Is Competitive Equilibrium Unique?

Ram Singh

Lecture 7

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Questions

- Is Competitive/Walrasian equilibrium unique?
- Why is a unique equilibrium helpful?
- If WE is not unique, how many WE can be there?
- What are the conditions, for a unique WE?
- Do these conditions hold in the real world?

Example

Two consumers:

- $u^1(.) = x_1^1 - \frac{1}{8} \left( \frac{1}{x_2^1} \right)^8$ and $u^2(.) = -\frac{1}{8} \left( \frac{1}{x_2^2} \right)^8 + x_2^2$

- $e^1 = (2, r)$ and $e^2 = (r, 2)$; $r = 2^{\frac{8}{9}} - 2^{\frac{1}{9}}$

The equilibria are solution to

$$\left( \frac{p_2}{p_1} \right)^{-\frac{1}{9}} + 2 + r \left( \frac{p_2}{p_1} \right) - \left( \frac{p_2}{p_1} \right)^{\frac{8}{9}} = 2 + r, \text{ i.e.,}$$

there are three equilibria:

$$\frac{p_2}{p_1} = \frac{1}{2}, 1, \text{ and } 2.$$
Consider

- Two goods: food and cloth
- Let \((p_f, p_c)\) be the price vector.
- We know that for all \(t > 0\): \(z(tp) = z(p)\).
- Therefore, we can work with

\[
p = \left( \frac{p_f}{p_c}, 1 \right) = (p, 1).
\]

From Walras’s Law, we have

\[
pz_f(p) + z_c(p) = 0.
\]

Assumptions for existence of WE:

- \(z_i(p)\) is continuous for all \(p >> p\), i.e., for all \(p > 0\).
Multiple WE: Example II

- there exists small $p = \epsilon > 0$ s.t. $z_f(\epsilon, 1) > 0$, and
- there exists another $p' > \frac{1}{\epsilon}$ s.t. $z_c(p', 1) > 0$.

We can ensure above by assuming the utility functions to be continuous and strictly monotonic.

**Question**

- Do the above assumptions guarantee unique WE?
- Under what conditions the WE be unique?

Additional assumption for uniqueness of WE:

- $z'_f(p) < 0$ for all $p > 0$. 

Let,

- there be two goods - food and cloth.
- \( \mathbf{e}^1 = (e^1_f, e^1_c) \) and \( \mathbf{e}^2 = (e^2_f, e^2_c) \) be the initial endowment vectors.
- \( \mathbf{p} = (p, 1) \) be a price vector.

utility functions be continuous, strictly monotonic and strictly quasi-concave.

From Walras Law we have

\[ pz_f(p) + z_c(p) = 0. \]

Assume:

- \( z_i(p) \) is continuous for all \( p \gg 0 \), i.e., for all \( p > 0 \).
- there exists small \( p = \epsilon > 0 \) s.t. \( z_f(\epsilon, 1) > 0 \) and another \( p' > \frac{1}{\epsilon} \) s.t. \( z_c(p', 1) > 0 \).
By definition:

\[ z_f(p) = z_f^1(p) + z_f^2(p) \]
\[ = [x_f^1(p) - e_f^1] + [x_f^2(p) - e_f^2] \]

Since endowments are fixed, we get

\[
\frac{\partial z_f(p)}{\partial p} = \left( \frac{\partial x_f^1(p)}{\partial p} \right)_{du^1=0} - (x_f^1(p) - e_f^1) \left( \frac{\partial x_f^1(p)}{\partial I} \right) \\
+ \left( \frac{\partial x_f^2(p)}{\partial p} \right)_{du^2=0} - (x_f^2(p) - e_f^2) \left( \frac{\partial x_f^2(p)}{\partial I} \right)
\] (1)

Let

\[ p^* \] be an equilibrium price vector. We know that \( p^* \) exists. Why?
WLOG assume that at in equilibrium Person 1 is net buyer of food; i.e., \( x^1_f(p^*) - e^1_f > 0 \).

At equilibrium price, \( p^* \), we have

\[
\frac{\partial z_f(p^*)}{\partial p} = \left( \frac{\partial x^1_f(p^*)}{\partial p} \right)_{du^1=0} - (x^1_f(p^*) - e^1_f) \left( \frac{\partial x^1_f(p^*)}{\partial I} \right) \\
+ \left( \frac{\partial x^2_f(p^*)}{\partial p} \right)_{du^2=0} - (x^2_f(p^*) - e^2_f) \left( \frac{\partial x^2_f(p^*)}{\partial I} \right)
\]

(2)

In equi. (food) market clears. So,

\[
x^2_f(p^*) - e^2_f = -[x^1_f(p^*) - e^1_f].
\]

We can rearrange (2) to get
Normal Goods and Number of Equilibria IV

\[
\frac{\partial z_f(p^*)}{\partial p} = \left( \frac{\partial x_f^1(p^*)}{\partial p} \right)_{d u^1=0} + \left( \frac{\partial x_f^2(p^*)}{\partial p} \right)_{d u^1=0} + \left( x_f^1(p^*) - e_f^1 \right) \left( \frac{\partial x_f^2(p^*)}{\partial l} - \frac{\partial x_f^1(p^*)}{\partial l} \right),
\]

Now, even if both goods are normal,

- Person 2 might have large income effect that can offset the negative substitution effects.
- \( \frac{\partial z_f(p^*)}{\partial p} < 0 \) might not hold.
- So, we cannot be sure of uniqueness of WE.
Let,

- $x(p)$ denote the bundle demanded at price $p$.
- $x' = x(p')$ denote the bundle demanded at price $p'$.

Therefore,

- $x = p.x(p)$ is the expenditure incurred at price $p$.
- $p'.x' = p'.x(p')$ is the expenditure incurred at price $p'$.

- $p.x' \leq p.x$ implies that the bundle $x'$ was affordable at price $p$.
- $p'.x > p'.x'$ implies that the bundle $x = x(p)$ is strictly more expensive (than $x' = x(p')$) at price $p'$. 
Definition
The demand satisfies WARP, if

\[ px' \leq px \implies p'x' < p'x. \]
\[ p(x(p') - x(p)) \leq 0 \implies p'(x(p') - x(p)) < 0. \]

Theorem
*If the aggregate demand satisfies the WARP, then the WE is unique.*
WARP and no of WE III

Proof: Suppose, WE is not unique. If possible, suppose \( p, p' \in E \), and \( p \neq p' \).

Note for any price vectors \( p \) and \( p' \), we have:

\[
\begin{align*}
p.z(p) & = 0, \text{ i.e.,} \\
p.(x(p) - e) & = 0.
\end{align*}
\]

Since \( p' \) is an equi. price vector, \( z(p) = x(p') - e = 0 \), i.e., \( x(p') = e \).

Therefore, the above gives us

\[
\begin{align*}
p.(x(p) - x(p')) & = 0, \text{ i.e.,} \\
p.(x(p') - x(p)) & = 0. \quad (3)
\end{align*}
\]

From WARP, we know that

\[
p(x(p') - x(p)) \leq 0 \Rightarrow p'(x(p') - x(p)) < 0. \quad (4)
\]

(3) and (4) give us,

\[
p'.(x(p') - x(p)) < 0. \quad (5)
\]
Similarly, we get:

\[
p'z(p') = 0, \text{ i.e.,}
\]
\[
p'(x(p') - e) = 0
\]
\[
p'(x(p') - x(p)) = 0
\]  

which is a contradiction, since in view of (5), we have

\[
p'(x(p') - x(p)) < 0.
\]  

**Question**

What are the assumptions needed for the aggregate demand function to exist?
Note:

- Aggregate demand function $x(.)$ will exist iff if the individual demand function, i.e., $x^i(.)$, exists for all $i = 1, .., N$.

- $x^i(.)$ exists if the underlying utility function satisfies the assumption of continuity, strict monotonicity and strict quasi-concavity.