Equilibrium in Factor Markets

Ram Singh

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1 Basics

In this write up we discuss the following questions about the factors of production (FOPs):[1] In a competitive setting,

- What is the relationship between the output prices and the wage rates for the relevant FOPs?
- Do competitive markets provide 'fair' wage to all FOPs?
- How does a change in the distribution of wealth affect the wage rates for different FOPs?
- What market factors affect the wage rates for different FOPs?
- What non-market factors affect the wage rates for different FOPs?

To start with, let us study the nature of demand for the FOPs. To keep things simple, assume the FOPs cannot be produced by the firms. You can think of FOPs as labour or other natural resources like land, coal, crude oil, natural gas, etc. That is, the FOPs are provided by nature and owned by consumers. The firms use FOPs to produce consumption goods. Moreover, in the interest of simplicity assume that the consumption goods are produced by firms by using only the FOPs. That is, assume that there are no intermediate goods. In such a production scenario, we have pure inputs (factors of production) and pure consumptions goods. Suppose, there are L FOPs; $l = 1, ..., L$. So, the set of FOPs is $\mathbb{L} = \{1, .., L\}$. Total aggregate endowments of factors is $\bar{\mathbf{z}} = (\bar{z}_1, ..., \bar{z}_L) >> \mathbf{0}$. To repeat, FOPs are provided by the nature and are initially owned by consumers. Consumers do not derive any utility from direct consumption of these endowments. So, they are willing to sell their endowment of FOPs to firms to buy the consumption goods from firms.

As before, let the set of goods be $\mathbb{M} = \{1, .., M\}$ and the set of firms be $\mathbb{K} = \{1, .., K\}$.

[1] For more on this topic, you can read MWG.

Note that now we have L inputs (factors of production) and M outputs (the consumption goods). Also, for a firm a production plan is a vector of inputs and outputs. Therefore, a production plan for firm k is a vector $(y_1^k, ..., y_L^k, y_{L+1}^k, ..., y_{L+M}^k)$. Formally, let the vector

$\mathbf{y}^k$ denote a production plan for firm k.

That is, $\mathbf{y}^k = (y_1^k, ..., y_L^k, y_{L+1}^k, ..., y_{L+M}^k)$, i.e, $\mathbf{y}^k \in \mathbb{R}^{L+M}$. Let $\mathbb{Y}^{L+M}$ denote the set of feasible production plans. For the vector $\mathbf{y}^k = (y_1^k, ..., y_L^k, y_{L+1}^k, ..., y_{L+M}^k)$, the first L components, i.e., $y_1^k, ..., y_L^k$, denote of quantities of inputs (FOPs) used. The last M components, i.e., $y_{L+1}^k, ..., y_{L+M}^k$ denote the levels of outputs produced.

To keep things simple, let us assume that one firm produces only one good. Let, good j be produced by firm j. So, the number of firms is equal to number of goods, i.e., $K = M$. So, the set of firms and also the consumption goods is $\mathbb{M} = \{1, .., M\}$. Now, since jth firms produces only the jth good. Therefore, production plan of jth firm can be written as $\mathbf{y}^j = (y_1^j, ..., y_L^j, 0, ..., y_{L+j}^j, ..., 0)$.

Also, since $y_1^j, ..., y_L^j$, denote of quantities of inputs (FOPs) used, these components of the production plan $\mathbf{y}^j$ are non-positive - since it is not possible to produce an output without using any input, some of these components have to be strictly negative. To separate the notations for the outputs and inputs, we can let

z_l^j denote the quantity of lth FOP used by firm j; $z_l^j \geq 0$.

Therefore, we can re-write the first L components of the production plan $\mathbf{y}^j$, as $(-z_1^j), ..., (-z_L^j)$. With these clarifications, we continue to work with the general notation for production plan $\mathbf{y}^j = (y_1^j, ..., y_k^j, y_{k+1}^j, ..., y_{L+M}^j)$. In that case, keeping in mind that:

$$y_k^j = \begin{cases} 0, & \text{if } \ k > L \text{and } k \neq j; \\ y_j^j, & \text{if } \ k > L \text{and } k = j; \\ -z_k^j, & \text{if } k \leq L \end{cases}$$

We want to analyze the demand for FOP by price-taking firms. Consider any given output price vector $\bar{\mathbf{p}} = (\bar{p}_1, ..., \bar{p}_M)$ and input price vector $\bar{\mathbf{w}} = (\bar{w}_1, ..., \bar{w}_L)$ for the FOPs. Firm j will choose profit maximizing production plan. That is, it will choose $\bar{\mathbf{y}}^j \in \mathbb{Y}^{L+M}$ to solve:

$$\max_{\mathbf{y}^j \in \mathbb{Y}^{L+M}} \left\{ -\sum_{k=1}^{L} \bar{w}_k . z_k^j + \bar{p}_j . y_j^j \right\}, i.e.,$$

$$\max_{\mathbf{y}^j \in \mathbb{Y}^{L+M}} \left\{ \bar{p}_j f^j(\mathbf{z}^j) - \sum_{k=1}^{L} \bar{w}_k . z_k^j \right\},$$

where $f^j(\mathbf{z}^j) = y_j^j$ is the 'production function' for the jth firm (and also for good j), i.e., it is the level of consumption good produced by the firm, when input vector used is $\mathbf{z}^j = (z_1^j, ..., z_L^j)$.

2 Equilibrium in Factor Markets

For the ease of illustration, assume that the economy is a small open economy. Prices are determined in the international market. So, for the economy under analysis, we can take the prices to be given. Given output price vector $\bar{\mathbf{p}} = (\bar{p}_1, ..., \bar{p}_M)$, equilibrium in the factors market is a price vector $\mathbf{w}^* = (w_1^*, ..., w_L^*)$, and a factor allocation (demand) across firms, $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$, where $\mathbf{z}^{*1} = (z^{*1}_1, ..., z^{*1}_L)$,..., $\mathbf{z}^{*M} = (z^{*M}_1, ..., z^{*M}_L)$. That is, $\mathbf{z}^{*1} = (z^{*1}_1, ..., z^{*1}_L)$ denotes the equilibrium demand of factors of production by the 1st firm, and so on. Being part of the equilibrium, $\mathbf{w}^* = (w_1^*, ..., w_L^*)$, and $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$ are such that: The total demand for each FOP is equal to its supply (initial endowment), i.e.,

$$(\forall l \in \mathbb{L}) \left[\sum_{j=1}^{M} z^{*j}_l = \bar{z}_l \right],$$

and $\mathbf{z}^{*j}$ is the profit maximizing demand for FOPs by jth firm, i.e., $\mathbf{z}^{*j}$ solves

$$\max_{\mathbf{z}^j} \left\{ \bar{p}_j f^j(\mathbf{z}^j) - \sum_{k=1}^{L} w_k^* . z_k^j, \right\}, i.e., \qquad \max_{\mathbf{z}^j} \left\{ \bar{p}_j f^j(\mathbf{z}^j) - \mathbf{w}^* . \mathbf{z}^j . \right\} \tag{1}$$

Now, if we make the usual assumption that $f^j(.)$ is strictly increasing and strictly concave for all $j = 1, ..., M$, then the above optimization problem will have unique solution. That is, for any given output price vector, $\bar{\mathbf{p}} = (\bar{p}_1, ..., \bar{p}_M)$, and the factor price vector, $\mathbf{w}^* = (w_1^*, ..., w_L^*)$, there is a unique solution to the firm's optimization problem. That is, $\mathbf{z}^{*j} = (z^{*j}_1, ..., z^{*j}_L)$ uniquely maximizes the firm's profits.

Being profit maximizing demand for FOPs, the vectors $\mathbf{z}^{*j} = (z^{*j}_1, ..., z^{*j}_L)$, satisfies the following FOCs for firms optimization problem:

$$\bar{p}_j \frac{\partial f^j(\mathbf{z}^j)}{\partial z_l^j} = w_l^* \text{ for all } l = 1, ..., L, \tag{2}$$

$$\sum_{j=1}^{M} z^{*j}_l = \sum_{j=1}^{M} \frac{\partial c^j(.)}{\partial w_l} = \bar{z}_l \text{ for all } l = 1, ..., L. \tag{3}$$

The first equality in (3) follows from the fact that $z^{*j}_l = \frac{\partial c^j(.)}{\partial w_l}$ (Shephard's Lemma). From (2) and (3) you can see that the demands for FOPs by firms, i.e., $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$ depend on $(w^{*1}, ..., w^{*L})$ and $\bar{\mathbf{z}} = (\bar{z}^1, ..., \bar{z}^L)$, among other things.

Remark: In the general equilibrium framework, the equilibrium output price vector, $\mathbf{p} = (\bar{p}_1, ..., \bar{p}_M)$, and the input/FOP price vector, $(w^*{}_1, ..., w^*{}_L)$, will be determined simultaneously. Both will depend on the given endowment of factors, i.e., $\bar{\mathbf{z}} = (\bar{z}_1, ..., \bar{z}_L)$, production technologies as well as the preferences of the consumers. However, since we want to focus on the equilibrium demand for FOPs, i.e., $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$, we have taken the output price vector, $\mathbf{p} = (\bar{p}_1, ..., \bar{p}_M)$, to be given.

3 Maximizing the Cake-size

Note that the vector of demands for FOPs by firms, i.e., the vector $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$, is also an allocation of FOPs across the firms. Obviously, the firms are interested only in maximizing their profits - as such they do not care whether the factor demand is efficient for the entire economy or not. Therefore, the allocation $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$ may or may not be the 'best' allocation from social view point. In this section we show that the factor allocation induced by the profit maximizing demand for FOPs by the firms, i.e., $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$, also happens to be efficient for the overall economy. In particular, the allocation $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$ maximizes the total social revenue.

Theorem 1 *The equilibrium factor allocation,* $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$*, maximizes the aggregate/total revenue for the economy.*

Given the output price vector, $\bar{\mathbf{p}} = (\bar{p}_1, ..., \bar{p}_M)$ and the input price vector $\mathbf{w}^* = (w_1^*, ..., w_L^*)$, consider any arbitrary allocation of FOPs across firms, say $(\mathbf{z}^1, ..., \mathbf{z}^M)$. At this allocation, the sum of the profits across all firms is given by

$$\begin{aligned}\sum_{j=1}^{M}\left(\bar{p}_j f^j(\mathbf{z}^j) - \mathbf{w}^*.\mathbf{z}^j\right) &= \left(\bar{p}_1 f^1(\mathbf{z}^1) - \mathbf{w}^*.\mathbf{z}^1\right) + ... + \left(\bar{p}_j f^j(\mathbf{z}^j) - \mathbf{w}^*.\mathbf{z}^j\right) \\ &\quad +... + \left(\bar{p}_M f^M(\mathbf{z}^M) - \mathbf{w}^*.\mathbf{z}^M\right) \qquad (4)\end{aligned}$$

where $\mathbf{w}^*.\mathbf{z}^j = \sum_{l=1}^{L} w_l^*.z_l^j$, for all $j = 1, ..., M$. By re-writing the RHS, (4) becomes

$$\sum_{j=1}^{M}\left(\bar{p}_j f^j(\mathbf{z}^j) - \mathbf{w}^*.\mathbf{z}^j\right) = \sum_{j=1}^{M}\bar{p}_j f^j(\mathbf{z}^j) - \sum_{j=1}^{M}\mathbf{w}^*.\mathbf{z}^j \qquad (5)$$

From (1) $\mathbf{z}^{*j}$ solves: $\max_{\mathbf{z}^j}\left\{\bar{p}_j f^j(\mathbf{z}^j) - \sum_{l=1}^{L} w_l^*.z_l^j\right\}$. Therefore, the RHS of the expression (4) takes maximum value at factor allocation $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$. Formally speaking, in view of (4) and (5), $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$ solves

$$\max_{\mathbf{z}^1,...,\mathbf{z}^M}\left\{\sum_{j=1}^{M}\left(\bar{p}_j f^j(\mathbf{z}^j) - \mathbf{w}^*.\mathbf{z}^j\right)\right\}, i.e.,$$

$$\max_{\mathbf{z}^1,...,\mathbf{z}^M}\left\{\sum_{j=1}^{M}\bar{p}_j f^j(\mathbf{z}^j) - \sum_{j=1}^{M}\mathbf{w}^*.\mathbf{z}^j\right\} \qquad (6)$$

Next, note that in equilibrium the total demand for FOPs by all firms should be equal the endowment of the FOPs, i.e., $\sum_j \mathbf{z}^j = \bar{\mathbf{z}}$ must hold. Therefore, $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$ solves:

$$\max_{\mathbf{z}^1,...,\mathbf{z}^M} \left\{ \sum_{j=1}^{M} \bar{p}_j f^j(\mathbf{z}^j) - \sum_{j=1}^{M} \mathbf{w}^*.\mathbf{z}^j, \right\} \tag{7}$$

subject to further constraint that $\sum_j \mathbf{z}^j = \bar{\mathbf{z}}$.

However, when $\sum_j \mathbf{z}^j = \bar{\mathbf{z}}$ must hold, we have $\sum_{j=1}^M \mathbf{w}^*.\mathbf{z}^j = \mathbf{w}^*. \sum_{j=1}^M \mathbf{z}^j = \mathbf{w}^*.\bar{\mathbf{z}}$. That is, $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$ essentially maximizes the total revenue, i.e., solves the following optimization problem:

$$\max_{\mathbf{z}^1,...,\mathbf{z}^M} \left\{ \sum_{j} \bar{p}_j f^j(\mathbf{z}^j), \right\} \tag{8}$$

such that $\sum_j \mathbf{z}^j = \bar{\mathbf{z}}$.

The above result shows that the Competitive equilibrium allocation also happens to be total Revenue maximizing allocation. In other words, the (efficient) allocation of FOPs can be determined without considering price/wage of the individual FOP. This result is used by some 'pro-market' economists to argue the following: In a control and command economy, the objective of the planning authority would be to maximize the total revenue. Since, it is equivalent to maximizing the purchasing power of the economy. However, under competitive markets the decisions of the profit maximizing firms also maximizes the total social revenue. So, the outcome is the same. Specifically, the firms' decisions are in the best social interests, and there is no need to worry if the wage rate received by different FOPs is not fair.

What are your views on the following questions: Should a country focus only on the Revenue maximizing allocation of FOPs? While deciding on allocation of FOPs, can we ignore the issue of equity in distribution of gains from growth?

4 Are Market Wages Fair ?

Recall, $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$ is the Revenue maximizing allocation of FOPs across firms. Let $(\mathbf{z}^1, ..., \mathbf{z}^M)$ be any other allocation of FOPs. Clearly, $\mathbf{z}^* = \sum_j^M \mathbf{z}^{*j}$ denotes the aggregate equilibrium demand, and $\mathbf{z} = \sum_j^M \mathbf{z}^j$ denotes any general level of aggregate demands. Denote

$$F(\mathbf{z}) \;\; = \;\; \bar{p}_1 f^1(\mathbf{z}^1) + ... + \bar{p}_M f^M(\mathbf{z}^M)$$

Note that $F(\mathbf{z}) = \sum_{j=1}^M \bar{p}_j f^j(\mathbf{z}^j)$, i.e., $F(\mathbf{z})$ denotes the total aggregate revenue for the entire economy. Therefore, in view of (6), we know that $(\mathbf{z}^{*1}, ..., \mathbf{z}^{*M})$ solves:

$$\max_{\mathbf{z} \geq \mathbf{0}} \{F(\mathbf{z}) - \mathbf{w}^*.\mathbf{z}\}$$

This optimization problem has the following FOCs:

$$\begin{aligned} w_1^* &= \frac{\partial F(\mathbf{z})}{\partial z_1} \\ \vdots &= \vdots \\ w_L^* &= \frac{\partial F(\mathbf{z})}{\partial z_L} \end{aligned}$$

Since, in equilibrium $\mathbf{z} = \bar{\mathbf{z}}$, we get

$$(\forall l \in \mathbb{L}) \left[w_l^* = \frac{\partial F(\bar{\mathbf{z}})}{\partial z_l} \right]$$

This result shows that in a competitive setting, each FOP is paid equal to its marginal social productivity (which is the marginal revenue product). In this sense, the market wage is said to be 'fair' - each factor is paid according to its marginal contribution to the society. In neoclassical economics this result is known as the 'marginal revenue productivity theory of wages'. The theory says that in competitive markets, wages are equal to the marginal revenue product of the factor of production - which is the increment to revenues produced by the last factor/laborer employed.

Do you remember our critique for this somewhat misleading claim? Can you argue how the distribution of wealth affects the market prices for FOPs?