Competitive Equilibria: Uniqueness and Stability

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Microeconomic Theory

Lecture 7
Questions

- Is Competitive/Walrasian equilibrium unique?
- Why is a unique equilibrium helpful?
- If WE is not unique, how many WE can be there?
- What are the conditions, for a unique WE?
- Do these conditions hold in the real world?
- Is Competitive/Walrasian equilibrium stable?
- Why is stability of an equilibrium important?

Background Readings:

Example

From MWG*(1995): Two consumers:

- \( u^1(.) = x_1^1 - \frac{1}{8} \left( \frac{1}{x_1^1} \right)^8 \) and \( u^2(.) = -\frac{1}{8} \left( \frac{1}{x_2^2} \right)^8 + x_2^2 \)

- \( e^1 = (2, r) \) and \( e^2 = (r, 2); \ r = 2^{\frac{8}{9}} - 2^{\frac{1}{9}} \)

- The equilibria are solution to

\[
\left( \frac{p_2}{p_1} \right)^{-\frac{1}{9}} + 2 + r \left( \frac{p_2}{p_1} \right) - \left( \frac{p_2}{p_1} \right)^{\frac{8}{9}} = 2 + r, \ i.e.,
\]

there are three equilibria:

\[
\frac{p_2}{p_1} = \frac{1}{2}, 1, \text{ and } 2.
\]
Consider 2 × 2 economy:

- Two goods: food and cloth
- Let \((p_f, p_c)\) be the price vector.
- We know that for all \(t > 0\): \(z(tp) = z(p)\).
- Therefore, we can work with

\[
p = (\frac{p_f}{p_c}, 1) = (p, 1).
\]

From Walras’s Law, we have \(p.z(p) = 0\), i.e.,

\[
pz_f(p) + z_c(p) = 0.
\]

Conditions for existence of WE:

- \(z_i(p)\) is continuous for all \(p \gg p\), i.e., for all \(p > 0\).
there exists small $p = \epsilon > 0$ s.t. $z_f(\epsilon, 1) > 0$, and
there exists another $p' > \frac{1}{\epsilon}$ s.t. $z_c(p', 1) > 0$.

We can ensure above, by assuming the utility functions to be continuous, strongly monotonic and strictly quasi-concave.

**Question**

- *Do the above assumptions guarantee unique WE?*
- *Under what conditions the WE be unique?*

An additional assumption can ensure uniqueness of WE:

- $z'_f(p) < 0$ for all $p > 0$. 
Do the above assumptions on utility functions ensure $z'_f(p) < 0$ for all $p > 0$?
Normal Goods and Number of Equilibria

Let,

- there be two goods - food and cloth.
- \( \mathbf{e}^1 = (e^1_f, e^1_c) \) and \( \mathbf{e}^2 = (e^2_f, e^2_c) \) be the initial endowment vectors.
- \( \mathbf{p} = (p, 1) \) be a price vector.

Assumption: Assume utility functions to be

- continuous, strongly monotonic and strictly quasi-concave.

From Walras Law we have \( \mathbf{p}.\mathbf{z}(\mathbf{p}) = 0 \), i.e.,

\[
p z_f(p) + z_c(p) = 0.
\]

By definition:

\[
z_f(p) = z_f^1(p) + z_f^2(p) = [x_f^1(p) - e_f^1] + [x_f^2(p) - e_f^2].
\]
In view of our assumptions on utilities,

- $z_f(p)$ and $z_c(p)$ are continuous for all $p >> 0$, i.e., for all $p > 0$.
- there exists small $p = \epsilon > 0$ s.t. $z_f(\epsilon, 1) > 0$ and another $p' > \frac{1}{\epsilon}$ s.t. $z_c(p', 1) > 0$.

Let $p^*$ denote an equilibrium price vector.

We know that for the above economy at least one $p^*$ exists. Why?

Let

- $l^1(p, e^1) = p.e^1 = p e^1_f + e^1_c$
- $l^2(p, e^2) = p.e^2 = p e^2_f + e^2_c$
Note

\[
\frac{dz_f(p)}{dp} = \frac{dz_1^1(p)}{dp} + \frac{dz_2^2(p)}{dp}
\]

Price effect (total)  Price effect (total)  Price effect (total)

Since endowments are fixed, we get

\[
\frac{dz_f(p)}{dp} = \left( \frac{\partial x_1^1(p)}{\partial p} \right)_{du^1=0} - (x_1^1(p) - e_1^1) \left( \frac{\partial x_1^1(p)}{\partial I_1} \right) \\
+ \left( \frac{\partial x_2^2(p)}{\partial p} \right)_{du^2=0} - (x_2^2(p) - e_2^2) \left( \frac{\partial x_2^2(p)}{\partial I_2} \right)
\]

(1)
WLOG assume that

- in equilibrium (at $p^*$). Person 1 is net buyer of food; i.e., $x_f^1(p^*) - e_f^1 > 0$.
- In equi. (food) market clears. So,

$$x_f^2(p^*) - e_f^2 = -[x_f^1(p^*) - e_f^1].$$

At equilibrium price, $p^*$, we have

$$\frac{\partial z_f(p^*)}{\partial p} = \left( \frac{\partial x_f^1(p^*)}{\partial p} \right)_{du^1=0} - (x_f^1(p^*) - e_f^1) \left( \frac{\partial x_f^1(p^*)}{\partial I^1} \right) + \left( \frac{\partial x_f^2(p^*)}{\partial p} \right)_{du^2=0} - (x_f^2(p^*) - e_f^2) \left( \frac{\partial x_f^2(p^*)}{\partial I^2} \right)$$

(2)

We can rearrange (2) to get
Now, even if both goods are normal,

- Person 2 might have large income effect that can offset the negative substitution effects.
- \( \frac{\partial z_f(p^*)}{\partial p} < 0 \) might not hold.
- So, we cannot be sure of uniqueness of WE.
WARP and no of WE I

Let,

- \( x = x(p) \) denote the bundle demanded at price \( p \); where \( x = (x_1, ..., x_m) \).
- \( x' = x(p') \) denote the bundle demanded at price \( p' \);
- \( x' = (x'_1, ..., x'_m) \).

Therefore,

- \( p.x = p.x(p) \) is the expenditure incurred at price \( p \).
- \( p'.x' = p'.x(p') \) is the expenditure incurred at price \( p' \).

The demand satisfies Weak Axiom of Revealed Preference (WARP), if

\[
p.x' \leq p.x \Rightarrow p'.x' < p'.x.
\]

- \( p.x' \leq p.x \) implies that the bundle \( x' \) was affordable at price \( p \).
- \( p'.x > p'.x' \) implies that the bundle \( x = x(p) \) is strictly more expensive (than \( x' = x(p') \)) at price \( p' \).
WARP and no of WE II

Restating: The demand satisfies WARP, if

\[ p \cdot x' \leq p \cdot x \Rightarrow p' \cdot x' < p' \cdot x. \]
\[ p(x(p') - x(p)) \leq 0 \Rightarrow p' (x(p') - x(p)) < 0. \]

Theorem

*If the aggregate demand function satisfies the WARP, then the WE is unique.*

Question

*What are the assumptions needed for the aggregate demand function to exist?*

- Aggregate demand function \( x(\cdot) \) will exist iff if the individual demand function, i.e., \( x^i(\cdot) \), exists for all \( i = 1, \ldots, N \).
x^i(.) exists if the underlying utility function satisfies the assumption of continuity, strong monotonicity and strict quasi-concavity.

Define (aggregate) vectors:

\[ x = (\sum_{i=1}^{N} x_1^i, \sum_{i=1}^{N} x_2^i, \ldots, \sum_{i=1}^{N} x_M^i) \]

\[ e = (\sum_{i=1}^{N} e_1^i, \sum_{i=1}^{N} e_2^i, \ldots, \sum_{i=1}^{N} e_M^i) \]

\[ z = x - e = (\sum_{i=1}^{N} (x_1^i - e_1^i), \ldots, \sum_{i=1}^{N} (x_M^i - e_M^i)) \]

\[ z = x - e = (z_1, \ldots, z_M) \]
WARP and no of WE IV

Proof: Suppose, WE is not unique. If possible, suppose $p, p' \in \mathbb{E}$, and $p \neq p'$. Note for any price vectors $p$ and $p'$, we have:

$$p \cdot z(p) = 0, \text{ i.e., } p \cdot (x(p) - e) = 0.$$  \hspace{1cm} (3)

Since $p'$ is an equi. price vector, $z(p') = x(p') - e = 0$, i.e., $x(p') = e$. Therefore, the assumption $p' \in \mathbb{E}$ gives us

$$p \cdot (x(p) - x(p')) = 0, \text{ i.e., } p \cdot (x(p') - x(p)) = 0.$$  \hspace{1cm} (4)

From WARP, we know that

$$p \cdot (x(p') - x(p)) \leq 0 \Rightarrow p' \cdot (x(p') - x(p)) < 0.$$  \hspace{1cm} (5)

(4) and (5) give us,

$$p' \cdot (x(p') - x(p)) < 0.$$  \hspace{1cm} (6)
Similarly, we get:

\[ p'.z(p') = 0, \text{ i.e.,} \]
\[ p'.(x(p') - e) = 0 \]
\[ p'.(x(p') - x(p)) = 0 \quad (7) \]

which is a contradiction, since in view of (6), we have

\[ p'(x(p') - x(p)) < 0. \]

- The assumption that there are two price vectors \( p, p' \in \mathbb{E} \) leads to a contradiction.
- There cannot be two or more equilibrium price vectors.