=0}^\infty \right\} \sum_{t=0}^\infty \beta^t u\left(c^h_t\right)
\]

subject to

(i) \(c^h_t \leq w_t + r_t a^h_t\) for all \(t \geq 0\);

(ii) \(a^h_{t+1} = w_t + r_t a^h_t + (1 - \delta) a^h_t - c^h_t\); \(a^h_t \geq 0\) for all \(t \geq 0\); \(a^h_0\) given.

However in specifying the problem, we assumed that there is no intra-household borrowing.

This assumption of no borrowing is too strong, and we do not really need it for the results that follow.

So let us relax that assumption to allow households to borrow from one another if they so wish.
Allowing for intra-household borrowings means that constraint (i) would no longer hold. A household can now consume beyond its current income at any point of time - by borrowing from others.

Allowing for intra-household borrowings also means that a household now has two forms of assets that it can invest its savings into:

1. physical capital \((k^h_t)\);
2. financial capital, i.e., lending to other households \((l^h_t \equiv -b^h_t)\).

Let the gross interest rate on financial assets be denoted by \((1 + \hat{r}_t)\).

Let physical capital depreciate over time at a constant rate \(\delta\). Then the gross interest rate on investment in physical capital is given by \((r_t + 1 - \delta)\).
Arbitrage in the asset market ensures that in equilibrium two interest rates are the same:

\[ 1 + \hat{r}_t = 1 + r_t - \delta \Rightarrow \hat{r}_t = r_t - \delta. \]

Thus we can define the total asset stock held by the household in period \( t \) as \( a^h_t \equiv k^h_t + l^h_t \).

Notice that \( l^h_t < 0 \) would imply that the household is a net borrower.

Hence the aggregate budget constraint of the household is now given by:

\[ c^h_t + s^h_t = w_t + \hat{r}_t a^h_t, \text{where } s^h_t \equiv a^h_{t+1} - a^h_t. \]

Re-writing to eliminate \( s^h_t \):

\[ a^h_{t+1} = w_t + (1 + \hat{r}_t)a^h_t - c^h_t. \]
Household’s Choice Problem: Ponzi Game

- But allowing for intra-household borrowing brings in the possibility of households’ playing a **Ponzi game**, as explained below.
- Consider the following plan by a household:
  - Suppose in period 0, the household borrows a huge amount $\bar{b}$ - which would allow him to maintain a very high level of consumption at all subsequent points of time. Thus
    \[ b_0 = \bar{b}. \]
  - In the next period (period 1) he pays back his period 0 debt with interest by borrowing again (presumably from a different lender). Thus his period 1 borrowing would be:
    \[ b_1 = (1 + \hat{r}_0)b_0. \]
  - In period 2 he again pays back his period 1 debt with interest by borrowing afresh:
    \[ b_2 = (1 + \hat{r}_1)b_1 = (1 + \hat{r}_1)(1 + \hat{r}_0)b_0. \]
    and so on.
Notice that proceeding this way, the household effectively never pays back its initial loan $\bar{b}$; he is simply rolling it over period after period.

In the process he is able to perpetually maintain an arbitrarily high level of consumption (over and above his current income).

His debt however grows at the rate $\hat{r}_t$:

$$b_{t+1} = (1 + \hat{r}_t) b_t$$

which implies that $\lim_{t \to \infty} a^h_t \sim - \lim_{t \to \infty} b^h_t \to -\infty$.

This kind scheme is called a **Ponzi finance scheme**.

If a household is allowed to play such a Ponzi game, then the household’s budget constraint becomes meaningless. There is effectively no budget constraint for the household any more; it can maintain any arbitrarily high consumption path by playing a Ponzi game.

To rule this out, we impose an additional constraint on the household’s optimization problem - called the **No-Ponzi Game Condition**.
One Version of **No-Ponzi Game (NPG) Condition**:

\[ \lim_{t \to \infty} \frac{a^h_t}{(1 + \hat{r}_0)(1 + \hat{r}_1)\ldots(1 + \hat{r}_t)} \geq 0. \]

This No-Ponzi Game condition states that as \( t \to \infty \), the present discounted value of an household’s asset must be non-negative.

Notice that the above condition rules out Ponzi finance scheme for sure.

- If you play Ponzi game then \( \lim_{t \to \infty} a^h_t \simeq - \lim_{t \to \infty} b^h_t \), when the latter term is growing at the rate \((1 + \hat{r}_t)\).
- For simplicity, let us assume interest rate is constant at some \( \bar{r} \). Then \( b^h_t = (1 + \bar{r})^t \bar{b} \).
- Plugging this in the LHS of the NPG condition above:

\[ \lim_{t \to \infty} \frac{a^h_t}{(1 + \bar{r})^t} \simeq \lim_{t \to \infty} \frac{(-b^h_t)}{(1 + \bar{r})^t} = \lim_{t \to \infty} \frac{-(1 + \bar{r})^t \bar{b}}{(1 + \bar{r})^t} = -\bar{b} < 0. \]

This surely violates the NPG condition specified above.
At the same time the NPG condition specified above is lenient enough to allow for some borrowing.

In fact the condition even permits perpetual borrowing as long as borrowing grows at a rate less than the corresponding interest rate.

To see this, suppose the household’s borrowing is growing at some rate $\bar{g} < \bar{r}$ such that

$$b_t^h = (1 + \bar{g})^t \bar{b}.$$ 

Plugging this in the LHS of the NPG condition above:

$$\lim_{t \to \infty} \frac{a_t^h}{(1 + \bar{r})^t} \sim \lim_{t \to \infty} \frac{-(b_t^h)}{(1 + \bar{r})^t} = \lim_{t \to \infty} -\frac{(1 + \bar{g})^t \bar{b}}{(1 + \bar{r})^t} = -\bar{b} \lim_{t \to \infty} \left(\frac{1 + \bar{g}}{1 + \bar{r}}\right)^t.$$ 

Notice that $\bar{g} < \bar{r}$ implies that the term $\left(\frac{1 + \bar{g}}{1 + \bar{r}}\right)$ is a positive fraction and as $t \to \infty, \left(\frac{1+\bar{g}}{1+\bar{r}}\right)^t \to 0.$
Since $\bar{b}$ is finite, this implies that in this case

$$\lim_{t \to \infty} \frac{a_t^h}{(1 + \bar{r})^t} \to 0.$$  

In other words, the NPG condition is now indeed satisfied - albeit at the margin!

Economically, this kind of borrowing behaviour implies that the debt of the agent is not exploding and the agent must have started repaying at least some part of it (though not all) from his own pocket!
Household’s Choice Problem - Revisited:

- After imposing the No-Ponzi Game condition, the household’s optimization problem now becomes:

\[
\begin{align*}
\text{Max.} & \quad \sum_{t=0}^{\infty} \beta^t u(c_t^h) \\
\{c_t^h\}_{t=0}^{\infty}, \{a_{t+1}^h\}_{t=0}^{\infty} & \quad \text{subject to} \\
(i) & \quad a_{t+1}^h = w_t + (1 + \hat{r}_t) a_t^h - c_t^h; \quad a_t^h \in \mathbb{R} \text{ for all } t \geq 0; \quad a_0^h \text{ given.} \\
(ii) & \quad \text{The NPG condition.}
\end{align*}
\]

- Here \(c_t^h\) is the control variable and \(a_t^h\) is the state variable.
We can now apply the dynamic programming technique to solve the household’s choice problem.

First let us use constraint (i) to eliminate the control variable and write the above dynamic programming problem in terms of the state variable alone:

\[
\text{Max. } \sum_{t=0}^{\infty} \beta^t u \left( \left\{ w_t + (1 + \hat{r}_t) a^h_t - a^h_{t+1} \right\} \right)
\]

Corresponding Bellman equation relating \( V(a^h_0) \) and \( V(a^h_1) \) is given by:

\[
V(a^h_0) = \text{Max} \left[ u \left( \left\{ w_0 + (1 + \hat{r}_0) a^h_0 - a^h_1 \right\} \right) + \beta V(a^h_1) \right].
\]
More generally, we can write the Bellman equation for any two time periods $t$ and $t+1$ as:

$$V(a^h_t) = \max_{a^h_{t+1}} \left[ u \left( \left\{ w_t + (1 + \hat{r}_t) a^h_t - a^h_{t+1} \right\} \right) + \beta V(a^h_{t+1}) \right].$$

Maximising the RHS above with respect to $a^h_{t+1}$, from the FONC:

$$u' \left( \left\{ w_t + (1 + \hat{r}_t) a^h_t - a^h_{t+1} \right\} \right) = \beta V'(a^h_{t+1}) \quad (6)$$

Notice that $V(a^h_{t+1})$ and $V(a^h_{t+2})$ would be related through a similar Bellman equation:

$$V(a^h_{t+1}) = \max_{a^h_{t+2}} \left[ u \left( \left\{ w_{t+1} + (1 + \hat{r}_{t+1}) a^h_{t+1} - a^h_{t+2} \right\} \right) + \beta V(a^h_{t+2}) \right].$$

Applying Envelope Theorem on the latter:

$$V'(a^h_{t+1}) = u' \left( \left\{ w_{t+1} + \hat{r}_{t+1} a^h_{t+1} - a^h_{t+2} \right\} \right) (1 + \hat{r}_{t+1}). \quad (7)$$
Combining (5) and (6):

\[
\begin{align*}
    u' \left( \left\{ w_t + \hat{r}_t a^h_t - a^h_{t+1} \right\} \right) \\
    = \beta u' \left( \left\{ w_{t+1} + \hat{r}_{t+1} a^h_{t+1} - a^h_{t+2} \right\} \right) (1 + \hat{r}_{t+1}).
\end{align*}
\]

The above equation implicitly defines a 2nd order difference equation is \( a^h_t \).

However we can easily convert it into a \( 2 \times 2 \) system of first order difference equations in the following way.
Noting that the terms inside the $u'(.)$ functions are nothing but $c^h_t$ and $c^h_{t+1}$ respectively, we can write the above equation as:

$$u'(c^h_t) = \beta u'(c^h_{t+1})(1 + \hat{r}_{t+1}). \tag{8}$$

We also have the constraint function:

$$a^h_{t+1} = w_t + (1 + \hat{r}_t)a^h_t - c^h_t; \ a^h_0 \text{ given.} \tag{9}$$

Equations (7) and (8) represents a $2 \times 2$ system of difference equations which implicitly defines the ‘optimal’ trajectories $\{c^h_t\}_{t=0}^{\infty}$ and $\{a^h_{t+1}\}_{t=0}^{\infty}$.

The two boundary conditions are given by the initial condition $a^h_0$, and the NPG condition.
Optimal Solution Path to Household’s Problem: An Example

- Let us look at an explicitly characterisation of the household’s optimal paths for a specific example.
- Suppose
  \[ u(c) = \log c \]
- Let us also assume that \( w_t = \bar{w} \) and \( r_t = \bar{r} \) for all \( t \).
- Then we can immediately get two difference equations characterizing the optimal trajectories for the household as:

  \[ c_{t+1}^h = \beta (1 + \bar{r}) c_t^h \]  \( (10) \)
  
  and

  \[ a_{t+1}^h = \bar{w} + (1 + \bar{r}) a_t^h - c_t^h; \quad a_0^h \text{ given.} \]  \( (11) \)

- The two equations along with the two boundary conditions can be solved explicitly to derive the time paths of \( c_t^h \) and \( a_t^h \).
Equation (9) is a linear autonomous difference equation, which can be directly solved (by iterating backwards) to get the optimal consumption path as:

$$c_t^h = \beta^t (1 + \bar{r})^t c_0^h.$$ (12)

However, we still cannot completely characterise the optimal path because we still do not know the optimal value of $c_0^h$. (Recall that $c_0^h$ is not given; it is to be chosen optimally).

Here the NPG condition comes in handy in identifying the optimal $c_0^h$. Note that the NPG condition in this case is given by:

$$\lim_{t \to \infty} \frac{a_t^h}{(1 + \bar{r})^t} \geq 0.$$
Now let us take the budget constraint of the household at any future date \( T > 0 \):

\[
a^h_{T+1} = \bar{w} + (1 + \bar{r})a^h_T - c^h_T.
\]

Iterating backwards,

\[
a^h_{T+1} = \bar{w} + (1 + \bar{r})a^h_T - c^h_T = \bar{w} + (1 + \bar{r}) \left[ \bar{w} + (1 + \bar{r})a^h_{T-1} - c^h_{T-1} \right] - c^h_T = \ldots
\]

\[
= \sum_{t=0}^{T} \left( \bar{w}(1 + \bar{r})^{T-t} \right) - \sum_{t=0}^{T} \left( c^h_t (1 + \bar{r})^{T-t} \right) + (1 + \bar{r})^{T+1}a^h_0.
\]

Rearranging terms:

\[
\frac{a^h_{T+1}}{(1 + r)^T} = \sum_{t=0}^{T} \left( \frac{\bar{w}}{(1 + r)^t} \right) + (1 + \bar{r})a^h_0 - \sum_{t=0}^{T} \left( \frac{c^h_t}{(1 + r)^t} \right)
\]
Now let $T \to \infty$. Then applying the NPG condition to the LHS, we get:

$$\sum_{t=0}^{\infty} \left( \frac{\bar{w}}{(1 + \bar{r})^t} \right) + (1 + \bar{r})a^h_0 - \sum_{t=0}^{\infty} \left( \frac{c^h_t}{(1 + \bar{r})^t} \right) \geq 0$$

i.e.,

$$\sum_{t=0}^{\infty} \left( \frac{c^h_t}{(1 + \bar{r})^t} \right) \leq \sum_{t=0}^{\infty} \left( \frac{\bar{w}}{(1 + \bar{r})^t} \right) + (1 + \bar{r})a^h_0. \quad (13)$$

Equation (12) represents the lifetime budget constraint of the household. It states that when the NPG condition is satisfied, then the discounted life-time consumption stream of the household cannot exceed the sum-total of its discounted life-time wage earnings and the returns on its initial wealth holding.

It is easy to see that even though we have specified the NPG condition in the form of an inequality, the households would always satisfy it at the margin such that it holds with strict equality.
Given that equation (12) holds with strict equality, we can now identify the optimal value of $c_0^h$.

We had already derived the optimal time path of $c_t^h$ as:

$$c_t^h = \beta^t (1 + \bar{r})^t c_0^h.$$  

Using this in equation (12) above, we get:

$$\sum_{t=0}^{\infty} \left( \frac{\beta^t (1 + \bar{r})^t c_0^h}{(1 + \bar{r})^t} \right) = \sum_{t=0}^{\infty} \left( \frac{\bar{w}}{(1 + \bar{r})^t} \right) + (1 + \bar{r})a_0^h$$

\[ \Rightarrow \sum_{t=0}^{\infty} (\beta^t) c_0^h = \left[ \sum_{t=0}^{\infty} \left( \frac{\bar{w}}{(1 + \bar{r})^t} \right) + (1 + \bar{r})a_0^h \right] \]

\[ \Rightarrow c_0^h = (1 - \beta) \left[ \sum_{t=0}^{\infty} \left( \frac{\bar{w}}{(1 + \bar{r})^t} \right) + (1 + \bar{r})a_0^h \right]. \]
So for this particular example, we have been able to explicitly solve for the optimal consumption path of the households.

But there is a problem that we still need to sort out.

Recall that while discussing the dynamic programming problem we had specified a transversality condition (TVC) as one of our boundary condition (Refer to equation (5) specified earlier).

Then in defining the household’s problem with intra-household borrowing, we have introduced the NPG condition as another boundary condition.

So we now have a problem of plenty: for a $2 \times 2$ dynamic system, it seems that we have three boundary conditions!!

Between the TVC and the NPG condition, which one should we use to characterise the solution?

As it turns out, along the optimal path the NPG condition and the TVC become equivalent.
Household’s Problem: Optimal Solutions (Contd.)

- To see this, let us take a closer look at the TVC as had been specified earlier in equation (5)

- In the context of the current problem, this transversality condition would be given by (derive this yourself):

  \[
  \lim_{t \to \infty} \beta^t u'(c_t^h)(1 + \hat{r}_t) a_t^h = 0
  \]

- For our specific example with log utility and constant factor prices, this condition reduces to

  \[
  \lim_{t \to \infty} \beta^t \frac{1}{c_t^h} (1 + \bar{r}) a_t^h = 0
  \]

- Now given the solution path of \( c_t^h \), we can further simplify the above condition to:

  \[
  \lim_{t \to \infty} \beta^t \frac{1}{\beta^t (1 + \bar{r}) t c_0^h} (1 + \bar{r}) a_t^h = 0 \Rightarrow \lim_{t \to \infty} \frac{a_t^h}{(1 + \bar{r})^t} = 0 .
  \]

- But this is nothing but our earlier NPG condition - now holding with strict equality!