Walrasian Equilibrium: Proof of Existence

Ram Singh

Lecture 6

September 27, 2017
Definition

Walrasian Equilibrium Price: A price vector $p^*$ is equilibrium price vector, if for all $j = 1, ..., M,$

$$z_j(p^*) = \sum_{i=1}^{N} x_j^i(p^*, p^*.e^i) - \sum_{i=1}^{N} e_j^i = 0,$$  i.e., if

$$z(p^*) = 0 = (0, ..., 0).$$

Proposition

If $p^*$ is equilibrium price vector, then $p' = tp^*$, $t > 0$, is also an equilibrium price vector.

If $p^*$ is equilibrium price vector, then $p' \neq tp^*$, $t > 0$, may or may not be an equilibrium price vector.
References: Jehle and Reny (2008).

Also see: Arrow and Debreu (1954), and McKenzie (2008); Arrow and Hahn (1971).

Recall, we have

- For all $i = 1, 2, ..., N$, we have: $x^i(tp) = x^i(p)$, for all $t > 0$.
- $z(tp) = z(p)$, for all $t > 0$.

Let

- $\hat{p} = (\hat{p}_1, ..., \hat{p}_M) > (0, ..., 0)$ be a price vector
- $t = \frac{1}{\hat{p}_1, ..., \hat{p}_M}$
Then $p = t \hat{p}$ is such that

\[ p_j \geq 0 \text{ and } \sum_{j=1}^{M} p_j = 1 \]

Without any loss of generality, we can restrict attention to the following set

\[ \mathbb{P}_\epsilon = \left\{ p = (p_1, \ldots, p_M) \mid \sum_{j=1}^{M} p_j = 1 \text{ and } p_j \geq \frac{\epsilon}{1 + 2M} \right\}, \]

where $\epsilon \geq 0$.

**Remark**

Note that $\mathbb{P}_\epsilon$ is non-empty, bounded, closed and convex set for all $\epsilon \in (0, 1)$. 

Theorem

Suppose \( u^i(.) \) satisfies the above assumptions, and \( \mathbf{e} \gg \mathbf{0} \). Let \( \{ \mathbf{p}^s \} \) be a sequence of price vectors in \( \mathbb{R}^M_+ \), such that

- \( \{ \mathbf{p}^s \} \) converges to \( \bar{\mathbf{p}} \), where
- \( \bar{\mathbf{p}} \in \mathbb{R}^M_+ \), \( \bar{\mathbf{p}} \neq \mathbf{0} \), but for some \( j \), \( \bar{p}_j = 0 \).

Then, for some good \( k \) with \( \bar{p}_k = 0 \), the sequence of excess demand (associated with \( \{ \mathbf{p}^s \} \)), say \( \{ z_k(\mathbf{p}^s) \} \), is unbounded above.

Theorem

Under the above assumptions on \( u^i \), there exists a price vector \( \mathbf{p}^* \gg \mathbf{0} \), such that \( z(\mathbf{p}^*) = \mathbf{0} \).
Let,

$$\bar{z}_j(p) = \min \{z_j(p), 1\}. \quad (1)$$

Note

$$\bar{z}_j(p) = \min \{z_j(p), 1\} \leq 1.$$ Therefore,

$$0 \leq \max \{\bar{z}_j(p), 0\} \leq 1.$$ Next, let’s define the function \( f = (f_1(p), ..., f_M(p)) \) such that: For \( j = 1, .., M, \)

$$f_j(p) = \frac{\epsilon + p_j + \max \{\bar{z}_j(p), 0\}}{\epsilon M + 1 + \sum_{j=1}^{M} \max \{\bar{z}_j(p), 0\}} = \frac{N_j(p)}{D(p)},$$

Note that \( \sum_{j=1}^{M} f_j(p) = 1. \) Moreover, we have

$$f_j(p) \geq \frac{N_j(p)}{\epsilon M + 1 + M.1} \geq \frac{\epsilon}{\epsilon M + 1 + M.1} \geq \frac{\epsilon}{1 + 2M}.$$
Therefore,
\[ f(p) : \mathbb{P}_\epsilon \mapsto \mathbb{P}_\epsilon. \]

Since \( D(p) \geq 1 > 0 \), the above function is well defined and continuous over a compact and convex domain. Therefore, there exists \( p^\epsilon \) such that
\[ f(p^\epsilon) = p^\epsilon, \text{ i.e.,} \]
\[ f_j(p^\epsilon) = p_j^\epsilon, \quad \text{for all } j = 1, \ldots, M. \]

That is, for all \( j = 1, \ldots, M \),
\[
\frac{N_j(p^\epsilon)}{D(p^\epsilon)} = p_j^\epsilon, \text{ i.e.,}
\]
\[
p_j^\epsilon[M\epsilon + \sum_{j=1}^{M} \max\{\bar{z}_j(p^\epsilon), 0\}] = \epsilon + \max\{\bar{z}_j(p^\epsilon), 0\}. \tag{2}
\]

Next, we let \( \epsilon \to 0 \). Consider the sequence of price vectors \( \{p^\epsilon\} \), as \( \epsilon \to 0 \).
WE: Proof III

- Sequence \( \{p^\varepsilon\} \), as \( \varepsilon \to 0 \), has a convergent subsequence, say \( \{p^{\varepsilon'}\} \). Why?

- Let \( \{p^{\varepsilon'}\} \) converge to \( p^* \), as \( \varepsilon' \to 0 \).

- Clearly, \( p^* \geq 0 \). Why?

Suppose, \( p^*_k = 0 \). Recall, we have

\[
p^{\varepsilon'}_k \left[ M\varepsilon' + \sum_{j=1}^{M} \max\{\bar{z}_j(p^{\varepsilon'}), 0\} \right] = \varepsilon' + \max\{\bar{z}_k(p^{\varepsilon'}), 0\}. \tag{3}
\]

as \( \varepsilon' \to 0 \):

- The LHS converges to 0, since \( \lim_{\varepsilon' \to 0} p^{\varepsilon'}_k = 0 \) and term \( [M\varepsilon' + \sum_{j=1}^{M} \max\{\bar{z}_j(p^{\varepsilon'}), 0\}] \) on LHS is bounded.

- However, the RHS takes value 1 infinitely many times. Why?
This is a contradiction, because the equality in (3) holds for all values of $\epsilon'$. Therefore, $p^*_j > 0$ for all $j = 1, \ldots, M$. That is,

$$p^* \gg 0, \text{ i.e.,}$$

$$\lim_{\epsilon \to 0} p^\epsilon = p^* \gg 0.$$

In view of continuity of $\bar{z}(p)$ over $\mathbb{R}^M_{++}$, from (2) we get (by taking limit $\epsilon \to 0$):

...
For all $j = 1, \ldots, M$

\[
p_j^* \sum_{j=1}^{M} \max\{\bar{Z}_j(p^*), 0\} = \max\{\bar{Z}_j(p^*), 0\}, \text{ i.e.,}
\]

\[
z_j(p^*) \rho_j^* \left( \sum_{j=1}^{M} \max\{\bar{Z}_j(p^*), 0\} \right) = z_j(p^*) \max\{\bar{Z}_j(p^*), 0\}, \text{ i.e.,}
\]

\[
\sum_{j=1}^{M} z_j(p^*) \rho_j^* \left( \sum_{j=1}^{M} \max\{\bar{Z}_j(p^*), 0\} \right) = \sum_{j=1}^{M} z_j(p^*) \max\{\bar{Z}_j(p^*), 0\}, \text{ i.e.,}
\]

\[
\sum_{j=1}^{M} z_j(p^*) \max\{\bar{Z}_j(p^*), 0\} = 0. \quad (4)
\]
Recall, \( \bar{z}_j(p) = \min\{z_j(p), 1\} \). Therefore,

\[
\begin{align*}
z_j(p^*) > 0 & \implies \max\{\bar{z}_j(p^*), 0\} > 0; \\
z_j(p^*) \leq 0 & \implies \max\{\bar{z}_j(p^*), 0\} = 0.
\end{align*}
\]

Therefore,

\[
\begin{align*}
z_j(p^*) > 0 & \implies z_j(p^*) \max\{\bar{z}_j(p^*), 0\} > 0; \\
z_j(p^*) \leq 0 & \implies z_j(p^*) \max\{\bar{z}_j(p^*), 0\} = 0.
\end{align*}
\]

Suppose, for some \( j \), we have \( z_j(p^*) > 0 \), then we will have

\[
\sum_{j=1}^{M} z_j(p^*) \max\{\bar{z}_j(p^*), 0\} > 0. \tag{5}
\]

But, this is a contradiction in view of (4).
Therefore:
For any \( j = 1, \ldots, M \), we have
\[
z_j(p^*) \leq 0. \tag{6}
\]

Suppose, \( z_k(p^*) < 0 \) for some \( k \). From (4), we know
\[
p_1 z_1(p^*) + \ldots + p_k z_k(p^*) + \ldots + p_M z_M(p^*) = 0.
\]

Since \( p_j^* > 0 \) for all \( j = 1, \ldots, M \).

- \( z_k(p^*) < 0 \) implies \( p_k^* z_k(p^*) < 0 \)
- Therefore, there must exist \( k' \), such that
\[ z_k'(p^*) > 0, \]  
which is a contradiction in view of (6). Therefore,

For all \( j = 1, \ldots, M \), we have:

\[ z_j(p^*) = 0, \text{ i.e.,} \]

\[ z(p^*) = 0. \]

That is,

- \( p^* \) is a WE price vector.