Proof of Weitzman Theorem

Let MB = a-bq -- (1)

MC = $\alpha + \beta q + u$ where u~ $(0, \sigma^2)$ -- (2)

Regulator maximises W = E[Net social benefit] = $E\int_0^q \left[MB(q) - MC(q,u)\right]dq$ --(3)

Procedure: (i) Obtain optimal values of q* and p* substtg these successively into (3) get EWG_{quota}, EWG_{tax} (EWG = expected welfare gain)

(ii) Compare the two, EWG_{tax} EWG_{quota} to measure the expected net benefit of one policy over another.

Optimal quota: Choose q* to max E (NSB) $\Rightarrow$ FOC - $\dfrac{\partial E(NSB)}{\partial q^*}$ = 0

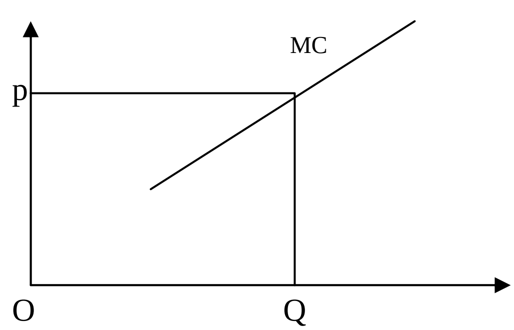
$\Rightarrow$ E $[MB - MC]$= 0 $\Rightarrow$ E$\left[a - bq^* - (\alpha + \beta q^* + u)\right]$ = 0 $\Rightarrow$ a-bq* = $\alpha + \beta q^*$

or q* = $\dfrac{a-\alpha}{b+\beta}$

So, EWG_{quota} = E$\int_o^{q^*=\frac{a-\alpha}{b+\beta}}$ $\left[a - bq - \alpha - \beta q - u\right]$ dq

$$= \mathrm{E}\left[(a-\alpha)q - \frac{(b+\beta)^{q2}}{2}\right]_o^{q^*} = \frac{1}{2}\left[\frac{(a-\alpha)2}{b+\beta}\right] \quad \text{--(4)}$$

<u>Optimal tax</u> (P*) : derive the firm's reaction function, i.e., the q that is produced by any P- MC (q,u) = P - firms treat u as certain since they <u>know</u> their MC -- invert MC - q = h (P,u) q = h (P,u) gives the qty abated (a random variable) as a function of P -- this is the firm's reaction function, i.e., the q produced by any P.

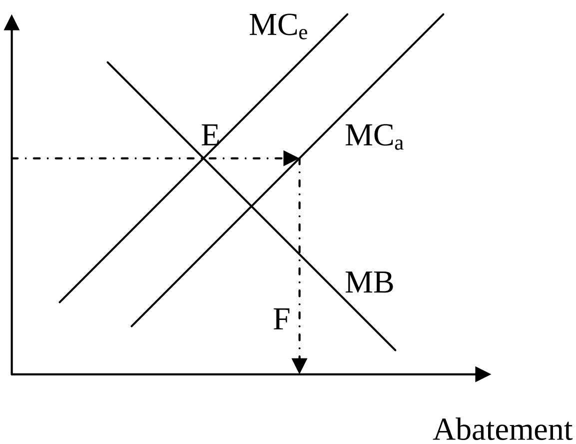


To pick optimal tax, set P to maximise NSB

FOC $\quad \frac{\partial E}{\partial P}(NSB) = \frac{\partial E}{\partial q}(NSB)\frac{\partial q}{\partial P} = \frac{\partial E}{\partial h}(NSB)\frac{\partial h}{\partial P} = 0$

$$= \text{E}\left[MB\frac{(h(p,u))}{q} - MC(h(p,u),u)\right]\frac{\partial h}{\partial P} = 0$$

Note, MB (q) = MB (h (p,u)) - MB is a function of u because the level of abatement is uncertain under a tax (<u>unlike the case w/ quotas</u>) e.g., set tax = p - you think you'll get to E on MB but actually you get to F



Now, since MC (q,u) = $\alpha + \beta q + u = P$

Solve for q, $\quad q = \frac{1}{\beta}(P - \alpha - u) = h(P,u)$

Substitute for q in (5): E $\left[a - \frac{b}{\beta}(P^* - \alpha - u) - \alpha - \frac{\beta}{\beta}(P^* - \alpha - u) - u\right]\frac{1}{\beta} = 0$

Or, $\quad (p^* - \alpha)\frac{(\beta + b)}{\beta} = a - \alpha \qquad\qquad \frac{\partial h}{\partial p} = \frac{1}{\beta}$

Optimal tax (P*): $\Rightarrow \frac{1}{\beta}(P^* - \alpha) = \frac{a - \alpha}{\beta + b}$ recall E(u) = 0 --(6)

So, EWG_{tax} = E $\int_0^{q=h(P^*,u)}[MB(q) - MC(q,u)]dq$

Aside : h (P*,u) = $\frac{1}{\beta}(p^* - \alpha - u) = \frac{1}{\beta}(p^* - \alpha) - \frac{u}{\beta}$

from (6) h (P*,u) = $\frac{a-\alpha}{\beta+b}-\frac{u}{\beta}$

So, EWG_{tax} = E $\int_0^{\frac{a-\alpha-\frac{\mu}{\beta}}{\beta+b}} \left[MB(q)-MC(q,u)\right]dq$

$$= E \int_0^{\frac{a-\alpha}{\beta+b}-\frac{u}{\beta}} \left(a-bq-\alpha-\beta q-u\right)dq$$

$$= E \left[(a-\alpha)q-uq-\frac{(b+\beta)q^2}{2}\right]_0^{\frac{a-\alpha}{\beta+b}-\frac{u}{\beta}}$$

Skipping a few steps--- Recall $E(u^2)$

$$= \frac{(a-\alpha)^2}{b+\beta}+\frac{\sigma^2}{\beta}-\frac{1}{2}.\frac{(a-\alpha)^2}{(b+\beta)^2}-\frac{\sigma^2}{2\beta^2}.(b+\beta)$$

$$= \frac{1}{2}.\left[\frac{(a-\alpha)^2}{b+\beta}\right]+\frac{\sigma^2}{2\beta^2}(2\beta-b-\beta)$$

same as (4) EWG_{quota}

So, EWGtax - EWGquota = $\frac{\sigma^2}{2\beta^2}(\beta-b)$ This is the key result. If β > b then

tax better & vice-verse. If $\beta=b$ EWG_{tax} = EWG_{quota}

Note that $E(u^2)$ = σ^2 affects the magnitude of the difference between EWG under the two policies (i.e., it reflects the distance b/w MAC_e & MAC_t), but it does not affect the choice of the policy instrument.