General Equilibrium with Production

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Microeconomic Theory

Lecture 11
Producer Firms I

- There are \( N \) individuals; \( i = 1, \ldots, N \)
- There are \( M \) goods; \( j = 1, \ldots, M \)
- There are \( K \) firms; \( k = 1, \ldots, K \)
- Each firm has a set of production plans, i.e., production set \( Y^k \subset \mathbb{R}^M \)
- The production set \( Y^k \) is the set of feasible production plans for firm \( k \)

Examples: Let \( M = 3 \) and \( K = 2 \). Suppose Firm 1 can produce six units of good 2 by using three units of good 1 and nine units of good 3, i.e.,

\[
\begin{align*}
\text{Firm 1: } & \quad y^1 = (-3, 6, -9) \\
\text{Firm 2: } & \quad y^2 = (8, -3 \frac{1}{2}, -14) \\
\text{Economy: } & \quad Y = (5, -2 \frac{1}{2}, -23)
\end{align*}
\]
A typical production plan for firm $k$ is

$$y^k = (y^k_1, \ldots, y^k_M),$$

where

- $y^k_j > 0$ if good $j$ is an output produced by firm $k$
- $y^k_j < 0$ if good $j$ is an input used by firm $k$.

Let $p = (p_1, \ldots, p_M)$ be the given price vector. For any $y^k \in Y^k$, profit for firm $k$ is

$$\pi(p, y^k) = p_1 y^k_1 + \ldots + p_M y^k_M = p \cdot y^k.$$

So, PMP for firm $k$ is:

$$\max_{y^k \in Y^k} \{p_1 y^k_1 + \ldots + p_M y^k_M\}, \text{ i.e.,}$$

$$\max_{y^k \in Y^k} \{p \cdot y^k\}.$$
Let

$$\Pi^k(p) = \max_{y^k \in Y^k} \pi(p, y^k)$$

$$\Pi^k(p)$$ homogenous function of degree 1 in $$p$$.

Assume, for each firm $$k$$,

- $$0 \in Y^k \subset \mathbb{R}^M$$, i.e., firm can always earn Zero profit
- So, profit is non-negative
- $$Y^k$$ is closed and bounded and strictly convex
  - For any given price vector $$p$$, the profit maximizing choice/production plan is unique
  - For different price vector $$p'$$, profit maximizing choice/production plan will be different, in general.
Let

- $y^k = (y^k_1, ..., y^k_M)$ be a feasible production plan for firm $k = 1, ..., K$.

Corresponding to the above production plan, the aggregate production plan for the economy is given by

$$Y = (y^1, ..., y^K), \text{ where } y^k = (y^k_1, ..., y^k_M). \quad (1)$$

Corresponding to the production plan (1), the total production of good $j$ is given by

$$\sum_{k=1}^{K} y^k_j.$$ 

So, corresponding to the production plan (1), the total ‘output’ vector is:

$$\left(\sum_{k=1}^{K} y^k_1, ..., \sum_{k=1}^{K} y^k_M\right) = \sum_{k=1}^{K} y^k = Y.$$
if $\sum_{k=1}^{K} y^k_j > 0$, good $j$ is a net output for the economy

if $\sum_{k=1}^{K} y^k_j < 0$, good $j$ is a net input for the economy.

The aggregate production possibility set (for the entire economy) is

$$\mathbb{Y} = \left\{ Y \mid Y = \sum_{k=1}^{K} y^k, \text{ where } y^k \in \mathbb{Y}^k \right\}.$$ 

That is, if $\hat{Y} \in \mathbb{Y}$, then there exist production plans $\hat{y}^1, ..., \hat{y}^k, ..., \hat{y}^K$ such that

- $\hat{y}^k \in \hat{\mathbb{Y}}^k$, i.e., $\hat{y}^k$ is a feasible plan for firm $k = 1, ..., K$; and
- $\hat{Y} = \sum_{k=1}^{K} \hat{y}^k$
Aggregate Production Plans III

Proposition

Given the above assumptions on $Y^k$ and PMP for firms,

- $0 \in Y \subset \mathbb{R}^M$
- $Y$ is closed and bounded and strictly convex

Question

Why $Y$ is a bounded set?

Proposition

Given the above assumptions on $Y^k$, for any given price vector $p = (p_1, \ldots, p_M) >> (0, \ldots, 0)$, the following PMP has a unique solution:

$$\max_{y^k \in Y^k} \{ p \cdot y^k \} \text{ for every } k=1,\ldots,K$$
Proposition

Given the above assumptions on $\mathbb{Y}$, for any given price vector $p = (p_1, \ldots, p_M) \gg (0, \ldots, 0)$, the following PMP has a unique solution:

$$\max_{\mathbb{Y}} \{ p \cdot Y \}$$
Efficient Production: Two definitions I

Definition

(Definition 1): Production Plan $Y = (y^1, ..., y^K) \in \mathbb{Y}$ is ‘efficient’ if there is no other plan $Z = (z^1, ..., z^K)$ such that: $Z$ is feasible, i.e., $Z \in \mathbb{Y}$, and

\[
\sum_{k=1}^{K} z_j^k \geq \sum_{k=1}^{K} y_j^k, \text{ for all goods } j
\]

\[
\sum_{k=1}^{K} z_j^k > \sum_{k=1}^{K} y_j^k, \text{ for at least one good } j
\]

Remark

Suppose $k$th good is a net input. In that case, $\sum_{k=1}^{K} z_j^k > \sum_{k=1}^{K} y_j^k$ implies that the production plan requires smaller quantity of this input.
Suppose $Y = (y^1, y^2)$:

firm 1: $y^1 = (-2, 4, -6)$
firm 2: $y^2 = (8, -4, -14)$

So, $Y = \sum_1^2 y^i = (\sum_1^2 y_1, \sum_1^2 y_2, \sum_1^2 y_3) = (6, 0, -20)$

Let $Z = (z^1, z^2)$:

firm 1: $z^1 = (-2, 4, -6)$
firm 2: $z^2 = (8, -3\frac{1}{2}, -14)$

So, $Z = \sum_1^2 z^i = (\sum_1^2 z_1, \sum_1^2 z_2, \sum_1^2 z_3) = (6, \frac{1}{2}, -20)$
Efficient Production: Two definitions III

Definition

(Definition 2): For given price vector $\mathbf{p} = (p_1, ..., p_M)$, production Plan $\mathbf{Y} = (y^1, ..., y^K) \in \mathbb{Y}$ is efficient if it solves

$$\max_{\mathbf{Y}' \in \mathbb{Y}} \{ \mathbf{p}.\mathbf{Y}' \}, \text{ i.e.,}$$

$$\mathbf{p}.\mathbf{Y} \geq \mathbf{p}.\mathbf{Y}' \text{ for all } \mathbf{Y}' \in \mathbb{Y}$$

Question

Does (D1) imply (D2)? Does (D2) imply (D1)?
Total supply is given by

\[ \sum_{i=1}^{N} e^i + \sum_{i=k}^{K} y^k \]

Let

- \( \bar{p} = (\bar{p}_1, ..., \bar{p}_M) \) be a price vector.
- \( \bar{Y} = (\bar{y}^1, ..., \bar{y}^K) \) be a production plan for the economy.

**Definition**

(\( \bar{Y}, \bar{p} \)) is a competitive ‘production’ equilibrium only if: For all \( k = 1, ..., K \),

\( \bar{y}^k \) solves \( \max_{y^k \in Y^k} \{ \bar{p} \cdot y^k \} \).
Proposition

Take any price vector $\bar{p} = (\bar{p}_1, ..., \bar{p}_M)$. If $\bar{y}^k \in \mathbb{Y}^k$ solves

$$\max_{y^k \in \mathbb{Y}^k} \{\bar{p}.y^k\} \text{ for } k = 1, ..., K.$$

Let $\bar{Y}$, where $\bar{Y} = \sum_{k=1}^{K} \bar{y}^k$. Then, there is NO other plan $Z = (z^1, ..., z^K)$ such that: $Z$ is feasible, i.e., $Z \in \mathbb{Y}$, and

\[
\sum_{k=1}^{K} z_j^k \geq \sum_{k=1}^{K} \bar{y}_j^k, \text{ for all goods } j
\]

\[
\sum_{k=1}^{K} z_j^k > \sum_{k=1}^{K} \bar{y}_j^k, \text{ for at least one good } j
\]
Proposition

Take any price vector $\bar{p} = (\bar{p}_1, ..., \bar{p}_M)$. Suppose production plans $\bar{y}^1, ..., \bar{y}^K$ are such that $\bar{y}^k \in Y^k$ solves

$$\max_{y^k \in Y^k} \{ \bar{p} \cdot y^k \} \text{ for all } k = 1, ..., K$$

Then, $\bar{Y} = \sum_{k=1}^{K} \bar{y}^k$ solves

$$\max_{Y \in Y} \{ \bar{p} \cdot Y \}$$

That is, individual profit maximization leads to total profit maximization. Why?

Question

What assumption is made about Externality?
Proposition

Take any price vector $\bar{p} = (\bar{p}_1, ..., \bar{p}_M)$. If $\bar{Y}$ solves

$$\max_{\bar{y} \in \bar{Y}} \{\bar{p}.\bar{y}\}$$

then there exist production plans $\bar{y}^1, ..., \bar{y}^K$ such that $\bar{y}^k \in Y^k$, $\bar{Y} = \sum_{k=1}^{K} \bar{y}^k$, and $\bar{y}^k$ solves

$$\max_{y^k \in Y^k} \{\bar{p}.y^k\} \text{ for all } k = 1, ..., K$$

Proof: Let $\bar{Y}$ solve $\max_{\bar{y} \in \bar{Y}} \{\bar{p}.\bar{y}\}$. That is,

$$\left(\forall \bar{y} \in \bar{Y}\right)[\bar{p}.\bar{Y} \geq \bar{p}.\bar{y}]$$

Let $\bar{y}^1, ..., \bar{y}^K$ are such that $\bar{y} = \sum_{k=1}^{K} \bar{y}^k$ and $\bar{y}^k \in Y^k$. If possible, suppose for some $k$, $\bar{y}^k$ does not solve

$$\max_{y^k \in Y^k} \{\bar{p}.y^k\}$$
So, there exists some $\hat{y}^k \in \mathbb{Y}^k$ such that

$$\bar{p}.\hat{y}^k > \bar{p}.\bar{y}^k$$

Now, consider the production plan $Z$ such that

$$Z = (z^1, \ldots, z^k, \ldots, z^K) = (\bar{y}^1, \ldots, \hat{y}^k, \ldots, \bar{y}^K)$$

Clearly, $Z \in \mathbb{Y}$. It is easy to show that

$$\bar{p}.Z > \bar{p}.\bar{Y},$$

a contradiction.
Privatized Economy I

Privatized Economy: \((u^i(\cdot), e^i(\cdot), Y^k, \theta^{ik})_{i \in \{1, \ldots, N\}, j \in \{1, \ldots, M\}, k \in \{1, \ldots, K\}}\)

- All firms are privately owned
- \(\theta^{ik}\) is the (ownership) share of \(k\)th firm owned by \(i\) the individual;
- \(0 \leq \theta^{ik} \leq 1\) for all \(i = 1, \ldots, N\) and \(k = 1, \ldots, K\)
- \(\sum_{i=1}^{N} \theta^{ik} = 1\) for all \(k = 1, \ldots, K\)

Now, for any given price vector, \(\mathbf{p} = (p_1, \ldots, p_M)\), individual \(i\) solves

\[
\max_{x^i \in \mathbb{R}^M_+} u^i(x^i), \quad \text{s.t.} \quad \mathbf{p} \cdot x^i \leq l^i(\mathbf{p}),
\]

where

\[
l^i(\mathbf{p}) = \mathbf{p} \cdot \mathbf{e}^i + \sum_{k=1}^{K} \theta^{ik} \pi^k(\mathbf{p}).
\]
Remark

In view of the above assumptions on Production sets,

1. $\pi(p) \geq 0$ is bounded above and continuous in $p$.
2. Moreover, $l^i(p)$ is a continuous function of $p$.
3. In view of the assumption on $u^i(.)$, the solution of the above UMP is unique.

Definition

The excess demand function for good $j$ is

$$z_j(p) = \sum_{i=1}^{N} x^j_i(p, l^i(p)) - \sum_{k=1}^{K} y^k_j(p) - \sum_{i=1}^{N} e^j_i.$$  

So, excess demand vector is $z(p) = (z_1(p), ..., z_M(p))$. 
Definition

Feasible Allocation: An allocation \((X, Y)\), where \(Y = (y^1, ..., x^K)\) is feasible, if:

- For all \(k\), \(y^k \in \mathbb{Y}^k\)

\[\sum_{i=1}^{N} x^i = \sum_{i=1}^{N} e^i + \sum_{i=k}^{K} y^k.\]
Existence of WE I

As before, in view of the assumption on \( u^i(.) \) and \( Y^k \),

- the excess demand function is homogeneous function of \( p \) of degree 0.
- For equilibrium to exist \( Y + \sum_{i=1}^{N} e^i >> 0 \) must hold for some \( Y \in Y \).
- As \( p_j \to 0 \) for some \( j \), the excess demand becomes unbounded for one of such commodities.

Theorem

Consider an economy \((u^i(.), e^i, \theta^{ik}, Y^j)\), where \( i = 1, ..., N, j = 1, ..., M \) and \( k = 1, ..., K \). If \( u^i(.) \) and \( Y^j \) satisfy above assumptions, then there exists a price vector \( p^* >> 0 \) such that \( z(p^*) = 0 \).
Efficiency of WE I

Definition

A feasible allocation \((\bar{X}, \bar{Y})\) is Pareto optimum if there is no other allocation \((X, Y)\) such that

\[
\sum_{i=1}^{N} x^i = \sum_{i=1}^{N} e^i + \sum_{i=k}^{K} y^k;
\]

\[
u^i(x^i) \geq \nu^i(\bar{x}) \text{ for all } i \in \{1, \ldots, N\} \text{ and }
\]

\[
u^i(x^i) > \nu^i(\bar{x}) \text{ for some } j \in \{1, \ldots, N\}
\]

Theorem

Consider an economy \((u^i(.), e^i, \theta^{ik}, y^k)\), where \(i = 1, \ldots, N\) and \(k = 1, \ldots, K\). If \(u^i(.)\) is strictly increasing, then every WE is Pareto optimum.

Proof: Suppose \((\bar{X}, \bar{Y})\) along with \(p^*\) is WE. Therefore,
Efficiency of WE II

\[
\sum_{i=1}^{N} \bar{x}^i = \sum_{i=1}^{N} e^i + \sum_{i=k}^{K} \bar{y}^k
\]

Suppose \((\bar{X}, \bar{Y})\) is not PO. So, there is some \((X, Y)\), such that

\[
\sum_{i=1}^{N} x^i = \sum_{i=1}^{N} e^i + \sum_{i=k}^{K} y^k;
\]

\[u^i(x^i) \geq u^i(\bar{x}^i)\] for all \(i \in \{1, \ldots, N\}\) and

\[u^i(x^i) > u^i(\bar{x}^i)\] for some \(j \in \{1, \ldots, N\}\)

Therefore,

\[p^*.x^i \geq p^*.\bar{x}^i\] for all \(i \in \{1, \ldots, N\}\) and

\[p^*.x^j > p^*.\bar{x}^i\] for some \(j \in \{1, \ldots, N\}\)
Efficiency of WE III

\[ p^* \cdot \sum_{i=1}^{N} x_i > p^* \cdot \sum_{i=1}^{N} \bar{x}_i \]

\[ p^* \cdot \left( \sum_{j=1}^{M} y^j + \sum_{i=1}^{M} e^i \right) > p^* \cdot \left( \sum_{j=1}^{M} \bar{y}^j + \sum_{j=1}^{M} e^i \right) \]

\[ p^* \cdot \sum_{j=1}^{M} y^j > p^* \cdot \sum_{j=1}^{M} \bar{y}^j \]

a contradiction. Why?
Efficiency of WE IV

Theorem

Consider an economy \((u^i(\cdot), e^i, \theta^{ik}, Y^j)\), where \(i = 1, \ldots, N\) and \(k = 1, \ldots, K\). Suppose, \(u^i(\cdot)\) and \(Y^j\) satisfy above assumptions, and \(y + \sum_{i=1}^{N} e^i >> 0\) for some \(y \in Y\). If \((\bar{x}, \bar{y})\) is any feasible PO allocation, then there exist

- 'Cash' transfers \(T^i, i = 1, \ldots, N\) such that \(\sum_{i=1}^{N} T^i = 0\)
- A price vector \(\bar{p} = (\bar{p}_1, \ldots, \bar{p}_M)\)

such that

1. \(\bar{x}^i\) maximizes \(u^i(x^i)\) s.t. \(\bar{p}.x^i \leq \bar{p}.e^i + \sum_{k=1}^{K} \theta^{ik}\pi^k(\bar{p}) + T^i\) for all \(i = 1, \ldots, N\)
2. \(\bar{y}^k\) maximizes \(\bar{p}.y^k\) for all \(k = 1, \ldots, K\)
3. \(z^j(\bar{p}) = 0\) for all \(j = 1, \ldots, M\)