Fundamental Theorems of Welfare Economics

Ram Singh*

This Write-up is available at photocopy shop. Not for circulation.

In this write-up we provide intuition behind the two fundamental theorems of welfare economics and discuss their properties. An economy is defined by: the number of individuals in the economy, preference/utility function, and the endowment vector for each individual in the economy.

1 First Fundamental Theorem

Consider an economy as specified in the following example:

Example. Suppose the economy consists of two individuals and two goods. The utility functions are $u^1(.) = x_1^1 + 2x_2^1$ for the first individual and $u^2(.) = x_1^2 x_2^2$ for the second, where x_j^i denotes the quantity of jth good consumed by ith individual; $i = 1, 2$ and $j = 1, 2$. Let the initial endowments be $e^1(.) = (1, \frac{1}{2})$ and $e^2(.) = (0, \frac{1}{2})$, respectively. Assume that individuals act as price-takers.

Faced with the price vector $\mathbf{p} = (p_1, p_2)$, individual i will demand the bundle, $(\hat{x}_1^i, \hat{x}_2^i)$, to maximizes $u^i(.)$. That is, the demanded bundle, $\hat{\mathbf{x}}^i = (\hat{x}_1^i, \hat{x}_2^i)$, will solve:

$$\max_{\mathbf{x}^i}\{u^i(\mathbf{x}^i)\} \tag{1}$$

subject to the budget constraint $p_1 x_1^i + p_2 x_2^i = p_1 e_1^i + p_2 e_2^i$, which can be written as $\mathbf{p}.\mathbf{x}^i = \mathbf{p}.\mathbf{e}^i$.

From introductory microeconomics you know that person 2 will choose the bundle where her indifference curve is tangent to her budget line, i.e., she will demand the bundle where the slope of her indifference curve is the same as the slope of her budget line. This means that she will demand bundle $(\hat{x}_1^2, \hat{x}_2^2)$ that equates her marginal rate of substitution with the price ratio. Mathematically put, the demanded bundle $(\hat{x}_1^2, \hat{x}_2^2)$ will be such that: At $(\hat{x}_1^2, \hat{x}_2^2)$ the marginal rate of substitution for second person, $MRS^2 = \frac{x_2^2}{x_1^2} = \frac{p_1}{p_2}$. Moreover, since the bundle will be on the budget line, i.e., she

*Not proof read. Please bring typos to my attention.

will spend all her income. Therefore, the demanded bundle $(\hat{x}_1^2, \hat{x}_2^2)$ will satisfy the following conditions:

$$\begin{aligned} p_2.\hat{x}_2^2 &= p_1.\hat{x}_1^2 \text{ and} \\ p_1\hat{x}_1^2 + p_2\hat{x}_2^2 &= p_1.0 + \frac{p_2}{2} \end{aligned}$$

Clearly $2p_1\hat{x}_1^2 = \frac{p_2}{2}$. You can verify that these equalities give us:

$$\hat{x}_1^2 = \frac{p_2}{4p_1}, \text{ and } \hat{x}_2^2 = \frac{1}{4}. \tag{2}$$

That is, at positive income level, her demand for 2nd good is fixed at $\frac{1}{4}$.

The same logic applies to the 1st person. You can verify that for him the following holds: $MRS^1 = \frac{1}{2}$, i.e., the (absolute) value of the slope of his indifference curve is $\frac{1}{2}$. Moreover, by drawing his ICs and budget line you can see that for given price vector (p_1, p_2), his demand has the following properties:

$$\begin{aligned} \frac{p_1}{p_2} > \frac{1}{2} &\Rightarrow \text{ only 2nd good is demanded} \\ \frac{p_1}{p_2} < \frac{1}{2} &\Rightarrow \text{ only 1st good is demanded} \\ \frac{p_1}{p_2} = \frac{1}{2} &\Rightarrow \text{ any } (x_1^1, x_2^1) \text{ on the budget line can be demanded.} \end{aligned}$$

In particular, he will demand both goods, i.e., will demand $(\hat{x}_1^1, \hat{x}_2^1) >> (0, 0)$ if the demanded bundle $(\hat{x}_1^1, \hat{x}_2^1)$ is such that: At $(\hat{x}_1^1, \hat{x}_2^1)$,

$$MRS^1 = \frac{1}{2} = \frac{p_1}{p_2}, \text{ and } p_1\hat{x}_1^1 + p_2\hat{x}_2^1 = p_1 + \frac{p_2}{2}; \tag{3}$$

otherwise, only one good will be demanded. In either case, the demanded bundle will lie on his budget line. You should also check that his IC will be tangent to his budget line when he demands both the goods, even though point of tangency is not unique - in that case, his IC coincide with the budget line. When he prefers to consume strictly one good, the tangency between his IC and the budget line does not hold.

To sum up, we have seen that for Person 1 the utility maximizing bundle $\hat{\mathbf{x}}^1 = (\hat{x}_1^1, \hat{x}_2^1)$ lies on his budget line, which is given by:

$$p_1x_1^1 + p_2x_2^1 = p_1e_1^1 + p_2e_2^1.$$

For Person 2 the utility maximizing bundle $\hat{\mathbf{x}}^2 = (\hat{x}_1^2, \hat{x}_2^2)$ lies on her budget line given by

$$p_1x_1^2 + p_2x_2^2 = p_1e_1^2 + p_2e_2^2.$$

It is useful to draw the two budget lines in the Edgeworth box. Start with the budget line for person 2, i.e., $p_1x_1^2 + p_2x_2^2 = p_1e_1^2 + p_2e_2^2$. We can re-write this equality as:

$$\begin{aligned} p_1e_1^2 + p_2e_2^2 &= p_1x_1^2 + p_2x_2^2, i.e., \\ p_1e_1^2 + p_2e_2^2 &= p_1(e_1^1 + e_1^2 - x_1^1) + p_2(e_2^1 + e_2^2 - x_2^1). \end{aligned}$$

The second equality follows from the fact that within the Edgeworth box, for each allocation $(\mathbf{x}^1, \mathbf{x}^2)$, we have

$$x_1^1 + x_1^2 = e_1^1 + e_1^2, \text{ and } x_2^1 + x_2^2 = e_2^1 + e_2^2.$$

After canceling the common terms on both sides, the above budget line for person 2 can be written as:

$$\begin{aligned} 0 &= p_1(e_1^1 - x_1^1) + p_2(e_2^1 - x_2^1), i.e., \\ p_1x_1^1 + p_2x_2^1 &= p_1e_1^1 + p_2e_2^1, \end{aligned}$$

which is the budget line for the 1 person. Therefore, the Edgeworth box the two budget lines coincide! The budget line will pass through the initial endowment point marked as $\mathbf{x}_0$ in Figure 1.

This means that the utility maximizing bundles for Person 1 and Person 2, i.e., bundles $\hat{\mathbf{x}}^1 = (\hat{x}_1^1, \hat{x}_2^1)$ and $\hat{\mathbf{x}}^2 = (\hat{x}_1^2, \hat{x}_2^2)$, will lie on the same (budget) line! Moreover, the IC of person 2 will be tangent to this line. The IC of person 1 will also be the tangent to his line if he demand both goods in positive qualities.

Next, let us try to find out the equilibrium for this economy. The demanded bundles $\hat{\mathbf{x}}^1$ and $\hat{\mathbf{x}}^2$ can form an equilibrium allocation only if

$$\hat{x}_1^1 + \hat{x}_1^2 = e_1^1 + e_1^2, \text{ and } \hat{x}_2^1 + \hat{x}_2^2 = e_2^1 + e_2^2, i.e.,$$

only if the total demand of each good equals its supply.[1] Moreover, from (2) note that as long as both prices are positive, the second person will demand only 1/4 units of the second good. So, in equilibrium, person 1 must demand exactly 3/4 units of the second good. Also, if the second person demand 1 unit of the first good, then we should have $p_2 = 4p_1$. However, $p_2 = 4p_1$ implies that $p_2 > 2p_1$, but then person 1 demands only the first good. So, total demand will exceed the total supply, which cannot be an equilibrium.

Therefore, in equilibrium the demand for first good by person 2 must be strictly less than one unit. This means that person 1 should demand strictly positive quantity of the first good. To sum up, in equilibrium, person 1 must demand both goods in positive quantity. But, this can happen only if the price vector $\mathbf{p} = (p_1, p_2)$ is such

[1] This also means that $(\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2)$ is an allocation in the Edgeworth box. Alternatively, if an allocation is not a WE, then it cannot be an allocation in the Edgeworth box (check this).

that $\frac{p_1}{p_2} = \frac{1}{2}$. I leave it to you to verify that for any price vector $\mathbf{p} = (p_1, p_2)$ such that $\frac{p_1}{p_2} \neq \frac{1}{2}$, the total demand (as derived above) cannot be equal to total supply.[2]

Let us take $(p_1, p_2) = (1, 2)$. Now the budget line has slope equal to $-1/2$. It is marked as BB in Figure 1. You can check that at such prices under any price vector $\mathbf{p} = (p_1, p_2)$ is such that $\frac{p_1}{p_2} = \frac{1}{2}$:

- For 1st person, the demanded (utility maximizing) bundle $\hat{\mathbf{x}}^1 = (\hat{x}_1^1, \hat{x}_2^1) = (1/2, 3/4)$. The
- For 2nd person, the demanded (utility maximizing) bundle $\hat{\mathbf{x}}^2 = (\hat{x}_1^2, \hat{x}_2^2) = (1/2, 1/4)$

Therefore, for the price vector such that $\frac{p_1}{p_2} = \frac{1}{2}$, the allocation $(\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2)$, where $\hat{\mathbf{x}}^1 = (1/2, 3/4)$ and $\hat{\mathbf{x}}^2 = (1/2, 1/4)$, satisfies the following properties:

1. $\hat{\mathbf{x}}^1$ maximizes utility for 1st person subject to the budget constraint:

   $p_1 x_1^1 + p_2 x_2^2 = p_1 e_1^1 + p_2 e_2^1$.

2. $\hat{\mathbf{x}}^2$ maximizes utility for 2nd person subject to the budget constraint:

   $p_1 x_1^2 + p_2 x_2^2 = p_1 e_1^2 + p_2 e_2^2$.

3. $\hat{x}_1^1 + \hat{x}_1^2 = e_1^1 + e_1^2 = 1$ and $\hat{x}_2^1 + \hat{x}_2^2 = e_2^1 + e_2^2 = 1$.

Therefore, the allocation $(\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2)$, where $\hat{\mathbf{x}}^1 = (1/2, 3/4)$ and $\hat{\mathbf{x}}^2 = (1/2, 1/4)$, along with $(p_1, p_2) = (1, 2)$ is a Walrasian (competitive) equilibrium. The equilibrium allocation is marked by point $\hat{\mathbf{x}}$ in Figure 1. Furthermore, for these three properties, note the following:

- 1 Implies: At $\hat{\mathbf{x}}^1 = (1/2, 3/4)$, the IC of person 1 is tangent to her budget line.
- 2 Implies: $\hat{\mathbf{x}}^2 = (1/2, 1/4)$, IC of person 2 is tangent to his budget line.

  We know that both of the demanded bundles, i.e., $\hat{\mathbf{x}}^1$ and $\hat{\mathbf{x}}^2$ lie on the same line. (Why?) Therefore, the ICs of person 1 and 2 are tangent to the same (budget) line.

- 3 Implies: The demanded bundles, i.e., $\hat{\mathbf{x}}^1$ and $\hat{\mathbf{x}}^2$ coincide. Why?

  That is the ICs of person 1 and 2 are tangent to the same (budget) line and at the same point; which means that the ICs are tangent to each other. That is, at allocation $\hat{\mathbf{x}} = (\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2)$ we have

  $$MRS^1 = MRS^2 = \frac{p_1}{p_2}.$$

  The first equality implies that the equilibrium allocation $\hat{\mathbf{x}} = (\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2)$ is Pareto Optimum.

[2] That is, $(\hat{\mathbf{x}}^1(\mathbf{p}), \hat{\mathbf{x}}^2(\mathbf{p}))$ cannot be an allocation in the Edgeworth box (check this).

Remark 1 In the above example we have demonstrated that Walrasian equilibrium can exist even when all of the preferences are not strictly quasi-concave. Moreover, it shows that for strictly increasing preferences, a Walrasian equilibrium is Pareto optimum.

We can easily extend the analysis to N-person and M-good economy. Now, for given price vector $\mathbf{p} = (p_1, ..., p_M)$ the bundle demanded by person i, $\hat{\mathbf{x}}^i = (\hat{x}_1^i, \hat{x}_2^i, ..., \hat{x}_M^i)$, will solve:

$$\max_{\mathbf{x}^i}\{u^i(\mathbf{x}^i)\} \tag{4}$$

subject to the budget constraint $p_1x_1^i + p_2x_2^i + ... + p_Mx_M^i = p_1e_1^i + p_2e_2^i + ... + p_Me_M^i$, i.e., subject to $\mathbf{p}.\mathbf{x}^i = \mathbf{p}.\mathbf{e}^i$.

For a 'well behaved' utility function, person i will equate her MRS between goods k and l with the price ratio for these goods, i.e., at the demanded bundle the following will hold:

$$MRS_{kl}^i = \frac{p_k}{p_l}, \tag{5}$$

for all goods $k, l = 1, ..., M$. Since all individuals are price-takers, (5) gives us:

$$MRS_{kl}^i = MRS_{kl}^j = \frac{p_k}{p_l}, \tag{6}$$

for all individuals $i, j = 1, ..., N$ and for all goods $k, l = 1, ..., M$. Assume that total demand is equal to total supply of each good - that is, the demanded bundles is an equilibrium allocation. Now, the first equality in (6) implies that the equilibrium is a Pareto optimum allocation. This result is known as *the First Fundamental Theorem of Welfare Economics.* Formally it is stated as follows.

Theorem 1 Consider an exchange economy $(u^i, \mathbf{e}^i)_{i\in N}$. If u^i is strictly increasing for all $i = 1, .., N$, then every Walrasian/Competitive equilibrium allocation is Pareto optimum.

For proof refer to our class room discussion. In the class we have proved actually a stronger claim. We have shown that $W((u^i, \mathbf{e}^i)_{i\in N}) \subseteq C((u^i, \mathbf{e}^i)_{i\in N})$. That is, every WE/Competitive equilibrium is in the Core. This, of course, means that every WE equilibrium is Pareto optimum.

2 Second Fundamental Theorem

Let us revert to the economy as described in the example in the earlier section. You can check that at the equilibrium allocation $(\mathbf{x}^1 = (1/2, 3/4), \mathbf{x}^2 = (1/2, 1/4))$, the first person gets a utility equal to 2 and the second person's utility is 1/8. Suppose, we want to improve welfare of the second person. How can we do so using the market mechanism? Alternatively, the question is: Can we use the market mechanism to arrive at a more egalitarian (socially desirable) allocation?

Specifically, consider an allocation, say $(\mathbf{y}^1, \mathbf{y}^2)$ where $\mathbf{y}^1 = (1/4, 5/8)$ and $\mathbf{y}^2 = (3/4, 3/8)$. You can check that at this allocation the individual 2 is strictly better off than at the above equilibrium allocation. Moreover, $MRS^1 = MRS^2$, so the allocation is Pareto optimum. See Figures 1 and 2. Can we induce it as Walrasian equilibrium? Yes, we can. The Second Fundamental Theorem is helpful in answering these questions. There are at least three different ways to achieve the objective.

The first option is to engage in direct transfer of goods/endowments from person 1 to person 2 so as to reach the above allocation, $(\mathbf{y}^1, \mathbf{y}^2)$, as a competitive equilibrium outcome. (The land reforms is one possible way of direct transfer of endowments). For the purpose, define the transfer vector $\acute{\mathbf{e}} = (\acute{\mathbf{e}}^1, \acute{\mathbf{e}}^2)$ such that: For all $i = 1, 2$

$$\mathbf{y}^i = \mathbf{e}^i + \acute{\mathbf{e}}^i.$$

After the transfer, individual i will end up with vector $\mathbf{y}^i$ as the new endowment. In Figure 2, this would mean that the endowment point shifts from $\mathbf{e}_0$ to the point marked as $\mathbf{y}$. It can be easily seen that to achieve the purpose, we need to transfer 3/4 units of first good from person 1 to person 2, and 1/8 units of second good from person 2 to person 1. That is, the transfer vectors are: $\acute{\mathbf{e}}^1 = (-3/4, 1/8)$ and $\acute{\mathbf{e}}^2 = (3/4, -1/8)$. Next, we want to be sure that the new endowment vector, $(\mathbf{y}^1, \mathbf{y}^2)$, with $\mathbf{y}^1 = (1/4, 5/8)$ and $\mathbf{y}^2 = (3/4, 3/8)$, is a competitive equilibrium; otherwise, in equilibrium the economic agents will end up consuming bundles different from what we want them to consume. However, the vector $(\mathbf{y}^1, \mathbf{y}^2)$ can be achieved as a competitive equilibrium allocation.

Note that at the allocation $(\mathbf{y}^1, \mathbf{y}^2)$ as defined above, we have $MRS^1 = MRS^2 = 1/2$. So, if we choose a price vector (p_1, p_2) such that $\frac{p_1}{p_2} = \frac{1}{2}$, at allocation $(\mathbf{y}^1, \mathbf{y}^2)$ as defined above, the following will hold

$$MRS^1 = MRS^2 = \frac{p_1}{p_2}.$$

In fact, you can verify that for a price vector (p_1, p_2) such that $\frac{p_1}{p_2} = \frac{1}{2}$, say $(p_1, p_2) = (1, 2)$, the allocation $(\mathbf{y}^1, \mathbf{y}^2)$ as defined above, satisfies all the condition of a competitive equilibrium. Therefore, the allocation $(\mathbf{y}^1, \mathbf{y}^2)$ along with the price vector $(p_1, p_2) = (1, 2)$ constitute a Walrasian equilibrium. Note that when price vector is $(p_1, p_2) = (1, 2)$, the budget line has slope of $-\frac{p_1}{p_2}$ and it passes through the allocation point $(\mathbf{y}^1, \mathbf{y}^2)$. (Take a note of how we have found, the equilibrium price vector.)

Therefore, one option to achieve any Pareto optimum allocation as competitive equilibrium is to directly transfer the endowments so that the new endowment vector (after transfer) is the same as the desired allocation. However, we may not have complete flexibility in transfer of endowments so as to move all the way to the desired allocation. For example, if one of the endowment is human capital then it is not possible to transfer it across individuals.

The second option is to engage in 'partial' transfer of endowments so as to reach any point on the budget line that has slope of $-\frac{p_1}{p_2}$ and passes through desired allocation

$(\mathbf{y}^1, \mathbf{y}^2)$. Note that we have taken the budget line as under the first option for improving welfare of the second person. Assume that only the first endowment is directly transferable. Now, consider the following transfers: $\acute{\mathbf{e}}^1 = (-1/2, 0)$ and $\acute{\mathbf{e}}^2 = (1/2, 0)$. The result will be new endowment vectors (1/2,1/2) for person 1 and (1/2,1/2) for person 2. In Figure 3, it means shift of endowment point $\mathbf{e}_0$ to the point marked as $\mathbf{e}_1$. Note that the new endowment vector is a point on the budget line considered in the first case above (, i.e., Figure 2). Since $(p_1, p_2) = (1, 2)$, we know that at allocation desired allocation $(\mathbf{y}^1, \mathbf{y}^2)$ as defined above, the following will hold:

$$MRS^1 = MRS^2 = \frac{p_1}{p_2}.$$

Hence, the allocation $(\mathbf{y}^1, \mathbf{y}^2)$, $\mathbf{y}^1 = (1/4, 5/8)$ and $\mathbf{y}^2 = (3/4, 3/8)$, emerges as a unique competitive equilibrium along with price vector $(p_1, p_2) = (1, 2)$.

As an exercise, you can consider the case when only the second endowment is transferable. If you consider the following transfers: $\acute{\mathbf{e}}^1 = (0, -1/4)$ and $\acute{\mathbf{e}}^2 = (0, 1/4)$ and take $(p_1, p_2) = (1, 2)$ as the price vector, you will find that the allocation $(\mathbf{y}^1, \mathbf{y}^2)$, $\mathbf{y}^1 = (1/4, 5/8)$ and $\mathbf{y}^2 = (3/4, 3/8)$, is still a unique competitive equilibrium!

Is it possible to achieve the desirable allocation, even if none of the endowments is directly transferable? Yes, it is. The third option involves cash transfers. Note that when the price vector is $\mathbf{p} = (p_1, p_2) = (1, 2)$, the transfer vectors considered above, $\acute{\mathbf{e}}^1 = (-1/2, 0)$ and $\acute{\mathbf{e}}^2 = (1/2, 0)$ entail a transfer of purchasing power equal to 1/2 from person 1 to person 2. It turns out that if keep the prices to be $\mathbf{p} = (p_1, p_2) = (1, 2)$, cash transfer will lead to the same outcome. Consider a transfer of T_i to the ith individual. Choose

$$T_1 = -\frac{1}{2} \text{ and } T_2 = \frac{1}{2}$$

Compared to the initial situation, such transfers clearly change the budget sets for the two individuals - it expands the budget set of person 2 but shrinks that of person 1. As a result, in equilibrium person 2 will consume (x_1^2, x_2^2) such that: $MRS^2 = \frac{x_2^2}{x_1^2} = \frac{p_1}{p_2}$, and (the new) budget constraint holds, i.e., $p_1 x_1^2 + p_2 x_2^2 = p_1.0 + p_2.\frac{1}{2} + T_2$. That is, her demanded bundle satisfies:

$$\begin{aligned} p_2.x_2^2 &= p_1.x_1^2 \text{ and} \\ p_1 x_1^2 + p_2 x_2^2 &= p_1.0 + p_2.\frac{1}{2} + T_2. \end{aligned}$$

See Figure 4. You can check that the unique solution to 2's problem is

$$(x_1^2, x_2^2) = (3/4, 3/8).$$

Similarly, you can verify that person 1 will demand $\mathbf{y}^1 = (1/4, 5/8)$. Yet again, the allocation $(\mathbf{y}^1, \mathbf{y}^2)$, $\mathbf{y}^1 = (1/4, 5/8)$ and $\mathbf{y}^2 = (3/4, 3/8)$, along with price vector $(p_1, p_2) = (1, 2)$ is a unique competitive equilibrium!

As an exercise, choose any other Pareto optimal allocation for the above economy. Following any the above approach, you can show that the chosen allocation can be achieved as competitive allocation through suitable transfers. This result is actually the Second Fundamental Theorem of Welfare Economics.

Theorem 2 If u^i is continuous, strictly increasing, and strictly quasi-concave for all $i = 1, .., N$, then *any* Pareto optimum allocation, $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$, such that $\mathbf{y}^i >> \mathbf{0}$, can be achieved as competitive equilibrium with suitable transfers.

That is, $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ is a WE with suitable transfer. With suitable transfers, market can achieve any of the socially desirable allocation as competitive equilibrium.

Question 1 We have demonstrated the claim in the second fundamental theorem can hold even if the utility functions are not strictly quasi-concave for all $i = 1, .., N$. However, the statement of the theorem requires u^i to be strictly quasi-concave for all $i = 1, .., N$, along with continuity and monotonicity for all functions. Why?

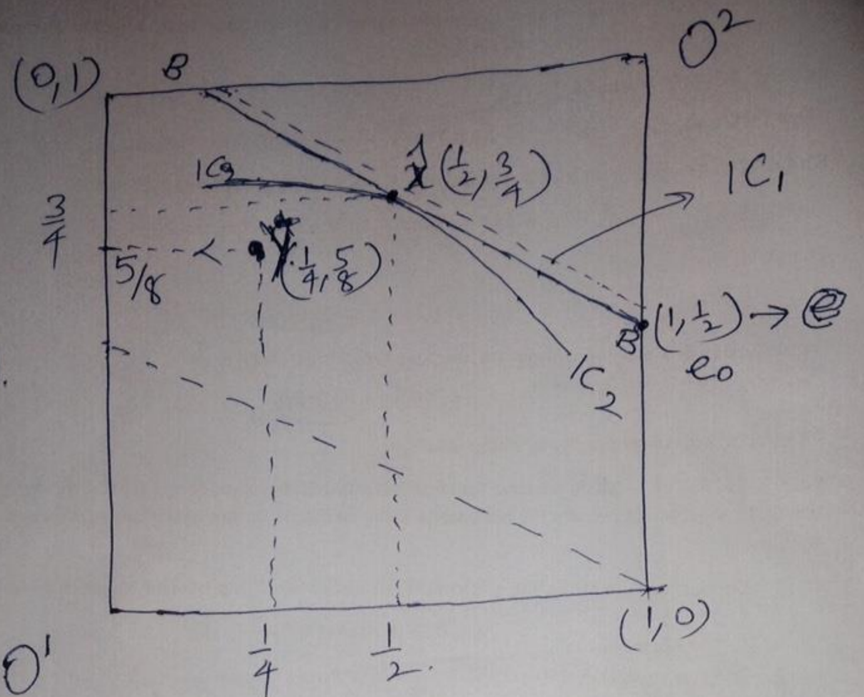

FIGURE 1.

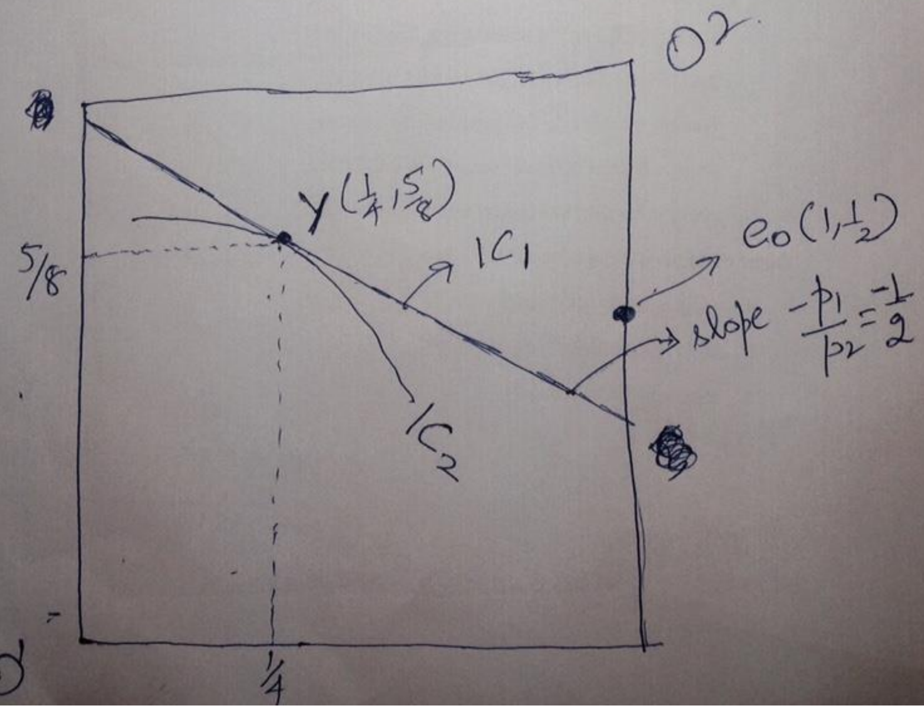

Figure 2.

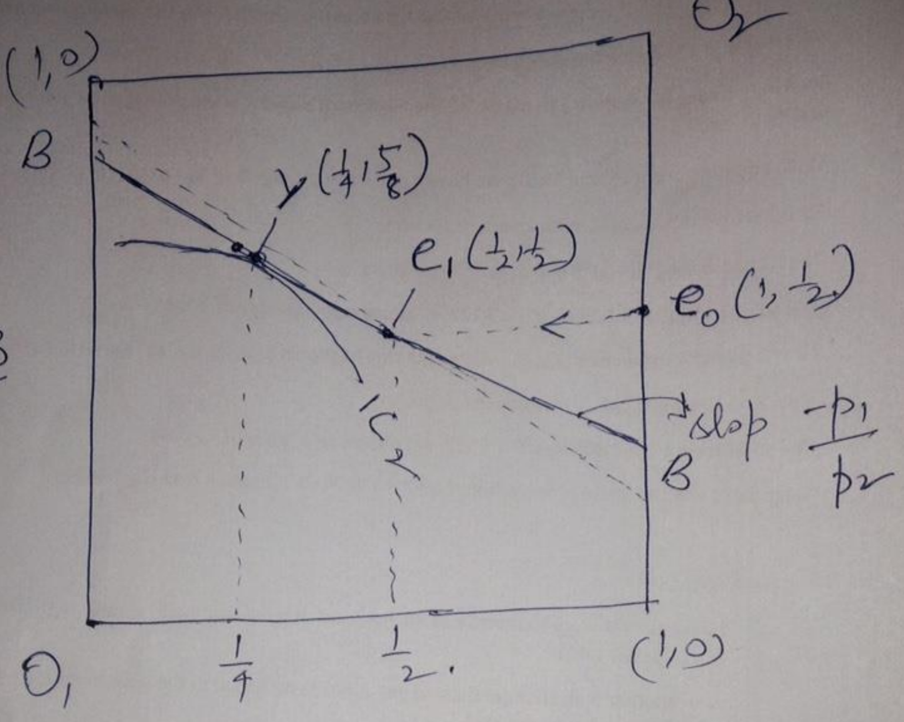

Figure 3

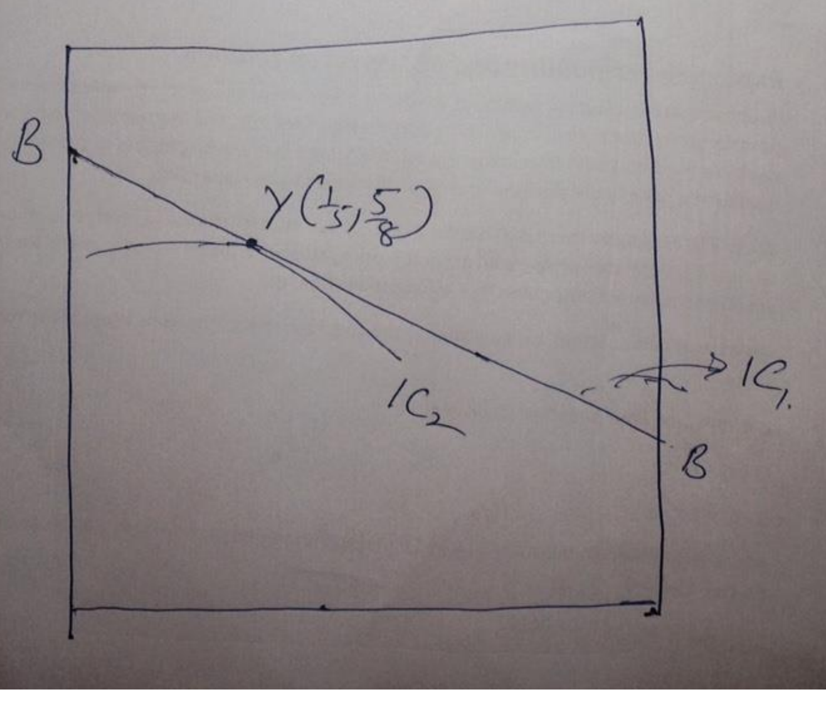

Figure 4