Barter Economy and the Core *†

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September 21, 2017

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1 Basics

Consider a pure exchange economy. That is, there is no production - either because it has already taken place or the initial endowments themselves are consumption goods. Individuals are free to trade with one another. Trade involves exchange of goods for goods. We want to find out the nature of equilibrium outcome under such an economy. In a 2×2 pure exchange economy the initial endowments can be written as $\mathbf{e}^1 = (e_1^1, e_2^1)$ for individual 1 and $\mathbf{e}^2 = (e_1^2, e_2^2))$ for individual 2. Suppose $(e_1^1, e_2^1) = (8, 2)$ and $(e_1^2, e_2^2) = (0, 7)$. That is, to start with, the first individual has 8 units of the first good and only 2 units of the second good. As to individual 2, she has 7 units of the second good only.

An allocation is a distribution of goods among the individuals. We will consider allocations that are consistent with the initial endowments. In principle, it is possible to distribute, i.e., allocate the two goods between these two individual differently from the initial allocation. For example, allocation $\mathbf{x}^1 = (3, 5)$ and $\mathbf{x}^2 = (5, 3)$ is one of the many allocations that are possible from the above endowments. This allocation gives to the first individual 3 units of first good and 5 units of the second good; for the second individual, it is the other way around.

Let us denote an allocation by $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$, where $\mathbf{x}^1 = (x_1^1, x_2^1)$, and $\mathbf{x}^2 = (x_1^2, x_2^2)$. By definition, allocation is a feasible distribution of good among the individuals. Therefore, for allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ we have: $x_1^1 + x_1^2 \leq e_1^1 + e_1^2$ and $x_2^1 + x_2^2 \leq e_2^1 + e_2^2$. Note that the second good is available in nine units. However, the allocation $\mathbf{x}^1 = (3, 5)$, and $\mathbf{x}^2 = (5, 3)$ 'wastes' one unit of the second good, as this unit is consumed by neither of the individuals. As intuition would suggest, an allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ is non-wasteful

*To learn more about the issues covered here, you can read Debreu and Scarf (1963), Aumann (1964), Scarf (1967) and Feldman (1974).

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if

$$\begin{aligned} x_1^1 + x_1^2 &= e_1^1 + e_1^2 \\ x_2^1 + x_2^2 &= e_2^1 + e_2^2 \end{aligned}$$

We can easily extend the above concepts to $N \times M$ economy, i.e., economy consisting of N individuals and M goods/commodities. Let an individual be indexed by i, $i = 1, 2, ..., N$; and a commodity be indexed by j, $j = 1, 2, ..., M$. Each individual i is endowed with a bundle of M commodities denoted by $\mathbf{e}^i = (e_1^i, e_2^i, ..., e_M^i)$.

So, for the $N \times M$ economy, an endowment is a vector of vectors: $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^N)$, where $\mathbf{e}^1 = (e_1^1, e_2^1, ..., e_M^1)$; $\mathbf{e}^i = (e_1^i, e_2^i, ..., e_M^i)$, etc. Using these notations, the total endowment of goods can be re-written as:

$$\text{good 1}: \quad e_1^1 + e_1^2 + ... + e_1^N = \sum_{i=1}^{N} e_1^i$$

$$\text{good j}: \quad e_j^1 + e_j^2 + ... + e_j^N = \sum_{i=1}^{N} e_j^i$$

For the $N \times M$ economy, an allocation can be denoted by:

$$\mathbf{x} = (\mathbf{x}^1, ..., \mathbf{x}^N),$$

where $\mathbf{x}^1 = (x_1^1, x_2^1, ..., x_M^1)$, $\mathbf{x}^i = (x_1^i, x_2^i, ..., x_M^i)$, etc. Moreover, we will call an allocation, $\mathbf{x} = (\mathbf{x}^1, ..., \mathbf{x}^N)$, to be non-wasteful w.r.t. good 1 if $\sum_{i=1}^{N} x_1^i = \sum_{i=1}^{N} e_1^i$. More generally,

Definition 1 Allocation $\mathbf{x} = (\mathbf{x}^1, ..., \mathbf{x}^N)$ is non-wasteful if

$$\sum_{i=1}^{N} x_j^i = \sum_{i=1}^{N} e_j^i$$

for all $j = 1, ..., M$

It makes no sense to go for wasteful allocations. That is, we can meaningfully restrict our attention to non-wasteful allocations. To sum up, we define an 'allocation' to be a feasible and non-wasteful distribution of the initial endowments among the individuals involved. Formally, we define the set of feasible allocations as follows.

Definition 2 Set of feasible (non-wasteful) allocations is the set

$$F(\mathbf{e}) = \left\{ \mathbf{x} = (\mathbf{x}^1, ..., \mathbf{x}^N) | \; x_j^i \geq 0 \; and \; \sum_{i=1}^{N} x_j^i = \sum_{i=1}^{N} e_j^i \right\}, \tag{1}$$

for all $i = 1, ..., N$ *and* $j = 1, ..., M$.

We will make the following assumptions about the choice sets and the individual preferences:

- The set of alternatives is the set of feasible (non-wasteful) allocations, i.e., the set defined in (1).
- Individuals have (self-interested) preferences defined over the set of allocations defined in (1). Specifically, an individuals ranking of the feasible allocations is based only on what she gets from each allocation.
- Each preference is/can be represented by a (self-interested) utility function. If a preference is complete, transitive and continuous, then it will be represented by a utility function.[1]
- Each preference relation and thereby each utility function is monotonic.

2 Outcome under Barter

In Barter individuals trade/exchange bundles of goods with one another, without using the 'money'. First of all, we want answer this questions. What are the achievable outcomes under Barter? We will answer this question assuming that individuals can freely decide whether to trade or not. Moreover, they have all the information relevant for the trade - who has what bundle of goods, who is willing to trade their bundle, etc. Moreover, there are no costs of engaging in the trade.

We can start by using the familiar Pareto efficiency criterion. Pareto efficiency, or Pareto optimality, is a state of allocation of resources in which it is impossible to make any one individual better off without making at least one individual worse off. For the above 2×2 economy, an allocation $(\mathbf{x^1}, \mathbf{x^2})$ is Pareto superior to the endowment, $(\mathbf{e^1}, \mathbf{e^2})$, if $u^i(\mathbf{x}^i) \geq u^i(\mathbf{e}^i)$ holds for $i = 1, 2$, and $u^i(\mathbf{x}^i) > u^i(\mathbf{e}^i)$ holds for at least one $i = 1, 2$. In other words, an allocation $(\mathbf{x^1}, \mathbf{x^2})$ is Pareto superior to the endowment, $(\mathbf{e^1}, \mathbf{e^2})$, if every individual is at least as well off with $(\mathbf{x^1}, \mathbf{x^2})$ as s/he was with $(\mathbf{e^1}, \mathbf{e^2})$ and at least one individual is strictly better off. Thus we have,

Remark 1 If $(\mathbf{x^1}, \mathbf{x^2})$ is Pareto superior to $(\mathbf{e^1}, \mathbf{e^2})$, then: $(\mathbf{e}^1, \mathbf{e}^2)$ will be Pareto inferior to $(\mathbf{x^1}, \mathbf{x^2})$; and $(\mathbf{e}^1, \mathbf{e}^2)$ CANNOT be Pareto optimum. However, $(\mathbf{x}^1, \mathbf{x}^2)$ may or may not be Pareto optimum.

Definition 3 Allocation $(\mathbf{x}^1, \mathbf{x}^2)$ is Pareto optimum, if there is no (feasible) allocation $(\mathbf{y}^1, \mathbf{y}^2)$ such that $(\mathbf{y}^1, \mathbf{y}^2)$ is Pareto superior to $(\mathbf{x}^1, \mathbf{x}^2)$.

Note, in general, there can be several Pareto optimum allocations.

When there are no restrictions on exchanges it seems plausible to argue that people would want to trade with one another so as to move from Pareto inferior allocations to

[1]Note: When the set of alternatives is finite then a preference relation can be represented by a utility function as long as it is complete and transitive. Why is this the case?

Pareto optimum ones. Therefore, the outcome under barter should be Pareto efficient. Formally speaking, the set of possible outcomes should be a subset of the set of Pareto optimal allocations. To see this, let's use the following example.

Example 1 Suppose the two-person two-goods economy consists of you (as first individual) and me (the second individual). Let endowments be: $\mathbf{e}^1 = (1, 9)$, and $\mathbf{e}^2 = (9, 1)$. Our preferences are 'similar' and are represented by utility functions given by, $u^1(x_1^1, x_2^1) = x_1^1 \times x_2^1 = x_1^1 x_2^1$ and $u^2(x_1^2, x_2^2) = x_1^2 \times x_2^2 = x_1^2 x_2^2$, respectively.

Consider the allocation: $\mathbf{x}^1 = (3, 3)$, and $\mathbf{x}^2 = (7, 7)$. Clearly, it is feasible. And, $u^1(\mathbf{x}^1) \geq u^1(\mathbf{e}^1)$, and $u^2(\mathbf{x}^2) \geq u^2(\mathbf{e}^2)$. In fact $u^2(\mathbf{x}^2) > u^2(\mathbf{e}^2)$. That is, allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ is Pareto superior to $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2)$

In the above example, allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2) = ((3, 3), (7, 7))$ makes both of us at least as well off as is the case at $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2)$, one of us is strictly better off at $\mathbf{x}$. Therefore nobody would mind moving from $\mathbf{e}$ to $\mathbf{x}$ and one of us would strictly prefer this shift. Therefore, we should move to $\mathbf{x}$ rather than staying put at $\mathbf{e}$. We can also restate this by saying that we will reject $\mathbf{e}$ in favour of $\mathbf{x}$. Alternatively, we can also say that the allocation $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2)$ is blocked by the allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ - i.e., $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2)$ cannot be a possible outcome under barter. Clearly, $\mathbf{x} = ((3, 3), (7, 7))$ is not the only allocation that blocks $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2)$. Try to find out the other allocations that can block $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2)$ in the sense just described.

Next, suppose I propose that we move from $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2)$ to $\mathbf{y} = (\mathbf{y^1}, \mathbf{y^2})$. If this allocation is such that $u^1(\mathbf{e}^1) > u^1(\mathbf{y}^1)$, you reject the proposal outright (since the trade is voluntary). So, $\mathbf{y} = (\mathbf{y^1}, \mathbf{y^2})$ cannot be an outcome under barter. Put differently, we can say that the allocation $\mathbf{y} = (\mathbf{y^1}, \mathbf{y^2})$ will be blocked from being a final outcome.

Formally, for a two-person two-goods economy, an allocation $\mathbf{y} = (\mathbf{y^1}, \mathbf{y^2})$ will be **blocked**, if any of the following is true:

$$\begin{aligned} u^1(\mathbf{e}^1) &> u^1(\mathbf{y}^1) \text{ or;} \\ u^2(\mathbf{e}^2) &> u^2(\mathbf{y}^2) \text{ or;} \end{aligned}$$

there exists a feasible allocation $(\mathbf{x^1}, \mathbf{x^2})$ such that

$$\begin{cases} u^i(\mathbf{z}^i) \geq u^i(\mathbf{y}^i) & \text{for every individual i=1,2. And;} \\ u^j(\mathbf{z}^j) > u^j(\mathbf{y}^j), & \text{for aleast one } \text{ j=1,2.} \end{cases}$$

The last condition says that $\mathbf{y} = (\mathbf{y^1}, \mathbf{y^2})$ will be blocked, i.e., rejected if there exists another allocation $(\mathbf{z^1}, \mathbf{z^2})$ that is Pareto superior to $(\mathbf{y^1}, \mathbf{y^2})$.

For a 2×2 economy, we say that an allocation belongs to the **Core**, if it cannot be blocked in the sense described above. To sum up the discussion so far, for a two-person two-goods economy, an allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ belongs to the Core, if and only if the following two conditions are satisfied: First, each individual i, $i = 1, 2$, must (weakly) prefer $\mathbf{x}^i$ at least as much as $\mathbf{e}^i$. That is,

$$u^i(\mathbf{x}^1) \geq u^i(\mathbf{e}^1) \text{ holds for each } i = 1, 2 \tag{2}$$

Second, the allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ must be Pareto optimum. That is, there should NOT be any feasible allocation, $\mathbf{z} = (\mathbf{z}^1, \mathbf{z}^2)$, such that

$$\begin{cases} u^i(\mathbf{z}^i) \geq u^i(\mathbf{x}^i) & \text{for every i=1,2. And;} \\ u^j(\mathbf{z}^j) > u^j(\mathbf{x}^j), & \text{for some at least one j=1,2} \end{cases} \tag{3}$$

The above conditions (2) and (3) are necessary and sufficient conditions for an allocation to be in the Core of 2×2 economy. In the above example, recall the allocation $(\mathbf{x^1}, \mathbf{x^2})$ with $\mathbf{x}^1 = (3, 3)$ and $\mathbf{x}^2 = (7, 7)$ cannot be blocked. Therefore, it belongs to the Core. In other words, it is Core allocation.

Here it will be useful to use an Edgeworth box (let's borrow one from Jehle and Reny, page 183). In this diagram, your origin is O^1, and my origin is O^2. You can check that if allocation $\mathbf{y} = (\mathbf{y^1}, \mathbf{y^2})$ lies below your (first person's) indifference curve passing through $\mathbf{e}$, you will find $\mathbf{y}$ unacceptable - you are better off holding on to whatever you have got. That is, the condition (2) stated above will not hold (for $i = 1$), and therefore, the allocation cannot be a Core allocation. On the other hand, if allocation $\mathbf{y} = (\mathbf{y^1}, \mathbf{y^2})$ lies above my indifference curve passing through $\mathbf{e}$, I will find $\mathbf{y}$ unacceptable. Now, condition (2) will not hold (for $i = 2$) again $\mathbf{y}$ cannot belong to the Core. Finally, suppose $\mathbf{y}$ lies within the area/lens formed by our ICs through $\mathbf{e}$. If $\mathbf{y}$ does not lie on the segment $c - c$ (suppose it is a point like B), in that case, two of us could gain by moving to a point like D. In that case, condition (3) will not hold. You can check that for any allocation $\mathbf{x}$ on the segment $c - c$, both of the above conditions hold.

By now, we know how to find the Core of a 2×2 economy. Note that if we start from a Core allocation, then no individual or a group of individuals would have objection against it. That is, no individual can do better by rejecting a Core allocation - in case of rejection an individual will end up with her endowment bundle. Moreover, acting jointly the individuals cannot be both better off if they move away from a core allocation to any other allocation. For the above example, what is the size of the Core? How many unblocked allocations are there?

Next question is: How can we find the Core of a $N \times M$ economy? Let us expand our 2×2 economy to make it $N \times M$. The conditions similar to conditions (2) and (3) are relevant for $N \times M$ economy as well. It is easy to see that for $N \times M$ economy, an allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$ will belong to the Core, only if

- Every i prefers $\mathbf{x}^i$ at least as much as $\mathbf{e}^i$, $i = 1, 2, ..., N$ - otherwise i will reject the proposed allocation $\mathbf{x}$, which is to say that individual i will block $\mathbf{x}$.[2]

[2]This can be called 'individual rationality' constraint.

- Allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$ is Pareto optimum - If $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$ is not Pareto optimum, then there will be some allocation $\mathbf{z} = (\mathbf{z}^1, \mathbf{z}^2, ..., \mathbf{z}^N)$ that is Pareto superior to $\mathbf{x}$. So, some individuals would want to move to $\mathbf{z}$ and such a move will not hurt anybody. That is, all individuals together will be okay moving from $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$ to $\mathbf{z} = (\mathbf{z}^1, \mathbf{z}^2, ..., \mathbf{z}^N)$. Formally, $\mathbf{z} = (\mathbf{z}^1, \mathbf{z}^2, ..., \mathbf{z}^N)$ will block $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$. So, $\mathbf{x}$ can be a Core allocation only if it is Pareto optimum allocation. In other words, Pareto optimality is a necessary condition for an allocation to be a Core allocation.

However, for $N \times M$ economy the above conditions are not sufficient to give us the set of Core allocations. To see why, let us modify our earlier example as follows:

Example 2 The 2×2 economy of Example 1 is expanded to make it the following three-person, two-goods economy: Endowments are $\mathbf{e}^1 = (1, 9)$, $\mathbf{e}^2 = (9, 1)$, and $\mathbf{e}^3 = (5, 5)$, respectively. Preferences are represented by $u^1(x_1^1, x_2^1) = x_1^1 x_2^1$, $u^2(x_1^2, x_2^2) = x_1^2 x_2^2$, and $u^3(x_1^3 x_2^3) = x_1^3 x_2^3$, respectively.

Now, consider the allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$, where

$$\mathbf{x}^1 = (3, 3),\ \mathbf{x}^2 = (7, 7), \text{ and } \mathbf{x}^3 = (5, 5).$$

Let us ask the following questions about allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$: Is it feasible? Does individual i prefer $\mathbf{x}^i$ over $\mathbf{e}^i$? Is allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ Pareto optimum? You should be able to show that answers to these questions are: 'Yes', 'Yes', and 'Yes'. Therefore, allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ satisfies the two conditions stated above. In fact, you can see that $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ is Pareto-superior to the endowment vector, $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$.

Still, allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ does NOT belong to the Core. To see why, consider a Coalition of 1 and 3, i.e., consider the set $S = \{1, 3\}$. Recall, $\mathbf{e}^1 = (e_1^1, e_2^1) = (1, 9)$ and $\mathbf{e}^3 = (e_1^3, e_2^3) = (5, 5)$. That is, individuals 1 and 3 together have 6 units of the first good and 14 units of the second good. Therefore, by pooling whatever is available between two of them, they can afford the following allocation: $\mathbf{y}^1 = (2, 5) = (y_1^1, y_2^1)$ for individual 1 and $\mathbf{y}^3 = (4, 9) = (y_1^3, y_2^3)$ for person 3. Note that for individuals in set $S = \{1, 3\}$, the following is true: $y_1^1 + y_1^3 = e_1^1 + e_1^3$ and $y_2^1 + y_2^3 = e_2^1 + e_2^3$. That is,

$$\sum_{i \in S} y_j^i = \sum_{i \in S} e_j^i \ \text{ for all } j = 1, 2$$

Moreover, $u^1(y_1^1, y_2^1) > u^1(x_1^1, x_2^1)$, i.e, $u^1(\mathbf{y}^1) > u^1(\mathbf{x}^1)$ and $u^3(y_1^3, y_2^3) > u^3(x_1^3, x_2^1)$, i.e., $u^3(\mathbf{y}^3) > u^3(\mathbf{x}^3)$. Using mathematical notations, we can say that in this example the following claims hold:

$$\begin{aligned} u^i(\mathbf{y}^i) = u^i(y_1^i, y_2^i) &\ \geq\ u^i(x_1^i, x_2^i) = u^i(\mathbf{x}^i) \ \text{ for all } \ i \in S = \{1, 3\} \\ u^j(\mathbf{y}^j) = u^j(y_1^j, y_2^j) &\ >\ u^i(x_1^j, x_2^j) = u^j(\mathbf{x}^j) \ \text{ for at least some } j \in S = \{1, 3\}. \end{aligned}$$

Now suppose someone asks individuals 1, 2 and 3 to give up their endowments and accept the above bundles ($\mathbf{x}^1$, $\mathbf{x}^2$ and $\mathbf{x}^3$), respectively. Individuals 1 and 3, i.e., those in the set $S = \{1, 3\}$ would not agree - they can serve their interests better by forming a 'coalition', pooling their endowments so as to consume ($\mathbf{y}^1$ and $\mathbf{y}^3$), respectively. In such a scenario, we will say that the set $S = \{1, 3\}$ is a blocking coalition against allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$.

Therefore, even if there is no individual deviation/objection against a Pareto optimum allocation, there can be subgroups who can profitably deviate from it. That is, a sub-group might be a blocking coalition against a Pareto optimum allocation. If this happens, the allocation cannot belong to the Core.

Now, we are ready to provide a general definition of a Blocking Coalition and the Core.

Definition 4 Blocking Coalition: A set $S \subseteq \{1, ..., N\}$ is called a blocking coalition against $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$ if there is some vector $\mathbf{y}$ such that[3]

$$\begin{aligned} \sum_{i \in S} y^i_j &= \sum_{i \in S} e^i_j \text{ for all } j = 1, ..., M \\ u^i(\mathbf{y}^i) = u^i(y^i_1, ..., y^i_M) &\geq u^i(x^i_1, ..., x^i_M) = u^i(\mathbf{x}^i) \text{ for all } i \in S \\ u^j(\mathbf{y}^j) = u^j(y^j_1, ..., y^j_M) &> u^j(x^j_1, ..., x^j_M) = u^j(\mathbf{x}^j) \text{ for at least some } j \in S \end{aligned}$$

Definition 5 Core: Consider a pure exchange economy $(u^i(.), \mathbf{e}^i)_{i \in N}$. The Core, $\mathbf{C}(u^i(.)_{i \in N}, \mathbf{e})$, is a subset of allocations that cannot be blocked. That is, $\mathbf{x} \in \mathbf{C}(u^i(.)_{i \in N}, \mathbf{e})$ if and only if there is no blocking coalition against $\mathbf{x}$.

3 The Core and its Size

How can we go about finding the Core of an economy? What can we say about its size? Let us address these issues.

In view of the above, to find the Core we just need to consider the set of Pareto optimal allocations. Further, we can safely restrict attention to only those Pareto optimal allocations that satisfy conditions described in (2) above. However, the set of Pareto optimal allocations satisfying these conditions can be large - in principle, there can be infinitely many allocations satisfying these conditions. Therefore, even for allocations that are Pareto optimum and satisfying these conditions in (2), we need to check possibility of a blocking coalition for each allocation. Discussion on Example 3 suggests that the set of possible blocking coalition increases with the number of individuals in the economy. When there are 2 players, the possible coalitions are given by the sets $\{1\}, \{2\}$, and $\{1, 2\}$. You can check that if there are just 25 individuals, there can be as many as $2^{25} - 1$ coalitions (non-empty subsets of individuals) that can potentially block an allocation.

[3] Vector $\mathbf{y}$ assigns a bundle of goods to each member of S.

The above discussion also suggests that the size of the Core changes with the number of individuals in the economy. Recall, the allocation that gives $(3, 3)$ to individual 1 and $(7, 7)$ to individual 2 belongs to the Core when $N = 2$, but does not belong to Core when $N = 3$. This inference is further corroborated by the following example, in which we replicate the economy described in Example 2.

Example 3 There are four individuals and two goods. Utility functions are: $u^i(x_1^i x_2^i) = x_1^i x_2^i$, for $i = 1, ..., 4$. Endowments are: $\mathbf{e}^1 = (1, 9)$, $\mathbf{e}^2 = (9, 1)$, $\mathbf{e}^3 = (1, 9)$, and $\mathbf{e}^4 = (9, 1)$, respectively.

That is, we have introduced two twins - individual 3 is twin of 1, and 4 is twin of 2! Now, consider the allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4)$, where

$$\mathbf{x}^1 = (3, 3) = \mathbf{x}^3 \text{ and } \mathbf{x}^2 = (7, 7) = \mathbf{x}^4.$$

Note that the allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4)$ is Pareto optimum. Moreover, we have already seen that allocation with $\mathbf{x}^1 = (3, 3)$ and $\mathbf{x}^2 = (7, 7)$ belongs to the Core when economy consists of individuals 1 and 2 only. In view of this, we may be tempted to think that the allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4)$ belongs to the Core of 4×2 economy of Example 4. But, this is not true.

Let $S = \{1, 2, 3\}$, $\mathbf{y}^1 = (3, 4) = \mathbf{y}^3$ and $\mathbf{y}^2 = (5, 11)$. I leave it to you to verify that individuals in set S can re-arrange their allocations to arrive at allocation $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3)$. Moreover, equipped with $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3)$, the set S satisfies all the conditions of a blocking coalition. So, set S can use allocation $\mathbf{y}$ to block allocation $\mathbf{x}$. The result? $\mathbf{x}$ is no more part of the Core.

Scarf (1963) showed that when indifference curves are convex, the Core is non-empty. However, as the above examples suggest, the size of the Core shrinks as N increases. Edgeworth (1881) himself had predicted this. However, the formal proofs were provided in the subsequent works of Debreu and Scarf (1963); Aumann (1964), among others.

To sum up, a particular allocation or a particular distribution of resources, say $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$ belongs to Core if it has the following property: No group or subgroup of individuals can do better by deviating from or refusing to accept the allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$. In other words, if individuals find themselves at the allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$, there will be no deviations from it. In that sense, the Core is set of possible equilibrium allocations.

We have seen that the location and the size of the Core depends on the nature of individual preferences, the initial distribution of endowment/wealth and the number of individuals in the economy. The above discussion also suggests that description of the Core is complex. Since, it is hard to fully describe the individual elements of the Core of Barter economy - beyond saying that they satisfy the conditions mentioned earlier.

Some of the complexity of the Core is due to the fact that in the above discussion utilities are 'non-transferable' across individuals. That is, once the endowments are used (consumed) and converted into utility, it is not possible to transfer the benefit/utility across individuals. In the following argument, we will show that for the contexts involving 'transferable' utilities the Core may be described more neatly.

4 The Core when Utilities are Transferable

Example 4 Production: Consider the following three firms/farmers, who are engaged in production of a good: The endowments of factors of production (FOP) are: $\mathbf{e}^1 = (e_1^1, e_2^1) = (1, 9)$, $\mathbf{e}^2 = (e_1^2, e_2^2) = (9, 1)$, and $\mathbf{e}^3 = (e_3^1, e_2^3) = (5, 5)$ Production Functions are: $q^1(x_1^1, x_2^1) = x_1^1 x_2^1$; $q^2(x_1^2, x_2^2) = x_1^2 x_2^2$; and $q^3(x_1^3 x_2^3) = x_1^3 x_2^3$. Assume that the FOPs can be shifted/employed across farms/plants as desired. Suppose, the farmers' (owners') consume the output produced, and utility of q units of output is q. That is, the utility functions are: $u^i(q^i) = q^i$, where $i = 1, 2, 3$.

Note that in Example 4, after endowments (FOPs) are converted into output level(s), it is still possible to transfer the resulting output/utility across individuals. In this sense we can say that, the utilities are transferable here. Let,

$\bar{e}_j^S$ denote the total endowment of FOP, j, jointly owned by a group or Coalition S. Therefore, we can write

$$\bar{e}_1^S = \sum_{i \in S} e_1^i \text{ } and \text{ } \bar{e}_2^S = \sum_{i \in S} e_2^i$$

Note that the maximum possible production level for Coalition, S, is $Q^S = \bar{e}_1^S . \bar{e}_2^S$. (Find out this quantity for each subgroup of firms/farmers.) Since output levels can be transferred across individuals, you can plausibly assume that each group or coalition, S, will produce $Q^S = \bar{e}_1^S . \bar{e}_2^S$ units. The maximum production occurs when the three farmers/firms go for joint production by pooling their endowments. The maximum output level is 225 units.

In this context, the Core is the set of those allocations (divisions of 225 units) that cannot be blocked/objected to by an individual farmer. You can see that allocation $(0, 0, 225)$ does not belong to the Core - since individual 1 and 2 both can enjoy higher utility by going alone, as well as by working together. Does allocation $(45, 45, 135)$ belong to the Core? You can check that no farmer can do better by individually refusing to accept this - the maximum farmers 1 and 2 individually can do is produce 9 units each, farmer three can produce maximum 25 units. However, farmers 1 and 2 together can produce 100 units and if they go for equal division they can get 50 units each. Therefore, $S = \{1, 2\}$ is a blocking coalition against allocation $(45, 45, 135)$, which in turn means that this allocation cannot belong to the Core.

You can show that an allocation (y_1, y_2, y_3) will be a Core allocation if and only if the following conditions are met with: $y_1 \geq 9$; $y_2 \geq 9$; $y_3 \geq 25$; $y_1 + y_2 \geq 100$; $y_1 + y_3 \geq$

84; $y_2 + y_3 \geq 84$. Formally, the set of Core allocations can be defined as:

$$\mathbf{C}(\mathbf{e}, q_i) = \{(y_1, y_2, y_3) |\ y_i \geq 0\ \&\ \sum_{i \in S} y_i \geq Q^S = \bar{e}_1^S . \bar{e}_2^S\}, \tag{4}$$

for each $S \subseteq \{1, 2, 3\}$ (note, we are talking only about the non-empty subsets).

In this context, you can immediately tell whether an allocation is Core allocation or not. Is (55,55,115) a Core allocation?

4.1 Does Core always exist?

(This Subsection is strictly optional)

The above discussion suggests that Core has many allocations - as a set it can be too big. As noted earlier, for the standard case of non-transferable utilities, Scarf (1963) showed that when indifference curves are convex, the Core is non-empty. However, depending on the rules of the game, in some contexts Core may not exist at all, i.e., the Core could be an empty set. As an illustration consider the following example.

Example 5 There are three farmers - 1, 2 and 3 - who own small but contiguous parcels of land. Individually they can use their parcels to produce a crop, say paddy. Let this outcome be denoted by (t_1, t_2, t_3) - t_i denotes the output level produced by farmer i using his land. Alternatively, they could plant any one of the three commercial crops; x, y, z. However, this will require them to pool their land. Further only one of the crops can be planted. If crop x is produced it will lead to the outcome allocation $\mathbf{x} = (x_1, x_2, x_3)$; similarly for the other two crops. Therefore, pooling of resources leads to any one of the following possible allocations: $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ and $\mathbf{z} = (z_1, z_2, z_3)$; Suppose, the production decision is to be taken unanimously. In case of any disagreement, farmers will use their land to produce t. The individual preference relations are as follows:

Table 1: (Preferences)

1	2	3
x_1	y_2	z_3
y_1	z_2	x_3
z_1	x_2	y_3
t_1	t_2	t_3

In this context, there are only four feasible allocations. Which of the above allocations are Pareto optimum? Which allocations are unblockable allocations? That is, which allocations belong to the Core? Here the Core is non-empty.

Let's change the rules of the game. Suppose, the production decision is to be taken unanimously. However, in case of partial disagreement the majority will prevail - in case of total disagreement farmers will use their land to produce t. Now, which of the above allocations are Pareto optimum? What is the Core?

5 Barter Vs Market

5.1 Informational and logistical requirements

- Barter requires
  - Search costs - you need to identify suitable trading partners.
  - Successful negotiations - after identification of suitable trading partners, you need to negotiate a mutually beneficial and acceptable deal. In Example 1: What is the number of mutually beneficial deals? Are the parties indifferent among the mutually beneficial deals?
- Market requires
  - No search costs - buyers and sellers can go to market to carry out transactions.
  - No cooperation or negotiations required - only decision making at individual level, given the prices of goods.

5.2 Relative Efficiency

- Under Barter, Pareto efficient outcome is unlikely, for large set of individuals. In that case, it will be very difficult to find suitable partners for mutually beneficial exchange of endowments. Even after partners have been identified, some of the negotiations are likely to fail. In real world, several factors can frustrate successful bargaining among individuals. Therefore, bargaining among individuals will fail to lead to one of the Pareto efficient and therefore Core allocations.
- Under Market, Pareto efficient outcome is more likely, especially for large set of individuals. Under market, there is no need to search and negotiate with other individuals. Also, when there are large number of buyers and sellers, they will be price-takers.

Later in the course we will show that under some conditions, the market outcome is Pareto efficient. The market can match Barter and do even better. These claims are valid with or without production. We will discuss the conditions under which these claims will be true.

5.3 Effect of Policy Interventions

Under Barter:

- Policy intervention only through reallocation of endowments

Under Market:

- Policy intervention through reallocation of endowments as well as direct transfers of 'purchasing power'

Remark: Neither Barter nor Market can guarantee the intended outcome. Further, some endowments are not transferable - E.g. human capital.

Also,

- The superiority of markets over Barter holds only for private goods.
- In case of Common Property Resources, the opposite can hold.

September 17

…that we have a uniq… …re strategy equilibrium in case of $p =$ …

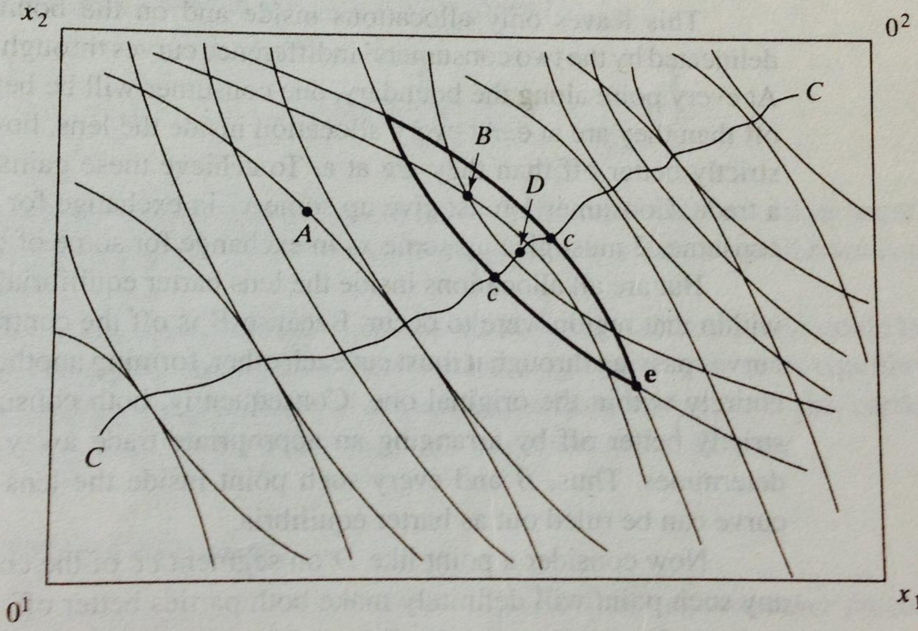

Figure 5.2. Equilibrium in two-person exchange.