Course 005
Midterm exam 1: Answer Key

Question 1

Discussed in class

Question 2a

Let q_i be the quantity produced by firm F_i, where $i = 0, 1, 2, 3..., n$.
We solve for SPNE using backward induction. After F_0 chooses q_0, followers' choice of $(q_1 \dots, q_n)$ must be a Nash equilibrium. That is given q_0 and q_{-i}, q_i must be best response for Firm i.
Profit of Firm i is denoted by π_i.

$$\pi_i = (1 - q - c)q_i = \big(1 - q_0 - \sum_{k=1}^{n} q_k - c\big)q_i \tag{1}$$

Differentiating w.r.t q_i, FOC for i-th firm, we get

$$1 - q_0 - \sum_{k=1}^{n} q_k - c = q_i \tag{2-i}$$

Solving Equation $(2-1)$ to $(2-n)$, we obtain,

$$q_i^*(q_0) = \frac{1 - q_0 - c}{n + 1} \tag{2}$$

Next, we focus on the choice of q_0 by F_0 at the beginning of the game. Note F_0 takes into account reaction function of followers while maximizing its profit function.

$$\pi_0 = (1 - q - c)q_0 = \left[1 - q_0 - n\left(\frac{1 - q_0 - c}{n + 1}\right)\right] q_0 \tag{3}$$

Differentiating w.r.t q_0 and rearranging we get, $q_0^* = \dfrac{1 - c}{2}$.
SPNE strategies are: $q_0^* = \dfrac{1 - c}{2}$ and $q_i^*(q_0) = \dfrac{1 - q_0 - c}{n + 1}$

[Note: Strategies must be properly specified. In the final examination marks will be deducted for such mistakes.]

Question 2b and 2c

Consider the following strategies,

F_0: $\hat{q}_0 = \dfrac{1-c}{n+2}$

(For all $i = 1, \ldots, n$) F_i: $q_i(\hat{q}_0) = \dfrac{1-c}{n+2}$, otherwise for all $q_0 \neq \hat{q}_0$, $q_i(q_0) = \dfrac{1-c}{n}$.

Given F_0's choice, $\hat{q}_0 = \dfrac{1-c}{n+2}$, all followers are playing Nash equilibrium strategy because from Equation (2), $q_i^*(\hat{q}_0) = \dfrac{1-c}{n+2}$.

Given strategies of the followers, $q_1(q_0), \ldots, q_n(q_0)$, F_0 earns zero profit at $q_0 \neq \hat{q}_0$ and earns positive profit at $\hat{q}_0$.

Thus the proposed strategies are mutual best response and hence is a Nash equilibrium.

However, this is not a SPNE because at subgames where $q_0 \neq \hat{q}_0$, followers are not playing Nash equilibrium response (given by Equation (2)).

Question 3

The game comprises of 2 stages. In the second stage, at any history, all present employers decide to hire the second candidate because $0 \leq v_2 \leq 10$ and payoff from not hiring is 0

In the first stage, if both employers decide to employ candidate 1, then they hire her with probability 0.5. In this case, the employer who loses the toss ends up hiring candidate 2 in the next period. Therefore, the expected payoff for each employer when both decide to hire in stage 1 is $0.5v_1 + 0.5\mathbb{E}[v_2]$. If one employer decides to hire then her payoff is v_1 and that of the other employer is $\mathbb{E}[v_2]$. Lastly, if both do not hire in the first stage then their expected payoff is $0.5\mathbb{E}[v_2]$, each.

Notice that $\mathbb{E}[v_2] = 5$. The first stage game can be expressed in the following way.

		Employer 2	
		Hire	Not hire
Employer 1	Hire	$0.5v_1 + 0.5\mathbb{E}[v_2]$, $0.5v_1 + 0.5\mathbb{E}[v_2]$	v_1, $\mathbb{E}[v_2]$
	Not hire	$\mathbb{E}[v_2]$, v_1	$0.5\mathbb{E}[v_2]$, $0.5\mathbb{E}[v_2]$

There are 4 different NE of the first stage game, depending on the value of v_1.

- If $v_1 \geq 5$, both employers decide to hire
- If $2.5 \leq v_1 \leq 5$, then one employer hires and the other does not
- If $v_1 < 2.5$, both employers do not hire

Thus SPNE strategies are:
Employer i: (Stage 1) Hire only if $v_1 \geq 2.5$, (Stage 2, at every history) Hire irrespective of v_2
Employer j: (Stage 1) Hire only if $v_1 \geq 5$, (Stage 2, at every history) Hire irrespective of v_2