General Equilibrium Analysis

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Lecture 4

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Individual UMP: Some Features I

Notations:

- $\mathbf{p} = (p_1, ..., p_M)$ is a M-component vector in $\mathbb{R}^M$.
- If $\mathbf{p} = (p_1, ..., p_M) \in \mathbb{R}^M_{++}$, then $p_j > 0$ for all $j = 1, ..., M$, i.e.,

$$(p_1, ..., p_M) > (0, ..., 0).$$

- If $\mathbf{p} = (p_1, ..., p_M) \in \mathbb{R}^M_{+}$, then $p_j \geq 0$ for all $j \in \{1, ..., M\}$ and $p_j > 0$ for some $j \in \{1, ..., M\}$, i.e.,

$$(p_1, ..., p_M) \geq (0, ..., 0) \text{ and } (p_1, ..., p_M) \neq (0, ..., 0).$$

- Let $\mathbf{x} = (x_1, ..., x_M)$ and $\mathbf{x}' = (x'_1, ..., x'_M)$. If $\mathbf{x}' \geq \mathbf{x}$, then $x_j \geq x_j$ for all $j \in \{1, ..., M\}$ and $x_j > x'_j$ for some $j \in \{1, ..., M\}$.
- Let $\mathbf{x} = (x_1, ..., x_M)$ and $\mathbf{x}' = (x'_1, ..., x'_M)$. If $\mathbf{x}' > \mathbf{x}$, then $x_j > x_j$ for all $j \in \{1, ..., M\}$.

Individual UMP: Some Features II

Take a price vector $\mathbf{p} = (p_1, ..., p_M) \in \mathbb{R}^M_{++}$. That is, $(p_1, ..., p_M) > (0, ..., 0)$. The consumer i's OP (UMP) is to solves:

$$\max_{\mathbf{x} \in \mathbb{R}^J_+} u^i(\mathbf{x}) \quad s.t. \quad \mathbf{p}.\mathbf{x} \leq \mathbf{p}.\mathbf{e}^i$$

Definition

u^i is strongly increasing if for any two bundles $\mathbf{x}$ and $\mathbf{x}'$

$$\mathbf{x}' \geq \mathbf{x} \Rightarrow u^i(\mathbf{x}') > u^i(\mathbf{x}).$$

Assumption

For all $i \in I$, u^i is continuous, strongly increasing, and strictly quasi-concave on $\mathbb{R}^M_+$

Individual UMP: Some Features III

In view of monotonicity, for given $\mathbf{p} = (p_1, ..., p_M) >> (0, ..., 0)$, consumer i solves:

$$\max_{\mathbf{x} \in \mathbb{R}^M_+} u^i(\mathbf{x}) \quad s.t. \quad \mathbf{p}.\mathbf{x} = \mathbf{p}.\mathbf{e}^i \tag{1}$$

Theorem

Under the above assumptions on $u^i(.)$, for every $(p_1, ..., p_M) > (0, ..., 0)$, (1) has a unique solution, say $\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i)$.

Note:

- Existence follows from Monotonicity and finiteness of the Budget set
- Uniqueness follows from ‘strictly quasi-concave’

Individual UMP: Some Features IV

Note:

- $\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i)$ is the (Marshallian) Demand Function for individual i.
- For each $i = 1, ..., N$,

$$\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) : \mathbb{R}^M_{++} \mapsto \mathbb{R}^M_+;$$

$$\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) = (x^i_1(\mathbf{p}, \mathbf{p}.\mathbf{e}^i), ..., x^i_j(\mathbf{p}, \mathbf{p}.\mathbf{e}^i), ..., x^i_M(\mathbf{p}, \mathbf{p}.\mathbf{e}^i)).$$

- In general, demand for jth good depends on price of kth good, $k = 1, ..., M$

Individual UMP: Some Features V

Theorem

Under the above assumptions on $u^i(.)$, for every $(p_1, ..., p_M) > (0, ..., 0)$,

- $\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i)$ *is continuous in* $\mathbf{p}$ *over* $\mathbb{R}^M_{++}$.
- *For all* $i = 1, 2, ..., N$, *we have:* $\mathbf{x}^i(t\mathbf{p}) = \mathbf{x}^i(\mathbf{p})$, *for all* $t > 0$. *That is, demand of each good* j *by individual* i *satisfies the following property:*

$$x^i_j(t\mathbf{p}) = x^i_j(\mathbf{p}) \text{ for all } t > 0.$$

Question

Given that $u^i(.)$ is strongly increasing,

- *is* $\mathbf{x}^i(\mathbf{p})$ *continuous over* $\mathbb{R}^M_+$*?*
- *is the demand function* $x^i_j(\mathbf{p})$ *defined at* $p_j = 0$*?*

Is a Cobb-Douglas utility function strongly increasing over $\mathbb{R}^M_+$?

Excess Demand Function I

Definition

The excess demand for *j*th good by the *i*th individual is give by:

$$\mathbf{z}_j^i(\mathbf{p}) = x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - e_j^i.$$

The aggregate excess demand for *j*th good is give by:

$$\mathbf{z}_j(\mathbf{p}) = \sum_{i=1}^{N} x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - \sum_{i=1}^{N} e_j^i.$$

So, Aggregate Excess Demand Function is a vector-valued function:

$$\mathbf{z}(\mathbf{p}) = (\mathbf{z}_1(\mathbf{p}), ..., \mathbf{z}_j(\mathbf{p}), ..., \mathbf{z}_M(\mathbf{p})),$$

Excess Demand Function II

Theorem

Under the above assumptions on $u^i(.)$, *for any* $\mathbf{p} >> \mathbf{0}$,

- $\mathbf{z}(.)$ *is continuous in* $\mathbf{p}$
- $\mathbf{z}(t\mathbf{p}) = \mathbf{z}(\mathbf{p})$, *for all* $t > 0$
- $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$. (the Walras' Law)

For any given price vector **p**, the individual UMP gives us

$$\begin{aligned}\mathbf{p}.\mathbf{x}^i(\mathbf{p},\mathbf{p}.\mathbf{e}^i) - \mathbf{p}.\mathbf{e}^i &= 0, i.e., \\ \sum_{j=1}^{M} p_j x_j^i(\mathbf{p},\mathbf{p}.\mathbf{e}^i) - \sum_{j=1}^{M} p_j e_j^i &= 0, i.e., \\ \sum_{j=1}^{M} p_j[x_j^i(\mathbf{p},\mathbf{p}.\mathbf{e}^i) - e_j^i] &= 0.\end{aligned}$$

Excess Demand Function III

This gives:

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{M} p_j[x_j^i(\mathbf{p},\mathbf{p}.\mathbf{e}^i) - e_j^i] &= 0, i.e., \\
\sum_{j=1}^{M}\sum_{i=1}^{N} p_j[x_j^i(\mathbf{p},\mathbf{p}.\mathbf{e}^i) - e_j^i] &= 0, i.e., \\
\sum_{j=1}^{M} p_j\left[\sum_{i=1}^{N} x_j^i(\mathbf{p},\mathbf{p}.\mathbf{e}^i) - \sum_{i=1}^{N} e_j^i\right] &= 0
\end{aligned}$$

That is,

$$\begin{aligned}
\sum_{j=1}^{M} p_j z_j(\mathbf{p}) &= 0, i.e., \\
\mathbf{p}.\mathbf{z}(\mathbf{p}) &= 0
\end{aligned}$$

Excess Demand Function IV

So,

$$p_1 z_1(\mathbf{p}) + p_2 z_2(\mathbf{p}) + \ldots + p_{j-1} z_{j-1}(\mathbf{p}) + p_{j+1} z_{j+1}(\mathbf{p}) + +p_M z_M(\mathbf{p}) = -p_j z_j(\mathbf{p})$$

For a price vector $\mathbf{p} >> \mathbf{0}$,

- if $z_k(\mathbf{p}) = 0$ for all $k \neq j$, then $z_j(\mathbf{p}) = 0$
- For two goods case
    - $p_1 z_1(\mathbf{p}) + p_2 z_2(\mathbf{p}) = 0.$, i.e.,

      $$p_1 z_1(\mathbf{p}) = -p_2 z_2(\mathbf{p}).$$

    - Therefore,

      $$\begin{aligned} z_1(\mathbf{p}) = 0 \quad &\Rightarrow \quad z_2(\mathbf{p}) = 0 \\ z_1(\mathbf{p}) > 0 \quad &\Rightarrow \quad z_2(\mathbf{p}) < 0. \end{aligned}$$

Walrasian Equilibrium I

Definition

Walrasian Equilibrium Price: A price vector $\mathbf{p}^*$ is equilibrium price vector, if for all $j = 1, ..., J$,

$$\mathbf{z}_j(\mathbf{p}^*) = \sum_{i=1}^{N} x_j^i(\mathbf{p}^*, \mathbf{p}^*.\mathbf{e}^i) - \sum_{i=1}^{N} e_j^i = 0, \text{ i.e., if}$$

$$\mathbf{z}(\mathbf{p}^*) = \mathbf{0} = (0, ..., 0).$$

Two goods: food and cloth
Let (p_f, p_c) be the price vector. Since, we know that for all $t > 0$:

$$\mathbf{z}(t\mathbf{p}) = \mathbf{z}(\mathbf{p})$$

Therefore, we can work with $\mathbf{p} = (\frac{p_f}{p_c}, 1) = (p, 1)$.

Walrasian Equilibrium II

From Walras's law, we have $p_f z_f(\mathbf{p}) + p_c z_c(\mathbf{p}) = 0$, i.e.,

$$pz_f(\mathbf{p}) + z_c(\mathbf{p}) = 0.$$

Let:

- $z_i(\mathbf{p})$ is continuous for all $\mathbf{p} >> \mathbf{0}$, i.e., for all $p > 0$.

Note

- When utility function is monotonic, $x_f(\mathbf{p})$ will explode as $p_f = p \to 0$.
- Therefore, there exists small $p = \epsilon > 0$ s.t. $z_f(p, 1) >> 0$.
- Also, there exists another $p' > \frac{1}{\epsilon}$ s.t. $z_f(p', 1) << 0$ (Why?).