Competitive Equilibrium and its Existence*

Ram Singh†

This Write-up is not for circulation

1 Basics

1.1 Individual UMP

Let us start with the utility maximization problem of the individuals in the economy. We assume that individual consumers are price-takers. Let the set of price vectors be $\mathbf{p} = (p_1, ..., p_M) \in \mathbb{R}^M_{++}$. That is, $(p_1, ..., p_M) > (0, ..., 0)$. Now, the consumer i will choose the her optimum bundle by solving:

$$\max_{\mathbf{x}\in\mathbb{R}^J_+} u^i(\mathbf{x}) \quad s.t. \quad \mathbf{p}.\mathbf{x} \leq \mathbf{p}.\mathbf{e}^i$$

Assumption 1 For all $i \in I$, u^i is continuous, strongly increasing, and strictly quasi-concave on $\mathbb{R}^M_+$.

The utility function, u^i, is said to be strongly increasing if for any two bundles $\mathbf{x}$ and $\mathbf{x}'$

$$\mathbf{x}' \geq \mathbf{x} \Rightarrow u^i(\mathbf{x}') > u^i(\mathbf{x}).$$

In view of monotonicity of the preferences, for given $\mathbf{p} = (p_1, ..., p_M) >> (0, ..., 0)$, consumer i solves:

$$\max_{\mathbf{x}\in\mathbb{R}^M_+} u^i(\mathbf{x}) \quad s.t. \quad \mathbf{p}.\mathbf{x} = \mathbf{p}.\mathbf{e}^i \tag{1}$$

From the first part of the course, you know that when $u^i(.)$ satisfies assumptions listed above, the following result holds.

Theorem 1 *Under the above assumptions on $u^i(.)$, for every $(p_1, ..., p_M) > (0, ..., 0)$, (1) has a unique solution, say $\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i)$.*

*References are: Arrow and Debreu (1954), and McKenzie (2008); Arrow and Hahn (1971). Jehle and Reny (2008).

†Delhi School of Economics, University of Delhi. Email:

Note: For each $i = 1, ..., N$,

$$\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) : \mathbb{R}^M_{++} \mapsto \mathbb{R}^M_+;$$

$\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) = (x^i_1(\mathbf{p}, \mathbf{p}.\mathbf{e}^i, x^i_M(\mathbf{p}, \mathbf{p}.\mathbf{e}^i)$.

Theorem 2 *Under the above assumptions on $u^i(.)$, for every $(p_1, ..., p_M) > (0, ..., 0)$,*

- $\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i)$ *is continuous in* $\mathbf{p}$ *over* $\mathbb{R}^M_+$.
- *For all $i = 1, 2, ..., N$, we have: $\mathbf{x}^i(t\mathbf{p}) = \mathbf{x}^i(\mathbf{p})$, for all $t > 0$. That is, demand of each good j by individual i satisfies the following property:*

$$x^i_j(t\mathbf{p}) = x^i_j(\mathbf{p}) \text{ for all } t > 0.$$

Question 1 *Given that $u^i(.)$ is strongly increasing,*

- *is $\mathbf{x}^i(\mathbf{p})$ continuous over $\mathbb{R}^M_+$?*
- *is the demand function $x^i_j(\mathbf{p})$ defined at $p_j = 0$?*

1.2 Excess Demand Function

Definition 1 *The excess demand for jth good by the ith individual is give by:*

$$\mathbf{z}^i_j(\mathbf{p}) = x^i_j(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - e^i_j.$$

The aggregate excess demand for jth good is give by:

$$\mathbf{z}_j(\mathbf{p}) = \sum_{i=1}^{N} x^i_j(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - \sum_{i=1}^{N} e^i_j.$$

So, Aggregate Excess Demand Function is:

$$\mathbf{z}(\mathbf{p}) = (\mathbf{z}_1(\mathbf{p}), ..., \mathbf{z}_M(\mathbf{p})),$$

Theorem 3 *Under the above assumptions on $u^i(.)$, for any $\mathbf{p} >> \mathbf{0}$,*

- $\mathbf{z}(.)$ *is continuous in* $\mathbf{p}$
- $\mathbf{z}(t\mathbf{p}) = \mathbf{z}(\mathbf{p})$, *for all $t > 0$*
- $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$. *(*the Walras' Law*)*

For any given price vector $\mathbf{p}$, we have

$$\begin{aligned} \mathbf{p}.\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - \mathbf{p}.\mathbf{e}^i &= 0, i.e., \\ \sum_{j=1}^{M} p_j[x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - e_j^i] &= 0. \end{aligned}$$

This gives:

$$\begin{aligned} \sum_{i=1}^{N}\sum_{j=1}^{M} p_j[x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - e_j^i] &= 0, i.e., \\ \sum_{j=1}^{M}\sum_{i=1}^{N} p_j[x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - e_j^i] &= 0, i.e., \\ \sum_{j=1}^{M} p_j\left[\sum_{i=1}^{N} x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - \sum_{i=1}^{N} e_j^i\right] &= 0 \end{aligned}$$

That is,

$$\begin{aligned} \sum_{j=1}^{M} p_j z_j(\mathbf{p}) &= 0, i.e., \\ \mathbf{p}.\mathbf{z}(\mathbf{p}) &= 0 \end{aligned}$$

So,

$$p_1 z_1(\mathbf{p}) + p_2 z_2(\mathbf{p}) + ... + p_{j-1} z_{j-1}(\mathbf{p}) + p_{j+1} z_{j+1}(\mathbf{p}) + +p_M z_M(\mathbf{p}) = -p_j z_j(\mathbf{p})$$

For a price vector $\mathbf{p} >> \mathbf{0}$,

- if $z_{j'}(\mathbf{p}) = 0$ for all $j' \neq j$, then $z_j(\mathbf{p}) = 0$
- For two goods case,
$$p_1 z_1(\mathbf{p}) = -p_2 z_2(\mathbf{p}).$$

So,

$$z_1(\mathbf{p}) > 0 \Rightarrow z_2(\mathbf{p}) < 0; \text{ and } z_1(\mathbf{p}) = 0 \Rightarrow z_2(\mathbf{p}) = 0$$

1.3 Walrasian Equilibrium

Definition 2 *Walrasian Equilibrium Price: A price vector $\mathbf{p}^*$ is equilibrium price vector, if for all $j = 1, ..., J$,*

$$\mathbf{z}_j(\mathbf{p}^*) = \sum_{i=1}^{N} x_j^i(\mathbf{p}^*, \mathbf{p}^*.\mathbf{e}^i) - \sum_{i=1}^{N} e_j^i = 0, \quad i.e., \text{ if}$$

$$\mathbf{z}(\mathbf{p}^*) = \mathbf{0} = (0, ..., 0).$$

Two goods: food and cloth

Let (p_f, p_c) be the price vector. We can work with $\mathbf{p} = (\frac{p_f}{p_c}, 1) = (p, 1)$. Since, we know that for all $t > 0$:

$$\mathbf{z}(t\mathbf{p}) = \mathbf{z}(\mathbf{p})$$

Therefore, we have

$$pz_f(\mathbf{p}) + z_c(\mathbf{p}) = 0.$$

Assumptions:

- $z_i(\mathbf{p})$ is continuous for all $\mathbf{p} >> \mathbf{0}$, i.e., for all $p > 0$.
- there exists small $p = \epsilon > 0$ s.t. $z_f(\epsilon, 1) >> 0$ and another $p' > \frac{1}{\epsilon}$ s.t. $z_f(p', 1) << 0$.

2 Existence of Walrasian Equilibrium: General Case

As demonstrated above, the individual demand functions are homogenous functions of degree zero. That is, for all $i = 1, 2, ..., N$, $\mathbf{x}^i(t\mathbf{p}) = \mathbf{x}^i(\mathbf{p})$, for all $t > 0$. Moreover, the excess demand function is also homogenous function of degree zero. So, it has the following property: $\mathbf{z}(t\mathbf{p}) = \mathbf{z}(\mathbf{p})$, for all $t > 0$.

Without any loss of generality, we can restrict attention to the following set of prices:

$$\mathbb{P}_\epsilon = \left\{ \mathbf{p} = (p_1, ..., p_M) | \sum_{j=1}^{M} p_j = 1 \text{ and } p_j \geq \frac{\epsilon}{1 + 2M} \right\},$$

where $\epsilon > 0$.

Note that the set $\mathbb{P}_\epsilon$ contains its boundaries. So, it is closed. Moreover, it is easily seen that the $\mathbb{P}_\epsilon$ is non-empty, bounded, and convex set for all $\epsilon \in (0, 1)$.

Theorem 4 Suppose $u^i(.)$ satisfies the above assumptions, and $\mathbf{e} >> \mathbf{0}$. Let $\{\mathbf{p}^s\}$ be a sequence of price vectors in $\mathbb{R}^M_{++}$, such that

- $\{\mathbf{p}^s\}$ converges to $\bar{\mathbf{p}}$, where
- $\bar{\mathbf{p}} \in \mathbb{R}^M_+$, $\bar{\mathbf{p}} \neq \mathbf{0}$, but for some j, $\bar{p}_j = 0$.

Then, for some good k with $\bar{p}_k = 0$, the sequence of excess demand (associated with $\{\mathbf{p}^s\}$), say $\{z_k(\mathbf{p}^s)\}$, is unbounded above.

Theorem 5 *Under the above assumptions on u^i, there exists a price vector $\mathbf{p}^* >> \mathbf{0}$, such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.*

2.1 WE: Proof

We are familiar with the properties of the excess demand function $z_j(\mathbf{p})$ for every good, $j = 1, ..., M$. In the proof we will use this function to derive some other functions that will be useful in proving the result. First of all, let us define a function,

$$\bar{z}_j(\mathbf{p}) = \min\{z_j(\mathbf{p}), 1\}. \tag{2}$$

Note by its specification, $\bar{z}_j(\mathbf{p}) = \min\{z_j(\mathbf{p}), 1\} \leq 1$. Therefore, we have

$$0 \leq \max\{\bar{z}_j(\mathbf{p}), 0\} \leq 1.$$

Next, we want to define a function $f(\mathbf{p}) = (f_1(\mathbf{p}), ..., f_M(\mathbf{p})) : \mathbb{P}_\epsilon \mapsto \mathbb{P}_\epsilon$. Note that $f(\mathbf{p}) : \mathbb{P}_\epsilon \mapsto \mathbb{P}_\epsilon$ if and only two conditions are met. First, $f_1(\mathbf{p}) \geq \geq \frac{\epsilon}{1+2M}$ should hold for every $j = 1, .., M$. Second, $\sum_{j=1}^{M} f_j(\mathbf{p}) = 1$.

Suppose, we specify a function such that: For $j = 1, .., M$,

$$f_j(\mathbf{p}) = \frac{\epsilon + p_j + \max\{\bar{z}_j(\mathbf{p}), 0\}}{\epsilon M + 1 + \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}), 0\}} = \frac{N_j(\mathbf{p})}{D(\mathbf{p})},$$

For this specification, we have $\sum_{j=1}^{M} f_j(\mathbf{p}) = 1$. Moreover, using the facts that $\max\{\bar{z}_j(\mathbf{p}), 0\} \leq 1$, $\epsilon < 1$ and $p_j > 0$, you can check that the following inequalities hold:

$$f_j(\mathbf{p}) \geq \frac{N_j(\mathbf{p})}{\epsilon M + 1 + M.1} \geq \frac{\epsilon}{\epsilon M + 1 + M.1} \geq \frac{\epsilon}{1 + 2M}.$$

Therefore, both of the above conditions are satisfied. So,

$$f(\mathbf{p}) = (f_1(\mathbf{p}), ..., f_M(\mathbf{p})) : \mathbb{P}_\epsilon \mapsto \mathbb{P}_\epsilon.$$

Also, since $D(\mathbf{p}) \geq 1 > 0$, the function $f(\mathbf{p})$ is a well defined and continuous function defined over a compact and convex domain. Therefore, by the Brouwer's fixed-point theorem, a 'Fixed Point' exists. That is, there exists $\mathbf{p}^\epsilon$ such that

$$f(\mathbf{p}^\epsilon) = \mathbf{p}^\epsilon, i.e.,$$

For all $j = 1, .., M$, we have: $f_j(\mathbf{p}^\epsilon) = p_j^\epsilon$. Using the full form of $f_j(.)$, this implies that for all $j = 1, .., M$,

$$\begin{aligned} \frac{\epsilon + p_j + \max\{\bar{z}_j(\mathbf{p}), 0\}}{\epsilon M + 1 + \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}), 0\}} &= p_j^\epsilon, i.e., \\ p_j^\epsilon[M\epsilon + \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^\epsilon), 0\}] &= \epsilon + \max\{\bar{z}_j(\mathbf{p}^\epsilon), 0\}. \end{aligned} \tag{3}$$

Next, we let $\epsilon \to 0$. Consider the sequence of price vectors $\{\mathbf{p}^\epsilon\}$, as $\epsilon \to 0$.

- Sequence $\{\mathbf{p}^{\epsilon}\}$, as $\epsilon \to 0$, has a convergent subsequence, say $\{\mathbf{p}^{\epsilon'}\}$. Why?
- Let $\{\mathbf{p}^{\epsilon'}\}$ converge to $\mathbf{p}^*$, as $\epsilon \to 0$.
- Clearly, $\mathbf{p}^* \geq \mathbf{0}$. Why?

Suppose, $p_k^* = 0$. Recall, we have

$$p_k^{\epsilon'}\left[M\epsilon' + \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^{\epsilon'}), 0\}\right] = \epsilon' + \max\{\bar{z}_k(\mathbf{p}^{\epsilon'}), 0\}. \tag{4}$$

as $\epsilon' \to 0$ while the LHS converges to 0, since $\lim_{\epsilon' \to 0} p_k^{\epsilon'} = 0$ and term $[M\epsilon' + \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^{\epsilon'}), 0\}]$ on LHS is bounded.

However, the RHS takes value 1 infinitely many times. Why? This is a contradiction, because the equality in (4) holds for all values of ϵ'. Therefore, $p_j^* > 0$ for all $j = 1, .., M$. That is,

$$\mathbf{p}^* >> \mathbf{0}, i.e.,$$

$$\lim_{\epsilon \to 0} \mathbf{p}^{\epsilon} = \mathbf{p}^* >> \mathbf{0}.$$

In view of continuity of $\bar{z}(\mathbf{p})$ over $\mathbb{R}^M_{++}$,
from (4) we get (by taking limit $\epsilon \to 0$):
For all $j = 1, .., M$

$$\begin{aligned}
p_j^* \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^*), 0\} &= \max\{\bar{z}_j(\mathbf{p}^*), 0\}, i.e., \\
z_j(\mathbf{p}^*) p_j^* \left(\sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^*), 0\}\right) &= z_j(\mathbf{p}^*) \max\{\bar{z}_j(\mathbf{p}^*), 0\}, i.e., \\
\sum_{j=1}^{M} z_j(\mathbf{p}^*) p_j^* \left(\sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^*), 0\}\right) &= \sum_{j=1}^{M} z_j(\mathbf{p}^*) \max\{\bar{z}_j(\mathbf{p}^*), 0\}, i.e.,
\end{aligned}$$

$$\sum_{j=1}^{M} z_j(\mathbf{p}^*) \max\{\bar{z}_j(\mathbf{p}^*), 0\} = 0. \tag{5}$$

You can verify that, given the definition of $\bar{z}_j(\mathbf{p}^*)$:

$$\begin{aligned}
z_j(\mathbf{p}^*) > 0 &\Rightarrow \max\{\bar{z}_j(\mathbf{p}^*), 0\} > 0; \\
z_j(\mathbf{p}^*) \leq 0 &\Rightarrow \max\{\bar{z}_j(\mathbf{p}^*), 0\} = 0.
\end{aligned}$$

Suppose, for some j, we have $z_j(\mathbf{p}^*) > 0$, then we will have

$$\sum_{j=1}^{M} z_j(\mathbf{p}^*) \max\{\bar{z}_j(\mathbf{p}^*), 0\} > 0. \tag{6}$$

But, this is a contradiction in view of (5). Therefore:

For any $j = 1, .., M$, we have

$$z_j(\mathbf{p}^*) \leq 0. \tag{7}$$

Suppose, $z_k(\mathbf{p}^*) < 0$ for some k. We know

$$p_1^* z_1(\mathbf{p}^*) + ... + p_k^* z_k(\mathbf{p}^*) + ... + p_M^* z_M(\mathbf{p}^*) = 0.$$

Since $p_j^* > 0$ for all $j = 1, .., M$.

$z_k(\mathbf{p}^*) < 0$ implies: There exists k', such that

$$z_{k'}(\mathbf{p}^*) > 0, \tag{8}$$

which is a contradiction in view of (7). Therefore,

$$\begin{aligned} \text{For all } \ j = 1, .., M, \ \text{we have:} z_j(\mathbf{p}^*) &= 0, i.e., \\ \mathbf{z}(\mathbf{p}^*) &= \mathbf{0}. \end{aligned}$$