Competitive Equilibria: Uniqueness and Stability

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Microeconomic Theory

Lecture 7

Questions

- Is Competitive/Walrasian equilibrium unique?
- Why is a unique equilibrium helpful?
- If WE is not unique, how many WE can be there?
- What are the conditions, for a unique WE?
- Do these conditions hold in the real world?
- Is Competitive/Walrasian equilibrium stable?
- Why is stability of an equilibrium important?

Background Readings:

Arrow and Hahn. (1971), Jehle and Reny* (2008), MWG*(1995)

Multiple WE: Example

Example

From MWG*(1995): Two consumers:

- $u^1(.) = x_1^1 - \frac{1}{8}\frac{1}{\left(x_2^1\right)^8}$ and $u^2(.) = -\frac{1}{8}\frac{1}{\left(x_1^2\right)^8} + x_2^2$
- $e^1 = (2, r)$ and $e^2 = (r, 2)$; $r = 2^{\frac{8}{9}} - 2^{\frac{1}{9}}$
- The equilibria are solution to

$$\left(\frac{p_2}{p_1}\right)^{-\frac{1}{9}} + 2 + r\left(\frac{p_2}{p_1}\right) - \left(\frac{p_2}{p_1}\right)^{\frac{8}{9}} = 2 + r, i.e.,$$

there are three equilibria:

$$\frac{p_2}{p_1} = \frac{1}{2}, 1, \text{ and } 2.$$

Unique WE: Conditions I

Consider 2×2 economy:

- Two goods: food and cloth
- Let (p_f, p_c) be the price vector.
- We know that for all $t > 0$: $\mathbf{z}(t\mathbf{p}) = \mathbf{z}(\mathbf{p})$.
- Therefore, we can work with

$$\mathbf{p} = (\frac{p_f}{p_c}, 1) = (p, 1).$$

From Walras's Law, we have $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$, i.e.,

$$pz_f(\mathbf{p}) + z_c(\mathbf{p}) = 0.$$

Conditions for existence of WE:

- $z_i(\mathbf{p})$ is continuous for all $\mathbf{p} >> \mathbf{p}$, i.e., for all $p > 0$.

Unique WE: Conditions II

- there exists small $p = \epsilon > 0$ s.t. $z_f(\epsilon, 1) > 0$, and
- there exists another $p' > \frac{1}{\epsilon}$ s.t. $z_c(p', 1) > 0$.

We can ensure above, by assuming the utility functions to be continuous, strongly monotonic and strictly quasi-concave.

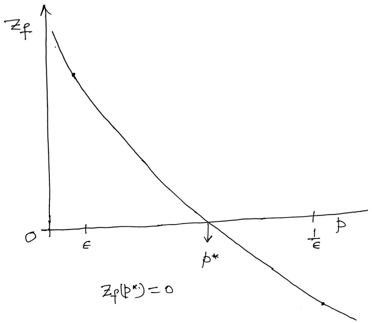
Question

- *Do the above assumptions guarantee unique WE?*
- *Under what conditions the WE be unique?*

An additional assumption can ensure uniqueness of WE:

- $z_f^{'}(\mathbf{p}) < 0$ for all $p > 0$.

Unique WE: Conditions III



Do the above assumptions on utility functions ensure $z_f'(\mathbf{p}) < 0$ for all $p > 0$?

Normal Goods and Number of Equilibria I

Let,

- there be two goods - food and cloth.
- $\mathbf{e}^1 = (e_f^1, e_c^1)$ and $\mathbf{e}^2 = (e_f^2, e_c^2)$ be the initial endowment vectors
- $\mathbf{p} = (p, 1)$ be a price vector.

Assumption: Assume utility functions to be

- continuous, strongly monotonic and strictly quasi-concave

From Walras Law we have $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$, i.e.,

$$pz_f(\mathbf{p}) + z_c(\mathbf{p}) = 0.$$

By definition:

$$\begin{aligned} z_f(\mathbf{p}) &= z_f^1(\mathbf{p}) + z_f^2(\mathbf{p}) \\ &= [x_f^1(\mathbf{p}) - e_f^1] + [x_f^2(\mathbf{p}) - e_f^2] \end{aligned}$$

Normal Goods and Number of Equilibria II

In view of our assumptions on utilities,

- $z_f(\mathbf{p})$ and $z_c(\mathbf{p})$ are continuous for all $\mathbf{p} >> \mathbf{0}$, i.e., for all $p > 0$.
- there exists small $p = \epsilon > 0$ s.t. $z_f(\epsilon, 1) > 0$ and another $p' > \frac{1}{\epsilon}$ s.t. $z_c(p', 1) > 0$.

- Let $\mathbf{p}^*$ denote an equilibrium price vector.
- We know that for the above economy at least one $\mathbf{p}^*$ exists. Why?

Let

- $I^1(\mathbf{p}, \mathbf{e}^1) = \mathbf{p}.\mathbf{e}^1 = pe_f^1 + e_c^1$
- $I^2(\mathbf{p}, \mathbf{e}^2) = \mathbf{p}.\mathbf{e}^2 = pe_f^2 + e_c^2$

Normal Goods and Number of Equilibria III

Note

$$\underbrace{\frac{dz_f(\mathbf{p})}{dp}}_{Price\ effect\ (total)} = \underbrace{\frac{dz_f^1(\mathbf{p})}{dp}}_{Price\ effect\ (total)} + \underbrace{\frac{dz_f^2(\mathbf{p})}{dp}}_{Price\ effect\ (total)}$$

Since endowments are fixed, we get

$$\underbrace{\frac{dz_f(\mathbf{p})}{dp}}_{Price\ effect\ (total)} = \left(\frac{\partial x_f^1(\mathbf{p})}{\partial p}\right)_{du^1=0} - (x_f^1(\mathbf{p}) - e_f^1)\left(\frac{\partial x_f^1(\mathbf{p})}{\partial I^1}\right)$$
$$+ \left(\frac{\partial x_f^2(\mathbf{p})}{\partial p}\right)_{du^2=0} - (x_f^2(\mathbf{p}) - e_f^2)\left(\frac{\partial x_f^2(\mathbf{p})}{\partial I^2}\right) \quad (1)$$

Normal Goods and Number of Equilibria IV

WLOG assume that

- in equilibrium (at $\mathbf{p}^*$). Person 1 is net buyer of food; i.e., $x_f^1(\mathbf{p}^*) - e_f^1 > 0$.
- In equi. (food) market clears. So,

$$x_f^2(\mathbf{p}^*) - e_f^2 = -[x_f^1(\mathbf{p}^*) - e_f^1].$$

At equilibrium price, $\mathbf{p}^*$, we have

$$\begin{aligned} \frac{\partial z_f(\mathbf{p}^*)}{\partial p} = & \left(\frac{\partial x_f^1(\mathbf{p}^*)}{\partial p}\right)_{du^1=0} - (x_f^1(\mathbf{p}^*) - e_f^1)\left(\frac{\partial x_f^1(\mathbf{p}^*)}{\partial I^1}\right) \\ + & \left(\frac{\partial x_f^2(\mathbf{p}^*)}{\partial p}\right)_{du^2=0} - (x_f^2(\mathbf{p}^*) - e_f^2)\left(\frac{\partial x_f^2(\mathbf{p}^*)}{\partial I^2}\right) \end{aligned} \tag{2}$$

We can rearrange (2) to get

Normal Goods and Number of Equilibria V

$$\frac{\partial z_f(\mathbf{p}^*)}{\partial p} = \left(\frac{\partial x_f^1(\mathbf{p}^*)}{\partial p}\right)_{du^1=0} + \left(\frac{\partial x_f^2(\mathbf{p}^*)}{\partial p}\right)_{du^1=0}$$
$$+ \quad (x_f^1(\mathbf{p}^*) - e_f^1)\left(\frac{\partial x_f^2(\mathbf{p}^*)}{\partial I^2} - \frac{\partial x_f^1(\mathbf{p}^*)}{\partial I^1}\right),$$

Now, even if both goods are normal,

- Person 2 might have large income effect that can offset the negative substitution effects.
- $\frac{\partial z_f(\mathbf{p}^*)}{\partial p} < 0$ might not hold.
- So, we cannot be sure of uniqueness of WE.

WARP and no of WE I

Let,

- $\mathbf{x} = \mathbf{x}(\mathbf{p})$ denote the bundle demanded at price $\mathbf{p}$; where $\mathbf{x} = (x_1, ...x_m)$.
- $\mathbf{x}' = \mathbf{x}(\mathbf{p}')$ denote the bundle demanded at price $\mathbf{p}'$; - $\mathbf{x}' = (x'_1, ...x'_m)$.

Therefore,

- $\mathbf{p}.\mathbf{x} = \mathbf{p}.\mathbf{x}(\mathbf{p})$ is the expenditure incurred at price $\mathbf{p}$.
- $\mathbf{p}'.\mathbf{x}' = \mathbf{p}'.\mathbf{x}(\mathbf{p}')$ is the expenditure incurred at price $\mathbf{p}'$.

The demand satisfies Weak Axiom of Revealed Preference (WARP), if

$$\mathbf{p}.\mathbf{x}' \leq \mathbf{p}.\mathbf{x} \Rightarrow \mathbf{p}'.\mathbf{x}' < \mathbf{p}'.\mathbf{x}.$$

- $\mathbf{p}.\mathbf{x}' \leq \mathbf{p}.\mathbf{x}$ implies that the bundle $\mathbf{x}'$ was affordable at price $\mathbf{p}$.
- $\mathbf{p}'.\mathbf{x} > \mathbf{p}'.\mathbf{x}'$ implies that the bundle $\mathbf{x} = \mathbf{x}(\mathbf{p})$ is strictly more expensive (than $\mathbf{x}' = \mathbf{x}(\mathbf{p}')$) at price $\mathbf{p}'$.

WARP and no of WE II

Restating: The demand satisfies WARP, if

$$\mathbf{p}.\mathbf{x}' \leq \mathbf{p}.\mathbf{x} \quad \Rightarrow \quad \mathbf{p}'.\mathbf{x}' < \mathbf{p}'.\mathbf{x}.$$
$$\mathbf{p}(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) \leq 0 \quad \Rightarrow \quad \mathbf{p}'(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) < 0.$$

Theorem

If the aggregate demand function satisfies the WARP, then the WE is unique.

Question

What are the assumptions needed for the aggregate demand function to exist?

- Aggregate demand function $\mathbf{x}(.)$ will exist iff if the individual demand function, i.e., $\mathbf{x}^i(.)$, exists for all $i = 1, .., N$.

WARP and no of WE III

- $\mathbf{x}^i(.)$ exists if the underlying utility function satisfies the assumption of continuity, strong monotonicity and strict quasi-concavity.

Define (aggregate) vectors:

$$
\begin{aligned}
\mathbf{x} &= \left(\sum_{i=1}^{N} x_1^i, \sum_{i=1}^{N} x_2^i, ..., \sum_{i=1}^{N} x_M^i\right) \\
\mathbf{e} &= \left(\sum_{i=1}^{N} e_1^i, \sum_{i=1}^{N} e_2^i, ..., \sum_{i=1}^{N} e_M^i\right) \\
\mathbf{z} &= \mathbf{x} - \mathbf{e} = \left(\sum_{i=1}^{N}(x_1^i - e_1^i), ..., \sum_{i=1}^{N}(x_M^i - e_M^i)\right) \\
\mathbf{z} &= \mathbf{x} - \mathbf{e} = (z_1, ..., z_M)
\end{aligned}
$$

WARP and no of WE IV

Proof: Suppose, WE is not unique. If possible, suppose $\mathbf{p}, \mathbf{p}' \in \mathbb{E}$, and $\mathbf{p} \neq \mathbf{p}'$. Note for any price vectors $\mathbf{p}$ and $\mathbf{p}'$, we have:

$$\begin{aligned} \mathbf{p}.\mathbf{z}(\mathbf{p}) &= 0, \textit{i.e.}, \\ \mathbf{p}.(\mathbf{x}(\mathbf{p}) - \mathbf{e}) &= 0. \end{aligned} \tag{3}$$

Since $\mathbf{p}'$ is an equi. price vector, $\mathbf{z}(\mathbf{p}') = \mathbf{x}(\mathbf{p}') - \mathbf{e} = 0$, i.e., $\mathbf{x}(\mathbf{p}') = \mathbf{e}$. Therefore, the assumption $\mathbf{p}' \in \mathbb{E}$ gives us

$$\begin{aligned} \mathbf{p}.(\mathbf{x}(\mathbf{p}) - \mathbf{x}(\mathbf{p}')) &= 0, \textit{i.e.}, \\ \mathbf{p}.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) &= 0. \end{aligned} \tag{4}$$

From WARP, we know that

$$\mathbf{p}.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) \leq 0 \Rightarrow \mathbf{p}'.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) < 0. \tag{5}$$

(4) and (5) give us,

$$\mathbf{p}'.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) < 0. \tag{6}$$

WARP and no of WE V

Similarly, we get:

$$\begin{aligned} \mathbf{p}'.\mathbf{z}(\mathbf{p}') &= 0, i.e., \\ \mathbf{p}'.(\mathbf{x}(\mathbf{p}') - \mathbf{e}) &= 0 \\ \mathbf{p}'.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) &= 0 \end{aligned} \tag{7}$$

which is a contradiction, since in view of (6), we have

$$\mathbf{p}'(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) < 0.$$

- The assumption that there are two price vectors $\mathbf{p}, \mathbf{p}' \in \mathbb{E}$ leads to a contradiction.
- There cannot be two or more equilibrium price vectors.