202: Dynamic Macroeconomic Theory

Public Investment, Political Economy & Growth: Barro (1990) to Alesina-Rodrk (1994)

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Why growth rates differ across countries: Differences in Infrastructure

- Recall that we started the course with the fundamental question: What explains the difference in levels as well as rates of growth of per capita income in the rich vis-a-vis the poor economies?
- The obvious answer is: differences in factor productivity. But this begs the following question: Why does factor productivity differ so much across the rich (developed) vis-a-vis the poor (developing) nations?
- A proximate cause for low productivity in poor economies is the lack of infrastructure (roads; irrigation; electricity; maintenance of law and order). But since most of these infrastructural inputs are typically provided by the government, why does not the governments in poor countries spend enough to improve the instrustructure which will in turn boost growth?
  - Is it because they lack resources?
    or
  - Is it because they lack the will?

Public Investment, Political Economy & Growth

- Barro; 1990 (Government Spending in a Simple model of Endogenous Growth; JPE) provides a framework which links the rate of growth of an economy to goverment spending on infrastructure.
- But the Barro model is a pure endogenous growth model **without any distributional considerartion**.
- Alesina & Rodrik; 1994 (Distributive Politics and Economic Growth; QJE) extends the Barro model to incorporate distributional conflict across agents and analyses its implication for the pattern of public investment and growth.
- We shall start with the description of the Barro model and then move on to Alesina-Rodrik.

The Barro Model of Endogenous Growth: Economic Structure

- In Barro model growth occurs due to infrastructural investment made by the government (in terms of improved roads, improved law and order etc.), which enhances labour productivity.
- However the infrastructural investment does not come for free. It has to be financed by taxing the household/firm income.
- This mode of financing the infrastructural inputs creates an externality across households/firms which (as we shall see) generates non-decreasing returns to capital for the aggregate economy - paving the way for perpetual (and endogenous) growth.

Barro Model: Economic Structure

- A single final commdity is produced - which can be either consumed or invested in physical capital.
- The economy consists of S identical firms and H **identical** households.
- Since all households are identical, one can immediately see that there is no scope for any distributional conflict in this model.
- Each household consists on a single **infinitely lived** member, with a given labour endowment of $\bar{l}$. They also have identical initial capital holding of k_0.
- There is no population growth, which implies that the size of labour force in every period is constant, given by: $L_t = H\bar{l}$.
- Also, $S = H$; so that there is no difference between the per capita, per household and per firm value of a variable.

Barro Model: Production Side Story

- Each firm is endowed with a production technology which uses labour(L), physical capital (K) and a governement- provided infrastructural input (g).
- We shall assume a specific functional form given by:

$$F(K_i, g, L_i) = (K_i)^{\alpha} (gL_i)^{1-\alpha}, \ 0 < \alpha < 1.$$

- Since g is provided by the government, the firms treat this as exogenous and choose the optimal level of the firm specific inputs (K_i and L_i) so as to maximise profit.
- Notice that the production function F is concave and CRS in the firm specific inputs, K_i and l_i, but actually exhibits IRS when we consider all the three factors: K, L and g .
- But the firms do not internalise the increasing returns; in their perception the production function is CRS.
- This allows perfect competition to prevail in the market economy.

Barro Model: Production Side Story (Contd.)

- Each firm take the market wage rate (w_t) and the rental rate for capital (r_t) as given and employ capital and labour such that

$$\begin{aligned} w_t &= (1-\alpha)\, g_t^{1-\alpha} (K_{i_t})^{\alpha} (L_{i_t})^{-\alpha} \\ r_t &= \alpha g_t^{1-\alpha} (K_{it})^{\alpha-1} (L_{it})^{1-\alpha} \end{aligned}$$

- Since all firms are identical and $S = H$, the per firm and per household values would be the same.
- Hence we shall ignore the firm-specific subscript (i) from now on and write:

$$\begin{aligned} L_{it} &\equiv \frac{L_t}{S} = \frac{L_t}{H} = \bar{l} \\ K_{it} &\equiv \frac{K_t}{S} = \frac{K_t}{H} = k_t \end{aligned}$$

where k_t is the per capita (average) capital holding at any point of time t (which is different from the *capital-labour* ratio of the economy, the latter being defined as $\tilde{k}_t \equiv \frac{K_t}{L_t} = \frac{k_t}{\bar{l}}$).

Barro Model: Production Side Story (Contd.)

- The market wage rate and the market rental rate on the other hand is determined by the demand and supply of each factors such that there is full employment of both the factors.
- Accordingly, aggregate output:

$$
\begin{aligned}
Y_t &= \sum_{i=1}^{S} F(K_i, g, l_i) = S\left(K_{it}\right)^{\alpha}\left(gl_{it}\right)^{1-\alpha} = g_t^{1-\alpha}\left(SK_{it}\right)^{\alpha}\left(Sl_{it}\right)^{1-\alpha} \quad (1)\\
&= g_t^{1-\alpha}\left(K_t\right)^{\alpha}\left(L_t\right)^{1-\alpha}
\end{aligned}
$$

- Hence the market wage rate and market rental rate are given by:

$$
\begin{aligned}
w_t &= (1-\alpha)\, g_t^{1-\alpha}\left(K_t\right)^{\alpha}\left(L_t\right)^{-\alpha} = (1-\alpha)\, g_t^{1-\alpha}\left(Hk_t\right)^{\alpha}\left(H\bar{l}\right)^{-\alpha}\\
&= (1-\alpha)\, g_t^{1-\alpha}\left(k_t\right)^{\alpha}\left(\bar{l}\right)^{-\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
r_t &= \alpha g_t^{1-\alpha}\left(K_t\right)^{\alpha-1}\left(L_t\right)^{1-\alpha} = \alpha g_t^{1-\alpha}\left(Hk_t\right)^{\alpha-1}\left(H\bar{l}\right)^{1-\alpha}\\
&= \alpha g_t^{1-\alpha}\left(k_t\right)^{\alpha-1}\left(\bar{l}\right)^{1-\alpha}
\end{aligned}
$$

Barro Model: Household's Income

- Notice that firms are competitive and therefore earn zero profits. Thus the entire output of a firm is distributed to a household in the form of wage and rental income.
- Thus income of a household:

$$y_t = w_t \bar{l} + r_t k_t$$

- Plugging back the actual values of w_t and r_t, one can see that a household's income and a firm's output are identical (which is consistent with the assumption that firms are competitive and earn zero profits.):

$$w_t \bar{l} + r_t k_t = F(k_t, g_t, \bar{l}) = (\bar{l})^{1-\alpha} g_t^{1-\alpha} (k_t)^{\alpha} \equiv f(k_t, g_t).$$

Barro Model: Financing of the Infrastructural Input

- The government provides the infrastructural input g **upfront** (before the production takes place) and imposes a tax **post-production** (in the same period) to recover the cost that it had incurred in providing the input.
- The government can either choose the level of g, and set the tax rate accordingly; or it can choose the tax rate, and determine the level of g to be provided residually. We shall assume the latter.
- We shall assume that the government finances g_t by taxing the household's income (post-production) y_t at some pre-determined tax rate τ.
- Since houshold's income and firm's output are identical, equivalently we can assume the the government imposes a production tax on the firms (post production). However we shall stick to the household income taxation interpretation.

Barro Model: Financing of the Infrastructural Input (Contd.)

- Notice that total investment on infrastructure by the government: $G_t = Sg_t$.
- On the other hand total tax revenue collected by the government: $T_t = \tau Y_t = \tau H y_t$.
- From the balanced budget condition of the governemnt, it therefore follows that

$$
\begin{aligned}
g_t &= \tau y_t = \tau \left(\bar{l}\right)^{1-\alpha} g_t^{1-\alpha} \left(k_t\right)^{\alpha} \\
\Rightarrow & \left(\frac{g_t}{k_t}\right)^{\alpha} = \tau \left(\bar{l}\right)^{1-\alpha} \\
\Rightarrow & \; g_t = \left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} \left(\tau\right)^{\frac{1}{\alpha}} k_t.
\end{aligned}
$$

Barro Model: Financing of the Infrastructural Input (Contd.)

- Notice that this relationship between g_t and $\bar{k}_t$is known to the **ominiscient** social planner, but not to the **atomistic** firms.
- Thus for firms the production function is still given by:

$$F(k_t, g_t, \bar{l}) = g_t^{1-\alpha} \left(\bar{l}\right)^{1-\alpha} \left(k_t\right)^{\alpha} \equiv f\left(k_t, g_t, \right).$$

- But for the social planner the per capita production function looks as follows:

$$\left(\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} \left(\tau\right)^{\frac{1}{\alpha}} k_t\right)^{1-\alpha} \left(\bar{l}\right)^{1-\alpha} \left(k_t\right)^{\alpha} = \left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} \left(\tau\right)^{\frac{1-\alpha}{\alpha}} k_t \equiv \phi(k_t).$$

Barro Model: Financing of the Infrastructural Input (Contd.)

- Accordingly, private (market) return to capital (under perfect foresight) is given by:

$$r_t = \alpha g_t^{1-\alpha} \left(\bar{l}\right)^{1-\alpha} \left(k_t\right)^{\alpha-1} = \alpha \left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} \left(\tau\right)^{\frac{1-\alpha}{\alpha}}$$

- And the corresponding social return is given by:

$$\phi'\left(k_t\right) = \left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} \left(\tau\right)^{\frac{1-\alpha}{\alpha}}$$

- Notice that the social return to capital formation is higher that the private (market) return.
- This is due to positive externality(complementarity) across the two inputs which works through the taxation scheme.
- An increase in capital stock generates more output - which in turn generates more tax revenue (through the proprtional income tax) - which creates more infrastructure - which in turn generates even more output.

Barro Model: Household Preferences

- Preferences of the single-member infinitely-lived representative household is denoted by the following life-time utility function:

$$U_0 = \int_{t=0}^{\infty} \log(c_t) \exp^{-\rho t} dt; \;\; \rho > 0.$$

- It is easy to very that the log specification of the utility function satisfies all the standard properties, namely,

$$u'(c) > 0; \;\; u''(c) < 0; \;\; \lim_{c \to 0} u'(c) = \infty; \;\; \lim_{c \to \infty} u'(c) = 0.$$

- In the market economy each household maximises the above utility function subject to its bugdet constraint.
- The social planner in benevolent; so he maximises the same utility function, but his budget constraint would be different than that of the household.

Romer Model: Social Planner's Problem

- The social planner carries out production using the given technology, and distributes the per capita output between per capita consumption and per capita investment.
- Thus the dynamic optimization problem of the social planner is given by:

$$\int_{t=0}^{\infty} \log\left(c_t\right) \exp^{-\rho t} dt \tag{I}$$

subject to its **post-tax** (per capita) budget constraint (after deducting the cost of provision of g from the final output):

$$\dot{k} = (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} (\tau)^{\frac{1-\alpha}{\alpha}} k_t - c_t;\ k_t \geqq 0,\ \bar{k}_0 \text{ given.}$$

Romer Model: Problem of the Market Economy

- The corresponding problem for a household operating in the market economy is given by:

$$\int_{t=0}^{\infty} \log\left(c_t\right) \exp^{-\rho t} dt \tag{II}$$

subject to its **post-tax** budget constraint:

$$\dot{k} = (1-\tau)\left[w_t \bar{l} + r_t k_t\right] - c_t; \;\; k(t) \geqq 0, \;\; k_0 \text{ given.}$$

Solution to the Social Planner's Problem:Characterization of the Optimal Path

- It can be easily shown from the FONCs that the dynamic equations for the Social Planner would be given by:

$$\frac{\dot{c}}{c} = (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \rho; \tag{2}$$

$$\frac{\dot{k}}{k} = (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \frac{c_t}{k_t} \tag{3}$$

$$\text{TVC: } \lim_{t\to\infty} \exp^{-\rho t} u'(c_t)k_t = 0$$

- These two equations along with the TVC now characterise the optimal path for the social planner.

Social Planner's Problem: Characterization of the Optimal Path (Contd.)

- We shall focus on the **balanced growth path**: the path where all variable in the economy grow at constant rates.
- Now from (1), $\frac{\dot{c}}{c}$ is already a constant. On the other hand, $\frac{\dot{k}}{k}$ would be a constant if and only if the last term in the RHS of (2) remains constant over time. But that can happen only if c_t and k_t grow at the same rate.
- Thus along a balanced growth path c_t and k_t *must* grow at the same rate.
- Hence for this planned economy, the balanced growth path is characterized by

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \rho.$$

(Verify that this balanced growth path indeed statisfies the TVC).

Corresponding Problem for the Competitive Market Economy:

- Recall that the only difference between the social planner problem and the household' problem in the market economy is in terms of the per capita production function: The social planner knows that the per capita output is given $\phi(k)$ while the household/firm reads the per capita output as $f(k, g)$.
- It can easily shown from the FONCs that the dynamic equations for the market economy would be given by:

$$\frac{\dot{c}}{c} = (1-\tau)r_t - \rho; \tag{4}$$

$$\frac{\dot{k}}{k} = (1-\tau)\frac{w_t \bar{l} + r_t k_t}{k_t} - \frac{c_t}{k_t} \tag{5}$$

The Competitive Market Economy: Characterization of the Optimal Path

- Assuming perfect foresight on the part of the households, and thereby substituting $g_t = (\tau)^{\frac{1}{\alpha}} k_t$ in the w_t and r_t equations we get,

$$\begin{aligned} w_t &= (1-\alpha)\left(\bar{l}\right)^{\frac{1-2\alpha}{\alpha}} (\tau)^{\frac{1-\alpha}{\alpha}} k_t; \\ r_t &= \alpha\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} (\tau)^{\frac{1-\alpha}{\alpha}} \end{aligned}$$

- Hence, from above:

$$\frac{\dot{c}}{c} = \alpha(1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} (\tau)^{\frac{1-\alpha}{\alpha}} - \rho; \tag{6}$$

$$\frac{\dot{k}}{k} = (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} (\tau)^{\frac{1-\alpha}{\alpha}} - \frac{c_t}{k_t} \tag{7}$$

$$\text{TVC: } \lim_{t\to\infty} \exp^{-\rho t} u'(c_t) k_t = 0$$

- These two equations along with the TVC now characterise the optimal path for the competitive market economy (under perfect foresight).

- Once again one can find out the balanced growth path for this economy.
- Arguing as before, it can be shown that along the optimal balanced growth trajectory for this competitive market economy c_t and k_t *must* grow at the same rate so that:

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \alpha(1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \rho.$$

- Clearly the growth rate of per capita consumption/output **is lower** in the competetive market economy than in the planned economy.
- However that the initial **level** of consumption would be **higher** in the competetive market economy than in the planned economy. (**How & Why?**)

Comparison between the Planned Economy and the Competitive Market Economy

- In fact one can precisely calculate the initial level of consumption (c_0) along the optimal path for the two economies in the following way:
  - RecallLike that for the planned economy:

$$\frac{\dot{k}}{k} = (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \frac{c_t}{k_t}$$

  - At the same time, by from the balanced growth condition:

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \rho \equiv \gamma^P(\tau) \quad \text{(say)}.$$

  - Then we can write:

$$\begin{aligned} (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \frac{c_t}{k_t} &= (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \rho \\ \Rightarrow \quad c_t &= \rho k_t. \end{aligned}$$

  - This relationship holds for all t along the optimal path. Hence it must hold for c_0 as well.

Planned Economy and Competitive Market Economy (Contd.)

- Thus for a given k_0, the *optimal* initial level of consumption for the planned economy is given by:

$$c_0^P = \rho k_0 \tag{8}$$

- Likewise, for the market economy,
  - $$\frac{\dot{k}}{k} = (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \frac{c_t}{k_t}$$
  - At the same time, by from the balanced growth condition:

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \alpha(1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \rho \equiv \gamma^M(\tau) \quad \text{(say)}.$$

  - Then we can write:

$$(1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \frac{c_t}{k_t} = \alpha(1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \rho$$

Planned Economy and Competitive Market Economy (Contd.)

- Simplifying:

$$c_t = \left[\rho + (1-\alpha)(1-\tau)(\bar{l})^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}}\right] k_t.$$

- This relationship holds for all t along the optimal path. Hence it must hold for c_0 as well.
- Thus for a given k_0, the *optimal* initial level of consumption for the market economy is given by:

$$c_0^M = \left[\rho + (1-\alpha)(1-\tau)(\bar{l})^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}}\right] k_0 \qquad (9)$$

- It is easy now to check (from (8) and (9)) that starting with the same k_0,

$$c_0^M > c_0^P \quad \text{while} \quad \gamma^P(\tau) > \gamma^M(\tau).$$

Planned Economy and Competitive Market Economy (Contd.)

- The intution for this result is straight-forward.
- Since the agents in the market economy under-estimate the marginal returns from savings, they under-invest and over-consume.
- As a result initially the consumption level in the market economy is high, although later the planned economy overtakes the market economy because consumption (as well as capital stock) in the latter economy is growing faster.

Barro Model: Growth Maximising Tax Rate

- Notice that the Barro model provides a more direct role of the government in the growth process.
- In the Barro model **the government can directly influence the growth rate** (both for the planned economy and well as for the market economy) by changing the tax rate τ.
- In fact the relationship between tax rate and growth rate is not monotonic.
- One can easily calculate the '**growth maximixing**' tax rate (for both the economies) as:

$$\tau^* = 1 - \alpha$$

- Economic explanation?

Barro Model: Some Additional Insights

- Notice that the growth rate in the Barro model also depends on agents' given labour endowment ($\bar{l}$).
- Indeed, the higher is $\bar{l}$, the greater is the growth rate. This feature is called the '**scale effect**'.
- The reason for the scale effect is as follows: as productivity per unit of labour improves due to complementary investment in infrastructure, it spreads automatically over all labour units. Thus the total labour endowment acts as a multiplier in improving total factor productivity.
- The higher is the labour endowment, the greater is the multiplier effect; hence scale effect shows up in the balanced growth equation.
- This is a slightly uncomfortable feature of many of the first generation endogenous growth models, which has been taken care of in later generation of models.

Barro Model: Welfare Maximising Tax Rate

- We have seen that the when the economy is on a balanced growth path, the growth-maximixing tax rate is given by $\tau^* = 1 - \alpha$.
- **But is this tax rate also welfare maximising?**
- Notice that along the balanced growth path the maximised value of the lifetime utility of an agent is given by:

$$U_0^*(\tau) = \int_{t=0}^{\infty} \log\left(c_0 \exp^{\gamma(\tau)t}\right) \exp^{-\rho t} dt;$$

  where $\gamma(\tau)$ is the balanced growth rate in the economy for any given τ such that

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \gamma(\tau).$$

- Now consider the social planner problem:
  - What tax rate should he choose so that $U_0^*(\tau)$ is maximised?
  - Will this welfare maximizing tax rate be the same as the growth maximising tax rate?

Barro Model: Welfare Maximising Tax Rate (Contd.)

- From above we can write themaximised value of the lifetime utility of an agent in a socially planned economy as

$$U_0^{*P}(\tau) = \int_{t=0}^{\infty} \log\left(c_0^P \exp^{\gamma^P(\tau)t}\right) \exp^{-\rho t}\, dt;$$

where

$$c_0^P = \rho k_0$$

and

$$\gamma^P(\tau) = (1-\tau)\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}(\tau)^{\frac{1-\alpha}{\alpha}} - \rho.$$

- Plugging these values and simplifyng the integral, one can easily show that $U_0^{*P}(\tau)$ is maximised at a tax rate

$$\tau^{**} = 1-\alpha$$

which is the same as the growth maximising tax rate. **(Verify)**

Barro Model: Welfare Maximising Tax Rate (Contd.)

- The economic intuition behind this result is as follows:
  - When the governement takes away τ proportion of a households income, there is a direct negative impact on the household's income by the margin of τ.
  - At the same time, the taxed amount is invested in provision on infrastructure, that in turn has an indirect positive impact on the household's income through g - because both $w_t = (1-\alpha)\, g_t^{1-\alpha} (k_t)^{\alpha}$ and $r_t = (1-\alpha)\, g_t^{1-\alpha} (k_t)^{\alpha-1}$ go up.
  - Notice that as g increases, the wage and the rental return increases symmetrically and the marginal impact of an increase in g (as a proportion of total income) is precisely $1-\alpha$.
  - Thus when $\tau = 1-\alpha$, the positive and the negative impact exactly balance each other.
- Thus $\tau = 1-\alpha$ is the *natural efficiency condition* whereby what the government takes away (as a proportion of total household income) exactly matches what it returns in terms of productivity gain (as a proportion of total household income). Since the gain exactly matches the corresponding loss, households' welfare is maximsed.

Bringing in Political Economy Consideration in Barro Model: Heterogeneous Households

- Recall that in the Barro model all households are identical - so there is no political economy/redistributive consideration.
- Let us now assume that **households differ in terms of their initial asset holdings.**
- Suppose households now have same labour endowment ($\bar{l}$), but different capital endowments, such that:

$$k_0^1 > k_0^2 > k_0^3 > ..... > k_0^H.$$

- Firms are still identical although we can no longer assign each household to a firm now, because households are no longer identical. However, for the sake of convenience, we shall continue to assume that $S = H$.
- However each firm still employs capital and labour by equating their marginal products to the given wage rate and rental rate:

$$w_t \;=\; (1-\alpha)(g)^{1-\alpha}\left(K_t^i\right)^{\alpha}\left(L_t^i\right)^{-\alpha}$$

Barro Model with Heterogeneous Households: (Contd.)

- Aggregating over all firms, we get the total demand for labour and capital in the economy. On the other hand, the supply of total capital stock and labour stock are historically given. Hence the market wage rate (w_t) and the rental rate for capital (r_t) adjust so as to have fullemployment of both factors in every period.
- Thus the equilibrium wage and rental rates in this economy are:

$$\begin{aligned} w_t &= (1-\alpha)\, g_t^{1-\alpha} (K_t)^{\alpha} (L_t)^{-\alpha} \\ r_t &= \alpha g_t^{1-\alpha} (K_t)^{\alpha-1} (L_t)^{1-\alpha} \end{aligned}$$

- For simplicity, let us assume that total labour supply in constant at unity:

$$\begin{aligned} L_t &= \bar{L} = 1 \\ &\Rightarrow H\bar{l} = 1 \Rightarrow \bar{l} = \frac{1}{H}. \end{aligned}$$

Barro Model with Heterogenous Households: (Contd.)

- Hence the equilibrium wage and rental rates in this economy can be written as:

$$\begin{aligned} w_t &= (1-\alpha)\, g_t^{1-\alpha} \left(Hk_t\right)^{\alpha} = (1-\alpha)\, g_t^{1-\alpha} \left(k_t\right)^{\alpha} \left(\bar{l}\right)^{-\alpha} \\ r_t &= \alpha g_t^{1-\alpha} \left(Hk_t\right)^{\alpha-1} = \alpha g_t^{1-\alpha} \left(k_t\right)^{\alpha-1} \left(\bar{l}\right)^{1-\alpha} \end{aligned}$$

  where $k_t = \frac{\sum k_t^h}{H} = \dfrac{K_t}{H}$ is the average capital holding (per capita capital stock) in the economy.
- Notice that the average capital holding in the economy (k_t) is no longer identified with either per firm capital employment or per household capital holding.
- As before, the average capital holding in the economy (k_t) is *different* from the economy-wide capital-labour ratio $(\tilde{k}_t)$ and the two are related in the following way:

$$k_t = \frac{K_t}{H} = \frac{\tilde{k}_t}{H} \quad (\text{since } L_t = 1).$$

Barro Model with Heterogenous Households: (Contd.)

- Let us now go back to the household h. The income of the h-th household is given by:

$$y_t^h = w_t \bar{l} + r_t k_t^h.$$

- Given that a part of that income is taxed away to provide for the infrastructural input, the post-tax budget contstraint of the h-th household is given by:

$$\dot{k}^h = (1 - \tau) \left[w_t \bar{l} + r_t k_t^h \right] - c_t^h.$$

- The corresponding optimization problem for household h operating in the market economy is given by:

$$\int_{t=0}^{\infty} \log\left(c_t^h\right) \exp^{-\rho t}\, dt \tag{II}$$

subject to

$$\dot{k}^h = (1-\tau)\left[w_t \bar{l} + r_t k_t^h\right] - c_t^h;\ \ k^h(t) \geqq 0,\ \ k_0^h \text{ given.}$$

- Solving we get the following two dynamic equations which (along with the TVC) charaterize the optimal path for household h:

$$\frac{\dot{c}^h}{c^h} = (1-\tau) r_t(k_t) - \rho; \tag{10}$$

$$\frac{\dot{k}^h}{k^h} = (1-\tau)\frac{w_t(k_t)\bar{l} + r_t k_t^h}{k_t^h} - \frac{c_t^h}{k_t^h} \tag{11}$$

- We can derive such equations for every household $h = 1, 2, ..., H$.

Barro Model with Heterogenous Households: (Contd.)

- What should be the corresponding balanced growth path for the average economy (i.e., in terms of **per capita** variables)?
- Recall that per capita consumption is defined as:

$$c_t = \frac{\sum c_t^h}{H}$$

- And though the households differ in terms of their initial asset holding, the rate of growth of consumption of each household is still the same, given by $(1-\tau)r_t - \rho$.
- Hence,

$$\begin{aligned}
\frac{dc_t}{dt} &= \frac{1}{H}\sum \frac{dc_t}{dt} = \frac{1}{H}\left[\frac{dc_t^1}{dt} + \frac{dc_t^2}{dt} + \dots + \frac{dc_t^H}{dt}\right] \\
\Rightarrow \quad \frac{\dot{c}}{c} &= \frac{1}{c_t}\frac{1}{H}\left[c_t^1 \frac{\dot{c}^1}{c^1} + c_t^2 \frac{\dot{c}^2}{c^2} + \dots + c_t^H \frac{\dot{c}^H}{c^H}\right] \\
&= (1-\tau)r_t - \rho.
\end{aligned}$$

- Now from our earlier analysis, we know that in the Barro model g_t and k_t are related in the following way:

$$g_t = \left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} \left(\tau\right)^{\frac{1}{\alpha}} k_t$$

- Further, in the equilibrium wage rate and the rental rate are given as:

$$\begin{aligned} w_t &= (1-\alpha)\, g_t^{1-\alpha} \left(k_t\right)^{\alpha} \left(\bar{l}\right)^{-\alpha} \\ r_t &= \alpha g_t^{1-\alpha} \left(k_t\right)^{\alpha-1} \left(\bar{l}\right)^{1-\alpha} \end{aligned}$$

- Thus implies (after simplification):

$$\begin{aligned} w_t &= (1-\alpha)\left(\tau\right)^{\frac{1-\alpha}{\alpha}} \left(\bar{l}\right)^{\frac{1-2\alpha}{\alpha}} k_t; \\ r_t &= \alpha \left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} \left(\tau\right)^{\frac{1-\alpha}{\alpha}}. \end{aligned}$$

Barro Model with Heterogeneous Households: (Contd.)

- This implies that along the balanced growth path, the rate of growth of average (per capita) consumption for this decentralized market economy with heterogenous agents will be given by:

$$\frac{\dot{c}}{c} = (1-\tau)r_t - \rho = \alpha(1-\tau)\left(\tau\right)^{\frac{1-\alpha}{\alpha}}\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} - \rho.$$

Barro Model with Heterogeneous Households: (Contd.)

- What about growth rate of average capital stock (asset stock)?
- Recall that per capita capita stock is defined as:

$$k_t = \frac{\sum k_t^h}{H}$$

- Thus

$$\begin{aligned}
\frac{dk_t}{dt} &= \frac{1}{H}\sum \dot{k}^h \\
&= \frac{1}{H}\sum \left[(1-\tau)\left(w_t\bar{l} + r_t k_t^h\right) - c_t^h\right] \\
&= \frac{H(1-\tau)w_t\bar{l}}{H} + (1-\tau)r_t\frac{\sum k_t^h}{H} - \frac{\sum c_t^h}{H} \\
&= (1-\tau)w_t\bar{l} + (1-\tau)r_t k_t - c_t
\end{aligned}$$

- Hence

$$\frac{\dot{k}}{k} = (1-\tau)\frac{w_t\bar{l} + r_t k_t}{k_t} - \frac{c_t}{k_t} = (1-\tau)\left(\tau\right)^{\frac{1-\alpha}{\alpha}}\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} - \frac{c_t}{k_t}.$$

Barro Model with Heterogeneous Households: (Contd.)

- As we had argued earlier, the average economy will grow along a balanced growth path iff c_t and k_t grow at the same rate:

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \alpha(1-\tau)\left(\tau\right)^{\frac{1-\alpha}{\alpha}}\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} - \rho$$

- But the balanced growth condition in this heterogenous agents framework implies that the growth paths of c_h and k_h must also be constant for *each household* h.
- Now from the optimal trajectory of household h we already know that,

$$\frac{\dot{c}^h}{c^h} = \alpha(1-\tau)\left(\tau\right)^{\frac{1-\alpha}{\alpha}}\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} - \rho = \frac{\dot{c}}{c}.$$

- On the other hand, $\dfrac{\dot{k}^h}{k^h}$ for each household would be constant iff

$$\frac{\dot{k}^h}{k^h} = (1-\tau)\frac{w_t(k_t)\bar{l} + r_t k_t^h}{k_t^h} - \frac{c_t^h}{k_t^h} = \text{ a constant}$$

Barro Model with Heterogeneous Households: (Contd.)

- This can happen if and only if (i) $\frac{c_t^h}{k_t^h}$ is a constant; **and** (ii) $\frac{w_t(k_t)\bar{l}}{k_t^h}$ is a constant.
- Note that
$$\frac{w_t(k_t)\bar{l}}{k_t^h} = \frac{(1-\alpha)(\tau)^{\frac{1-\alpha}{\alpha}}(\bar{l})^{\frac{1-\alpha}{\alpha}}k_t}{k_t^h}.$$
- Thus condition (ii) reduces to the condition that along the balanced growth path, $\frac{k_t}{k_t^h}$ is a constant.

Barro Model with Heterogeneous Households: (Contd.)

- Putting all these results together, in this decentralized market economy with heterogenous households the balanced growth path is characterized as follows :

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{c}^h}{c^h} = \frac{\dot{k}^h}{k^h} = \alpha(1-\tau)\left(\tau\right)^{\frac{1-\alpha}{\alpha}}\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} - \rho \text{ for all } h.$$

- In other words, in the Barro model with heterogenous households, consumption as well as asset stock of *all households* grow at the same (constant) rate along the balanced growth path.

Barro Model with Heterogeneous Households: (Contd.)

- However the initial **level** of consumption of each household would be different and therefore, their welfare level along the balanced growth path would also be different.
- In fact the level of consumption for a rich household (with higher initial capital stock) will be higher than that of a poorer households at all points of time. (**Prove this**).
- **A thought experiment:** Suppose a household h is asked to choose the tax rate τ so as to maximise its own welfare (subject to the balanced growth condition).
  - What tax rate would it choose?
  - Would the chosen tax rate be different across households?

Barro Model with Heterogeneous Households: Welfare Maximising Tax Rate

- Recall that along the balanced growth path, for household h :

$$\frac{\dot{c}^h}{c^h} = \frac{\dot{k}^h}{k^h} = \alpha(1-\tau)\left(\tau\right)^{\frac{1-\alpha}{\alpha}}\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} - \rho.$$

- Then again from the budget constarint of the $h-$th hosehold:

$$(1-\tau)\frac{\left[w_t\bar{l} + r_t k_t^h\right]}{k_t^h} - \frac{c_t^h}{k_t^h} = \alpha(1-\tau)\left(\tau\right)^{\frac{1-\alpha}{\alpha}}\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}} - \rho$$

- Noting that $w_t = (1-\alpha)\left(\tau\right)^{\frac{1-\alpha}{\alpha}}\left(\bar{l}\right)^{\frac{1-2\alpha}{\alpha}} k_t$ and $r_t = \alpha\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}\left(\tau\right)^{\frac{1-\alpha}{\alpha}}$, simplifying:

$$\frac{c_t^h}{k_t^h} = (1-\alpha)(1-\tau)\left(\tau\right)^{\frac{1-\alpha}{\alpha}}\left(\bar{l}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{k_t}{k_t^h}\right) + \rho. \qquad (12)$$

Welfare Maxinising Tax Rate: (Contd.)

- Let us define $\sigma_t^h \equiv \dfrac{l^h / \bar{L}}{k_t^h / K_t}$ as the '**relative factor endowment**' of the $h-$th household at time t.
- Since we have assumed that all households have the same endowment $\bar{l} = \dfrac{1}{H}$ and since total labour supply $\bar{L} = 1$, we can write

$$\sigma_t^h = \frac{\bar{l}}{k_t^h / K_t} = \frac{K_t / H}{k_t^h} = \frac{k_t}{k_t^h}.$$

- Thus in the current specification of the model, σ_t^h simply captures the relative capital endowment of a household h vis-a-vis the economy-wide average capital holding.
- Since the initial capital stocks differ across households, each household starts with a unique σ_0^h which identifies its initial position in the wealth hierarchy of the economy.
- A higher σ^h identifies a relatively capital-poor household (as compared to the average).

Welfare Maxinising Tax Rate: (Contd.)

- Applying this definition of σ_t^h in equation (12), one can see that for any household h, along the banaced growth path:

$$\frac{c_t^h}{k_t^h} = (1-\alpha)(1-\tau)(\tau)^{\frac{1-\alpha}{\alpha}} (\bar{l})^{\frac{1-\alpha}{\alpha}} \sigma_t^h + \rho. \qquad (13)$$

- Since along the balanced growth path, $\frac{\dot{k}}{k} = \frac{\dot{k}^h}{k^h}$ for all h, the relative factor endowment for each household remains **constant** along the balanced growth path, i.e.,

$$\sigma_t^h = \sigma_0^h \text{ for all } h.$$

Welfare Maxinising Tax Rate: (Contd.)

- Therefore, along the balanced growth path, the following relationship holds for all t :

$$c_t^h = \left[\left\{(1-\alpha)(1-\tau)(\tau)^{\frac{1-\alpha}{\alpha}}(\bar{l})^{\frac{1-\alpha}{\alpha}}\right\}\sigma_0^h + \rho\right]k_t^h.$$

- Also,

$$\frac{\dot{c}^h}{c^h} = \alpha(1-\tau)(\tau)^{\frac{1-\alpha}{\alpha}}(\bar{l})^{\frac{1-\alpha}{\alpha}} - \rho \equiv \gamma(\tau) \text{ say.}$$

- Thus,

$$\begin{aligned} c_t^h &= c_0^h \exp^{\gamma(\tau)t} \\ &= \left[\left\{(1-\alpha)(1-\tau)(\tau)^{\frac{1-\alpha}{\alpha}}(\bar{l})^{\frac{1-\alpha}{\alpha}}\right\}\sigma_0^h + \rho\right]k_0^h \exp^{\gamma(\tau)t} \end{aligned}$$

Welfare Maxinising Tax Rate: (Contd.)

- Hence the welfare of the $h-$the household (along the balanced growth path):

$$
\begin{aligned}
W^h(\tau) &= \int_{t=0}^{\infty} \log\left(c_t\right) \exp^{-\rho t} dt \\
&= \int_{t=0}^{\infty} \log\left[c_0^h \exp^{\gamma(\tau)t}\right] \exp^{-\rho t} dt \\
&= \int_{t=0}^{\infty} \left[\log c_0^h + \gamma(\tau)t\right] \exp^{-\rho t} dt \\
&= \log c_0^h \int_{t=0}^{\infty} \exp^{-\rho t} dt + \gamma(\tau) \int_{t=0}^{\infty} t \exp^{-\rho t} dt \\
&= \frac{\log c_0^h}{\rho} + \frac{\gamma(\tau)}{\rho^2}
\end{aligned}
$$

Welfare Maxinising Tax Rate: (Contd.)

- Plugging back the values of c_0^h and $\gamma(\tau)$, then maximising with respect to τ, it can be easily shown that the welfare maximising tax rate for household h (subject to the condition that the economy is on its balanced growth path) is given by

$$\tau_h^{**} = 1 - \alpha$$

  (Prove it)

- Notice that τ_h^{**} is **independent of the household's relative factor endowment**.
- Thus *even though households differ in terms of initial asset holding*, if you ask them to choose their respective welfare maximising tax rate (subject to the condition that the economy is on its balanced growth path), then *everybody chooses the same tax rate*.

Welfare Maxinising Tax Rate: (Contd.)

- In fact if the tax rate was chosen by the majority-voting then $\tau^{**} = 1 - \alpha$ would indeed be the majority-chosen tax rate (follows trivially).
- Thus *income distribution does not affect the majority-voted tax rate in the Barro structure even when we bring in heterogenous households*.
- Why is this hapenning?
- As we have noted, before when the governement takes away τ proportion of a households income, there is a direct negative impact on the household's income by the margin of τ.
- At the same time when it invests this amount in provision on infrastructure, that in turn has an indirect positive impact on the household's income through g - because both $w_t = (1-\alpha) g_t^{1-\alpha} (k_t)^{\alpha} (\bar{l})^{-\alpha}$ and $r_t = \alpha g_t^{1-\alpha} (k_t)^{\alpha-1} (\bar{l})^{1-\alpha}$ go up.

Welfare Maxinising Tax Rate: (Contd.)

- Notice that as g increases, both the wage and the rental return increase *symmetrically* and the marginal impact of an increase in g (on either income) is precisely $1 - \alpha$.
- Moreover, **with a proportional income tax**, the wange and the reantal income is also taxed *symmetrically;* both decrease at the margin by a proprtion τ.
- In other words, under a proportional income tax, both the incremental benefit and the increamental cost affect wage and rental income symmetricall.
- Hence even though households differ in terms of the distribution of their labour and rental income, with **proportional income tax**, most-preferred (i.e., welfare-maximizing) tax rate chosen by a household would be the same across all households, even though they differ in terms of the distribution of labour vis-a-vis rental income.

Welfare Maxinising Tax Rate: (Contd.)

- Note that given the production function, benefits of g will always be symetrically distributed over capital and labour. (Both marginal products get augmented by $g^{1-\alpha}$).
- However whether the associated cost (tax burden) is symmetric or not depends crucially on the **mode of taxation**.
- In fact the welfare-maximising tax rate would differ across households iff the cost/tax buren is not symmetric, i.e., if the labour income and rental income are taxed differently.
- This is precisely what Alesina-Rodrik (1994) does, which we now discuss.

Political Economy & Growth:Alesina-Rodrik (QJE,1994)

- The economic structure is **almost** identical to Barro (1990).
- **The only significant difference is in term of the mode of taxation**.
- As before, a single final commdity is produced - which can be either consumed or invested in physical capital.
- The economy has S identical firms and H households, which differ in terms of their labour and initial capital endowments. For convenience let us continue assume $S = H$ although we can no longer assign each household to a firm (since households are no longer identical).
- Each household consists on a single **infinitely lived** member.
- There is no population growth, which implies that the size of labour force is fixed every period: $L_t = \sum l^h = \bar{L}$.

Alseina-Rodrik Model: Production Side Story

- Each firm is endowed with a firm specigic technology given by

$$Y_{it} = (g_t)^{1-\alpha} (K_{it})^{\alpha} (L_{it})^{1-\alpha}$$

  where g as before is the publicly provided infrastructural input.

- Since all firms are identical, we can aggreagte over all firms to generate the aggreate production function:

$$\begin{aligned} Y &= (g_t)^{1-\alpha} (SK_{it})^{\alpha} (SL_{it})^{1-\alpha} \\ &= (g)^{1-\alpha} (K_t)^{\alpha} (L_t)^{1-\alpha}, \; 0 < \alpha < 1 \end{aligned}$$

  where K and L are the aggrerate capital stock and aggregate labour available in the economy.

Production Side Story: (Contd.)

- Each firm take the market wage rate (w_t) and the rental rate for capital (r_t) as given and employ capital and labour to maximise their respecive profits.
- Aggreagting over all firms, and equating the total demand for labour and capital in the economy with the historically given supply of capital stock and labour stock, we can derive the market wage rate (w_t) and the rental rate for capital (r_t) in equilibrium as follows:

$$
\begin{aligned}
w_t &= (1-\alpha)\, g_t^{1-\alpha} \left(K_{_t}\right)^{\alpha} \left(L_t\right)^{-\alpha} \\
w_t &= \alpha g_t^{1-\alpha} \left(K_{_t}\right)^{\alpha-1} \left(L_t\right)^{1-\alpha}
\end{aligned}
$$

- The are And adjust so as to have fullemployment of both factors in every period.
- Let us assume that total labour supply in constant at unity:

$$L_t = \bar{L} = 1$$

Alesina-Rodrik Model: Financing of the Infrastructural Input

- Then

$$\begin{aligned} w_t &= (1-\alpha)\, g_t^{1-\alpha} (K_t)^{\alpha} \\ r_t &= \alpha g_t^{1-\alpha} (K_t)^{\alpha-1} \end{aligned}$$

- As in Barro, the government provides the infrastructural input g upfront (before the production takes place).
- However, the mode of financing is now different. In the Alesina-Rodrik model the government imposes an asset/wealth tas rather than an income tax.
- Suppose the governement imposes an **tax on capital** (in the same period, after production has taken place) to recover the cost that it had incurred in providing the input:

$$\begin{aligned} G_t &= \tau K_t \\ \Rightarrow &\quad g_t \equiv \frac{G_t}{S} = \frac{G_t}{H} = \frac{\tau K_t}{H} = \tau k_t \end{aligned}$$

Production Side Story: (Contd.)

- Thus as in Barro, g_t and k_t are related, although the firms do not recongnise this link when they decide to employ capital.
- As before, this link generates an externality (higher $k_t \Rightarrow$ higher tax revenue $\Rightarrow$ higher $g_t \Rightarrow$ higher productivity) which counteracts with the law of diminishing returns to effectively generate an AK production for the economy as a whole, just as in Barro. (Of course the functional specification of the relationship is now a little different.)
- Plugging in the relationship between g_t and k_t in the wage and rental equations, we get

$$
\begin{aligned}
w_t &= \left(\frac{1}{H}\right)^{1-\alpha} (1-\alpha)\,\tau^{1-\alpha} k_t \equiv \omega(\tau) k_t; \\
r_t &= \left(\frac{1}{H}\right)^{1-\alpha} \alpha \tau^{1-\alpha} \equiv r(\tau)
\end{aligned}
$$

where $\frac{1}{H} = \frac{\bar{L}}{H}$ is the *average* labour endowment per household in the economy.

Alesina-Rodrik Model: Household Side Story

- Households differ in terms of their labour endowment (l^h) as well as capital endowment (k^h).
- The labour endowment remains constant over time, while the capital stock changes over time depending on the accumulation decision of the household.
- Income of the h-th household is given by:

$$y_t^h = w_t l^h + r_t k_t^h.$$

- Given that a part of the income is taken away and given that the taxation is proportional to the capital stock owned by the household, the post-tax budget contstraint of the h-th household is given by:

$$\begin{aligned} \dot{k}^h &= \left[w_t l^h + r_t k_t^h - \tau k_t^h \right] - c_t^h \\ &= \left[w_t l^h + (r_t - \tau)\, k_t^h \right] - c_t^h \end{aligned}$$

Household Side Story: (Contd.)

- As before, let us define the '**relative factor endowment**' of a household as

$$\sigma_t^h \equiv \frac{l^h / \bar{L}}{k_t^h / K_t}$$

- Notice that now households differ both in terms of the labour endowment as well as capital endowment.
- Thus σ_t^h now captures household h's relative ownership *share* in the economy's aggregate stocks of labour **and** capital.
- Equivalently, the relative factor endowment of a household can be defined in terms of the household's capital-labour ratio vis-a-vis the economy's average capital labour ratio:

$$\sigma_t^h \equiv \frac{K_t / \bar{L}}{k_t^h / l^h} = \frac{K_t}{k_t^h / l^h} = \frac{K_t l^h}{k_t^h}$$

- Notice that unlike the previous case, σ_0^h no longer *uniquely* identifies a household's initial position in the wealth/asset hierarchy of the economy.

Household Side Story: (Contd.)

- In fact two households - r and p - may have vastly different wealth stocks (such that $k_0^r \gg k_0^p$), but as long as household r also has a equally higher labour endowment compared to p, they will be placed in the same position in terms of the σ^h mapping (although household r is far richer than household p is every respect!)
- This somewhat unsatisfactory ranking emerges because we have clubbed together two different rankings of factor endowments (labour endowment ranking vis-a-vis capital endowment ranking).
- However, as we shall see later, it is this **relative** factor endowment ranking that really matters for optimal household decisions; not the absolute rankings of either factor**.**

Household Side Story: (Contd.)

- We have assumed that households are heterogenous with respect to both labour and capital endowment.
- This means no two households have the same initial capital endowments; likewise for labour endowments.
- But as we have just noted, each household having different capital and labour endowments (such the $k_0^h \neq k_0^{h'}$ **and** $l^h \neq l^{h'}$) does not necessarily ensure that they will all have different σs.
- However, for convenience, we shall assume that each household differ in terms of their initial relative factor endowments and we can index them in increasing order of $\sigma : \sigma_0^1 < \sigma_0^2 < \sigma_0^3 < .... < \sigma_0^H$.
- A higher σ^h identifies a relatively capital-poor household (as compared to the average capital/labour ratio in the economy).
- Notice that the **lowest** posible value of σ is 0. It identifies a person who has no labour endowment at all. He is relatively the most **capital-rich** in this economy.

Alesina-Rodrik: Household's Optimization Problem:

- The optimization problem for household h operating in the market economy is given by:

$$\int_{t=0}^{\infty} \log\left(c_t^h\right) \exp^{-\rho t}\, dt \qquad \text{(II)}$$

subject to

$$\dot{k}^h = \left[w_t l^h + (r_t - \tau)\, k_t^h\right] - c_t^h;\;\; k_t^h \geqq 0,\;\; k_0^h \text{ given.}$$

- Solving we get the following two dynamic equations :

$$\frac{\dot{c}^h}{c^h} = r_t - \tau - \rho; \qquad (14)$$

$$\frac{\dot{k}^h}{k^h} = \frac{w_t l^h + (r_t - \tau)\, k_t^h}{k_t^h} - \frac{c_t^h}{k_t^h} \qquad (15)$$

Alesina-Rodrik: Household's Optimization Problem (Contd.)

- Noting that $w_t = \omega(\tau)k_t$ and $r_t = r(\tau)$, and also, given the definition of σ^h, we can write these tow dynamic equations for household h as:

$$\frac{\dot{c}^h}{c^h} = r(\tau) - \tau - \rho; \quad (16)$$

$$\frac{\dot{k}^h}{k^h} = \omega(\tau)\sigma_t^h + (r(\tau) - \tau) - \frac{c_t^h}{k_t^h} \quad (17)$$

- We can derive such equations for every household $h = 1, 2, ..., H$.
- Notice that any household h would be on a balanced growth path if:

(i) σ_t^h remains constant;

(ii) $\frac{c_t^h}{k_t^h}$ remains constant.

Alesina-Rodrik: Characterization of the Balanced Growth Path

- As before, along the balanced growth path, consumption as well as capital stocks of *all* households grow at the same rate, and so do their average values, such that

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{c}^h}{c^h} = \frac{\dot{k}^h}{k^h} = r(\tau) - \tau - \rho \equiv \gamma(\tau).$$

- Moreover along such a balanced growth path the '**relative**' distribution of the factors would remain constant at the initially given level, i.e.,

$$\sigma_t^h \equiv \frac{K_t l^h}{k_t^h} = \sigma_0^h \text{ for all } h.$$

- Since σ_t^h remains constant along such a balanced growth path, henceforth we shall ignore the time subscript and denote it only by σ^h.

Characterization of the Balanced Growth Path: (Contd.)

- It is easy to verify that the growth-maximising tax rate in this case is given by

$$\tau^* = \left[\left(\frac{1}{H} \right)^{1-\alpha} \alpha \left(1 - \alpha\right) \right]^{\frac{1}{\alpha}}.$$

- We are however more interested in the welfare maximising tax rate for a household h (subject to the balanced growth condition).
- Preceeding as we did for the Barro model with heterogeneous agents, it is easy to show that the maximised value of welfare of household h along a balanced growth path would be given by:

$$\begin{aligned} W^h(\tau) &= \int_{t=0}^{\infty} \log\left(c_t\right) \exp^{-\rho t} dt \\ &= \log c_0^h \int_{t=0}^{\infty} \exp^{-\rho t} dt + \gamma(\tau) \int_{t=0}^{\infty} t \exp^{-\rho t} dt \end{aligned}$$

Welfare Maxinising Tax Rate:

- Noting that balanced growth condition implies

$$c_t^h = \left[\omega(\tau)\sigma^h + \rho\right] k_t^h$$

and plugging back the corresponding c_0^h value in the above expression we can derive the precise value of $W^h(\tau)$ in terms of τ.

- Maximising $W^h(\tau)$ in terms of τ, from the FONC:

$$\tau\left[1 - \left(\frac{1}{H}\right)^{1-\alpha} \alpha(1-\alpha)\tau^{-\alpha}\right] = \rho(1-\alpha)\frac{\omega(\tau)\sigma^h}{\omega(\tau)\sigma^h + \rho} \qquad (18)$$

- The above equation implicitly defined the welfare maximising tax rate τ_h^{**} for household h.
- It is not easy to slove this equation analytically; so we will follow the graphical approach.

Welfare Maxinising Tax Rate: (Contd.)

- Let the RHS of the above equation be represented by the following function:

$$\rho(1-\alpha)\frac{\omega(\tau)\sigma^h}{\omega(\tau)\sigma^h+\rho} \equiv f(\tau)$$

- It is easy to verify that

$$f(0)=0;\ \ f'(\tau)>0;\ \ f''(\tau)<0.$$

- Next, let the LHS of the above equation be represented by the following function:

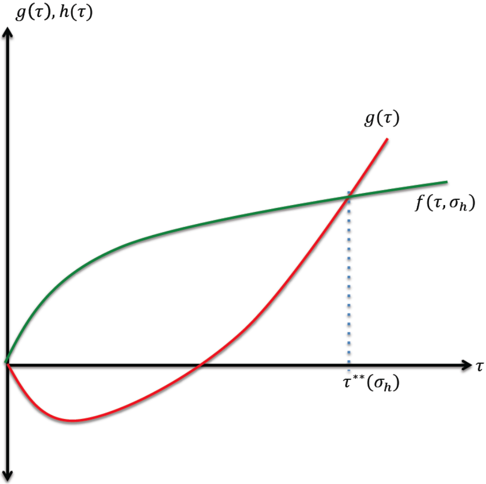
$$\tau\left[1-\left(\frac{1}{H}\right)^{1-\alpha}\alpha(1-\alpha)\tau^{-\alpha}\right] \equiv g(\tau)$$

- Once again it is easy to varify that

$$\begin{aligned} g(0) &= 0; \\ g'(\tau) &\gtreqless 0 \text{ according as } \tau \gtreqless \left[\left(\frac{1}{H}\right)^{1-\alpha}\alpha(1-\alpha)^2\right]^{\frac{1}{\alpha}} \equiv \underline{\tau} \text{ (say)}. \end{aligned}$$

Welfare Maxinising Tax Rate: (Contd.)

- Plotting the two functions with τ on the horizontal axis (for a given value of σ^h):



Welfare Maximising Tax Rate: (Contd.)

- A few interesting observations about the welfare maximising tax rate (τ_h^{**}):
  1. An increase in σ^h shifts the $f(\tau)$ function upward leaving the $g(\tau)$ function unchanged$\Rightarrow$ τ_h^{**} is increasing in σ^h .(We have to make some additional parametric assumptions to ensure that $\tau_h^{**} \leqq 1$)
  2. For a household such that $\sigma^h = 0$ (no labour edowment at all; **maximum possible capital-rich** - in a relative sense):

$$\tau_h^{**} = \left[\left(\frac{1}{H} \right)^{1-\alpha} \alpha \left(1 - \alpha \right) \right]^{\frac{1}{\alpha}} = \tau^*.$$

- In other words, relatively the **more capital-poor** a household is, the **higher** is its welfare-maximing or the **most preferred** tax rate. (**Intution?**)
- Moreover for the **richest possible household**, the welfare-maximising tax rate or the **most preferred** tax rate coincides with the growth-maximising tax rate for the economy.

Poitical Economy Consideration: Tax Rate chosen by Majority Voting

- Now suppose the tax rate in this economy is chosen by majority voting.
- We shall define the majority voting rule by the **Codorcet Rule**:
  - Consider a set of feasible options denoted by the following set: $C \equiv \{a, b, c, d, ...\}$
  - Suppose each option $a \in C$ is paired with another option $a' \in C$ and voters are asked to choose one over the other.
- Option a is said to be **majority-preferred** to a' if

$$\text{prop}^{\text{n}} \text{ of voters preferring } a \text{ to } a' > \text{prop}^{\text{n}} \text{ of voters preferring } a' \text{ to } a$$

- Or, equivalently, if

$$\text{prop}^{\text{n}} \text{ of voters preferring } a \text{ to } a' > \frac{1}{2}$$

Majority Voting Equilibrium (Condercet Winner):

- Suppose we carry out this exercise for all possible pairs in the set C.
- If there exists an option $a^* \in C$ such that a^* is **majority-preferred** to *every other* option $a \in C$, then we say that a^* represents a **majority voting equilibrium** (or a Condorcet winner).
- Notice that a majority voting equilibrium may not always exist.
- Moreover, even when it exists it may not be unique.
- In our problem of tax rate choice however a majority voting equilibrium does exist and is indeed unique.

Alesina-Rodrik: Majority Voting Equilibrium

- Recall that in our problem households differ in terms of the relative factor endowment (σ_h).
- Therefore each household has a most preferred tax rate τ_h^{**} which different from other households.
- The are two important features of τ_h^{**} :
  1. For a given σ_h, preferences $(W_h(\tau))$ are **single-peaked**: unique welfare-maximising τ_h^{**}
  2. When we vary σ_h, the corresponding most preferred tax rates are **monotonic**: higher $\sigma_h \Rightarrow$ higher τ_h^{**}
- These two features will ensure that a unique majority voting equilibrium will exist in this case.

Alesina-Rodrik: Majority Voting Equilibrium (Contd.)

- Let us assume that the households are odd in number; Let $H = 2n + 1$, where n is any positive real number.
- Recall that each household is identified by its respective relative factor endowment, σ_h.
- Since H (by assumption) is an odd number, there exsist a **unique median household**

$$\sigma^1 < \sigma^2 < ...\boldsymbol{\sigma}^M < .... < \sigma^H$$

  such that there are exactly n agents preceding him and exactly n agents following him.
- Let $T \equiv \{\tau_h^{**}\}_{h=1}^H$ represent the set of possible options.
- Let us now ask the households to play a Condorcet game over the choice set T.
- Can we identify a majority-voting equilibrium here?

Alesina-Rodrik: Majority Voting Equilibrium (Contd.)

- **Claim:** There indeed exists a majority-voting equilbrium here, represented by the most-preffered tax rate of the median household: τ_M^{**}
- **Proof:**
  - Consider any other tax rate $\tau \in T$ such that $\tau < \tau_M^{**}$. All households with $\sigma_h > \boldsymbol{\sigma}^M$ would prefer τ_M^{**} over τ (**why?**); and the median voter of course prefers τ_M^{**} over τ (it is **his** most-preferred tax rate after all!). That constitutes the majority.
  - Now consider any other tax rate $\tau' \in T$ such that $\tau > \tau_M^{**}$.All households with $\sigma_h < \boldsymbol{\sigma}^M$ would prefer τ_M^{**} over τ (**why?**); and the median voter of course prefers τ_M^{**} over τ. That again constitutes the majority.

Relationship between Inequality & Growth in Alesina-Rodrik:

- Recall that only the richest possible agent (with $\sigma^h = 0$) would most prefer the growth-maximising tax (τ^*).
- Any other agent (with even a small positive endowment of labour) would have most prefer a tax rate $\tau_h^{**} > \tau^*$; therefore the corresponding (potential) growth rate would be lower.
- Moreover the higher is σ_h, the higher the most-preferred tax rate and the lower is the corresponding (potential) growth rate.
- Now consider two economies: **A** and **B**, which have identical total endowment; identical preferences; same population; identical technology etc.
- **BUT suppose capital is more unequally distributed in Country A than Country B.**

Inequality & Growth in Alesina-Rodrik: (Contd.)

- In particular, let us assume that the *average* capital holding in both countries are same, but its distribution is more skewed to the left for country **A** than country **B.**
- This implies that the median voter in country **A** is more capital-poor than the median voter in country **B:**

$$\left(\sigma^M\right)_A > \left(\sigma^M\right)_B$$

- Now if the tax rate in the economy is chosen by majority voting (as happens in democracies), country **B** would fare better than country **A** in terms of growth.
- Moreover any redistribution of capital from the rich to the poor which improves the relative capital endowment of the median voter would increase the growth rate (in either country).
- Does this mean, **oligarchy (concentration of economic and political power in the hand of few) would fare better than democracy in terms of growth**?

A Few Simple Extensions of Alesina-Rodrik:

- What if the tax is imposed on rental income rather than capital?
- What if the tax is imposed on labour income instead?
  - To answer this later question, consider two possible scenarios. In one case the households have heterogenous capital holdings but identical labour holdings (which remain constant over time) - as we have assumed in the model above. In this set up examine what happens if g is financed through labour income taxation instead of capital taxation.
  - Next suppose households differ in terms of their innate abilities. In particular, assume that capital holding across households are identical and constant, but households' can augment their abilities over time by investing in education such that for a household h with some given innante ability a_0^h, $\frac{da^h}{dt} = w_t a_t^h + r_t \bar{k} - c_t^h$.In this set up examine what happens if G is financed through labour income taxation instead of capital taxation.