Market Equilibrium Price: Existence, Properties and Consequences

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Lecture 5

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Today, we will discuss the following issues:

- How does the Adam Smith’s *Invisible Hand* work?
- Is increase in Prices bad?
- Do some people want increase in prices?
- If yes, who would want an increase in prices and of what type?
- Does increase in prices have distributive consequences?
Individual UMP: Some Features I

Notations:

- \( \mathbf{p} = (p_1, \ldots, p_M) \) is a \( M \)-component vector in \( \mathbb{R}^M \).
- If \( \mathbf{p} = (p_1, \ldots, p_M) \in \mathbb{R}^{M}_{++} \), then \( p_j > 0 \) for all \( j = 1, \ldots, M \), i.e.,
  \[
  (p_1, \ldots, p_M) > (0, \ldots, 0).
  \]
- If \( \mathbf{p} = (p_1, \ldots, p_M) \in \mathbb{R}^M_{+} \), then \( p_j \geq 0 \) for all \( j \in \{1, \ldots, M\} \) and \( p_j > 0 \) for some \( j \in \{1, \ldots, M\} \), i.e.,
  \[
  (p_1, \ldots, p_M) \geq (0, \ldots, 0) \text{ and } (p_1, \ldots, p_M) \neq (0, \ldots, 0).
  \]
- Let \( \mathbf{x} = (x_1, \ldots, x_M) \) and \( \mathbf{x}' = (x'_1, \ldots, x'_M) \). If \( \mathbf{x}' \geq \mathbf{x} \), then \( x'_j \geq x_j \) for all \( j \in \{1, \ldots, M\} \) and \( x'_j > x_j \) for some \( j \in \{1, \ldots, M\} \).
- Let \( \mathbf{x} = (x_1, \ldots, x_M) \) and \( \mathbf{x}' = (x'_1, \ldots, x'_M) \). If \( \mathbf{x}' > \mathbf{x} \), then \( x'_j > x_j \) for all \( j \in \{1, \ldots, M\} \).
Individual UMP: Some Features II

Take a price vector \( p = (p_1, \ldots, p_M) \in \mathbb{R}_+^M \). That is, \((p_1, \ldots, p_M) > (0, \ldots, 0)\). The consumer \( i \)'s OP (UMP) is to solves:

\[
\max_{x \in \mathbb{R}_+^J} u^i(x) \quad \text{s.t.} \quad p.x \leq p.e^i
\]

**Definition**

\( u^i \) is strongly increasing if for any two bundles \( x \) and \( x' \)

\[
x' \geq x \Rightarrow u^i(x') > u^i(x).
\]

**Assumption**

For all \( i \in I \), \( u^i \) is continuous, strongly increasing, and strictly quasi-concave on \( \mathbb{R}_+^M \)
In view of monotonicity, for given $p = (p_1, \ldots, p_M) \gg (0, \ldots, 0)$, consumer $i$ solves:

$$\max_{x \in \mathbb{R}_+^M} u^i(x) \quad s.t. \quad p.x = p.e_i$$

(1)

**Theorem**

*Under the above assumptions on $u^i(.)$, for every $(p_1, \ldots, p_M) \gg (0, \ldots, 0)$, (1) has a unique solution, say $x^i(p, p.e_i)$.***

**Note:**

- Existence follows from Monotonicity and Boundedness of the Budget set
- Uniqueness follows from ‘strictly quasi-concavity’
Note:

- $x^i(p, p.e^i)$ is the (Marshallian) Demand Function for individual $i$.

For each $i = 1, \ldots, N$,

$$x^i(p, p.e^i) : \mathbb{R}^M_+ \rightarrow \mathbb{R}^M_+;$$

$$x^i(p, p.e^i) = (x^i_1(p, p.e^i), \ldots, x^i_j(p, p.e^i), \ldots, x^i_M(p, p.e^i)).$$

- In general, demand for $j$th good depends on price of $k$th good, $k = 1, \ldots, M$

- Demand depend for $j$th good depends on price of $k$th good relative to the other prices
Theorem

Under the above assumptions on $u^i(.)$, for every $(p_1, \ldots, p_M) > (0, \ldots, 0)$,

- $x^i(p, p.e^i)$ is continuous in $p$ over $\mathbb{R}^M_{++}$.
- For all $i = 1, 2, \ldots, N$, we have: $x^i(tp) = x^i(p)$, for all $t > 0$. That is, demand of each good $j$ by individual $i$ satisfies the following property:
  
  $$x^i_j(tp) = x^i_j(p) \text{ for all } t > 0.$$ 

Question

Given that $u^i(.)$ is strongly increasing,

- is $x^i(p)$ continuous over $\mathbb{R}^M_+$?
- is the demand function $x^i_j(p)$ defined at $p_j = 0$?

Is a Cobb-Douglas utility function strongly increasing over $\mathbb{R}^M_+$?
Excess Demand Function I

Definition
The excess demand for $j$th good by the $i$th individual is given by:

$$z^i_j(p) = x^i_j(p, p.e^i) - e^i_j.$$

The aggregate excess demand for $j$th good is given by:

$$z_j(p) = \sum_{i=1}^{N} x^i_j(p, p.e^i) - \sum_{i=1}^{N} e^i_j.$$

So, Aggregate Excess Demand Function is a vector-valued function:

$$z(p) = (z_1(p), ..., z_j(p), ..., z_M(p)).$$
Excess Demand Function II

Theorem

Under the above assumptions on $u_i(\cdot)$, for any $p \gg 0$,

- $z(\cdot)$ is continuous in $p$
- $z(tp) = z(p)$, for all $t > 0$
- $p.z(p) = 0$. (the Walras’ Law)

For any given price vector $p$, the individual UMP gives us

$$p.x^i(p, p.e^i) - p.e^i = 0, \text{ i.e.},$$

$$\sum_{j=1}^{M} p_jx_j^i(p, p.e^i) - \sum_{j=1}^{M} p_je_j^i = 0, \text{ i.e.},$$

$$\sum_{j=1}^{M} p_j[x_j^i(p, p.e^i) - e_j^i] = 0.$$
Excess Demand Function III

This gives:

\[
\begin{align*}
\sum_{i=1}^{N} \sum_{j=1}^{M} p_j [x_j^i(p, p.e^i) - e_j^i] &= 0, \text{i.e.,} \\
\sum_{j=1}^{M} \sum_{i=1}^{N} p_j [x_j^i(p, p.e^i) - e_j^i] &= 0, \text{i.e.,} \\
\sum_{j=1}^{M} p_j \left[ \sum_{i=1}^{N} x_j^i(p, p.e^i) - \sum_{i=1}^{N} e_j^i \right] &= 0
\end{align*}
\]

That is,

\[
\begin{align*}
\sum_{j=1}^{M} p_j z_j(p) &= 0, \text{i.e.,} \\
p.z(p) &= 0
\end{align*}
\]
Excess Demand Function IV

So,

\[ p_1 z_1(p) + p_2 z_2(p) + \ldots + p_{j-1} z_{j-1}(p) + p_{j+1} z_{j+1}(p) + \ldots + p_M z_M(p) = -p_j z_j(p) \]

For a price vector \( p \gg 0 \),

- if \( z_k(p) = 0 \) for all \( k \neq j \), then \( z_j(p) = 0 \)
- For two goods case
  - \( p_1 z_1(p) + p_2 z_2(p) = 0 \), i.e.,
    \[ p_1 z_1(p) = -p_2 z_2(p). \]
  - Therefore,
    \[ z_1(p) = 0 \Rightarrow z_2(p) = 0 \]
    \[ z_1(p) > 0 \Rightarrow z_2(p) < 0. \]
Walrasian Equilibrium

Definition
Walrasian Equilibrium Price: A price vector \( p^* \) is equilibrium price vector, if for all \( j = 1, \ldots, J \),

\[
z_j(p^*) = \sum_{i=1}^{N} x_{ij}(p^*, p^*, e^i) - \sum_{i=1}^{N} e_j = 0, \text{ i.e., if } z(p^*) = 0 = (0, \ldots, 0).
\]

Proposition
If \( p^* \) is equilibrium price vector, then \( p' = tp^* \), \( t > 0 \), is also an equilibrium price vector

If \( p^* \) is equilibrium price vector, then \( p' \neq tp^* \), \( t > 0 \), may or may not be an equilibrium price vector
**Two goods**: food and cloth

Let \((p_f, p_c)\) be the price vector.

We can work with \(p = (\frac{p_f}{p_c}, 1) = (p, 1)\). Since, we know that for all \(t > 0\):

\[ z(tp) = z(p) \]

Therefore, we have \(p_f z_f(p) + p_c z_c(p) = 0\), i.e.,

\[ p z_f(p) + z_c(p) = 0. \]

Assume:

- \(z_i(p)\) is continuous for all \(p \gg 0\), i.e., for all \(p > 0\).
Note

- When utility function is monotonic, \( x_f(p) \) will explode as \( p_f = p \rightarrow 0 \).
  Therefore,

- there exists small \( p = \epsilon > 0 \) s.t. \( z_f(p, 1) > 0 \) and \( z_c(p, 1) < 0 \) (Why?).

- there exists another \( p' > \frac{1}{\epsilon} \) s.t. \( z_f(p', 1) < 0 \) and \( z_c(p, 1) > 0 \). (Why?).

Therefore, for a two goods case we have:

- There is a value of \( p \) such that \( z_f(p, 1) = 0 \) and \( z_c(p, 1) = 0 \)

- That is, there exists a WE price vector.

In general, Equilibrium price is determined by *Tatonnement* process