Output Dynamics In the Long Run: Issues of Economic Growth

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Lecture Notes, DSE

Feb 23- Mar 9; 2018

Output Dynamics: Short Run to Long Run

- In the first part of the course, we have seen how output and employment are determined in the short run.
- In the Classical system (and its various extensions), these are determined by the supply side factors (production conditions); role of demand is limited to the detemination of the equilibrium price level.
- In the Keynesian system (and its various extensions), aggregate demand plays a direct role in determining equilibrium output and employment in the short run.
- As discussed before, both these systems are based on aggregative behavioural equations.
- One could provide micro-founded justifications for some of these equations, but they may not necessarily be consistent with one another.

Output Dynamics: Short Run to Long Run (Contd.)

- Alternatively, one could build an internally consistent, dynamic general equilibrium (DGE) framework, where forward-looking agents make their optimal decisions, taking both current and future variables into account.
- But this would entail dynamic equations involving today's and tomorrow's consumption; today's and tomorrow's capital stock etc.
- In other words this would bring us directly into the realm of output dyamics over time, i.e., economic growth.
- In the present module, we are going to explore issues pertaining to the output dynamics over time.
- In particular, we shall examine how the period-by-period output dynamics generate a growth path for the economy and what happens to this growth trajectory in the long run:
    - Does the economy reach a long run equilibrium with a constant growth rate, or does growth fizzle out in the long run?

Definition of Long Run: Steady State vis-a-vis Balanced Growth Path

- Before we proceed further, it is important to define the concept of 'long run equilibrium' in the context of growth models.
- Long run equilibrium in a growth model is typically defined as a **balanced growth path**, where all endogeneous variables grow at some *constant* rate.
- This constant growth rate may differ from variable to variable.
- More importantly, this constant growth rate could even be zero for some variables.
- The latter case is typically identified as the **steady state** in the conventional dynamic analysis (which is a special case of a balanced growth path).

Output Dynamics: Short Run to Long Run (Contd.)

- Must growth dynamics be necessarily based on the DGE framework?
- In other words, can the aggregative behavioural equations of the Keynesian or the Classical system also throw up some growth trajectories for the economy?
  - If yes, how would they differ from the growth trajectories predicted by the DGE framework?
- Indeed, one can develop growth models based on the short run (static) characterization of the macroeconomy in either the Keynesian system or the Classical system.
- And the long run charateristics of these growth models may differ substantially from that of the DGE framework.
- We start our discussion by analysing the dynamic versions of one of these aggregative models - namely the Classical one.

Dynamic Extension of the Classical System: the Neoclassical Growth Model

- The reason why we focus on the dynamic extension of the Classical sytem only and not the Keynesian one is because we want to discuss **long run** growth, not short/medium run business fluctuations.
- It is generally believed that in the long run, it is the supply side factors which are crucial in determining the output dynamics, not the demand side factors.
- We are also going to abstract away from price dynamics and indeed abstract from all nominal variables, including money itself.
- Moreover, unless stated otherwise, we shall generally assume that government is passive in the sense that it does not intervene in the functioning of the market economy via fiscal policies.

Neoclassical Growth Model: Solow

- Solow (QJE, 1956) extends the static Classical system to a dynamic 'growth' framework where both factors of production - capital and labour - grow over time.
- Capital stock grows due to the savings-cum-investment decisions of the households (all capital is owned by the households; there is no independent investment function coming from the firms).
- Labour supply grows due to the population growth (at an exogenous rate n).
- Growth of these two factors generates a growth rate of output in this economy.

Neoclassical Growth Model: Solow (Contd.)

- The questions that we are interested in are as follows:
    - What is the rate of growth of aggregate as well as per capita output in this economy?
    - Does the economy attain a balanced growth path in the long run?
    - What is the welfare level attained by the households in the long run?
    - Is there any role of the government in the growth process - either in terms of augmenting growth or in terms of ensuring maximum welfare?

Solow Growth Model: The Economic Environment

- Consider a closed economy producing a single final commodity which is used for consumption as well as for investment purposes (i.e, as capital.)
- At the beginning of any time period t, the economy starts with a given total endowment of labour (N_t) and a given aggregate capital stock (K_t).
- There are H identical households in the economy and the labour and capital ownership is equally distributed across all these households.
- At the beginning of any time period t, the households offer their labour and capital (inelastically) to the firms.
- The competitive firms then carry out the production and distribute the total output produced as factor incomes to the households at the end of the period
- The households consume a constant fraction of their total income and save the rest.
- The savings propensity is exogenously fixed, denoted by $s \in (0, 1)$.

Solow Growth Model: (Contd.)

- **A Crucial Assumption:** All savings are automatically invested, which augments the capital stock in the next period.
- As we had noted earlier, this asumption implies that:
  - it is the households who make the investment decisions; not firms.
  - Firms simply rent in the capital from the households for production and distributes the output as wage and rental income to the households at the end of the period.
- In other words, this assumption rules out the existence of an independent investment function - as we had assumed earlier in the static Neoclassical Model (recall the IS relationship)!
- In fact, rate of retun of capital (r_t) is determined here by the market clearing condition in the factor market; not in the goods market.
- Indeed in Solow, savings is always equal to investment (by assumption); hence the goods market is always in equilibrium.

Solow Growth Model: Production Side Story

- The economy is characterized by S identical firms.
- Since all firms are identical, we can talk in terms of a representative firm.
- The representative firm i is endowed with a standard '**Neoclassical**' production technology

$$Y_{it} = F(N_{it}, K_{it})$$

which satisfies all the standard properties e.g., diminishing marginal product of each factor (or law of diminishing returns), CRS and the Inada conditions.
- In addition, $F(0, K_{it}) = F(N_{it}, 0) = 0$, i.e., both inputs are essential in the production process.
- The firms operate in a competitive market structure and take the market wage rate (w_t) and rental rate for capital (r_t) in real terms as given.
- The firm maximises its current profit: $\Pi_{it} = F(N_{it}, K_{it}) - wN_{it} - rK_{it}$.

Production Side Story (Contd.)

- Static (period by period) optimization by the firm yields the following FONCs:

$$\begin{aligned} \text{(i) } F_N(N_{it}, K_{it}) &= w. \\ \text{(ii) } F_K(N_{it}, K_{it}) &= r. \end{aligned}$$

- Recall (from our previous analysis) that identical firms and CRS technology imply that firm-specific marginal products and economy-wide (social) marginal products (derived from the corresponding aggregate production function) of both labour and capital would be the same. Thus

$$\begin{aligned} F_N(N_{it}, K_{it}) &= F_N(N_t, K_t); \\ F_K(N_{it}, K_{it}) &= F_K(N_t, K_t). \end{aligned}$$

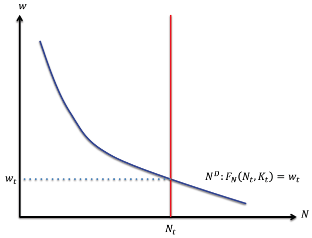
Production Side Story (Contd.)

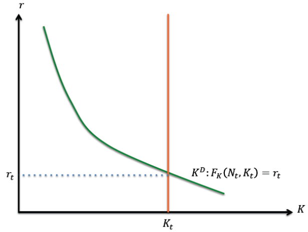
- Thus we get the familiar demand for labour schedule for the aggreagte economy at time t, and a similarly defined demand for capital schedule at time t as:

$$\begin{aligned} N^D &: \quad F_N(N_t, K_t) = w_t; \\ K^D &: \quad F_K(N_t, K_t) = r_t. \end{aligned}$$

- Recall that the supply of labour and that of capital at any point of time t is historically given at N_t and K_t respectively.
- **Assumption:** The market wage rate and the rental rate for capital, w_t and r_t, are fully flexible and they adjust so that the labour market and the capital market clear in every time period.

Determination of Market Wage Rate & Rental Rate of Capital at time t:





Distribution of Aggregate Output:

- Recall that the firm-specific production function is CRS; hence so is the aggregate production function.
- We know that for any constant returns to scale (i.e., linearly homogeneous) function, by Euler's theorem:

$$\begin{aligned} F(N_t, K_t) &= F_N(N_t, K_t)N_t + F_K(N_t, K_t)K_t \\ &= w_t N_t + r_t K_t. \end{aligned}$$

- This implies that after paying all the factors their respective marginal products, the entire output gets exhausted, confirming that firms indeed earn zero profit.
- Thus the total output currently produced goes to the households as income (Y_t) - of which they consume a fixed proportion and save the rest.

Dynamics of Capital and Labour:

- Recall that the capital stock over time gets augmented by the savings/investment made by the households.
- Also recall that households are identical and they invest a fixed proportion (s) of their income (which adds up to the aggregate output - as we have just seen).
- Hence aggegate savings (& investment) in the economy is given by :

$$S_t \equiv I_t = sY_t; \; 0 < s < 1.$$

- Let the existing capital depreciate at a constant rate $\delta : 0 \leqq \delta \leqq 1$.
- Thus the capital accumulation equation in this economy is given by:

$$\begin{aligned} K_{t+1} &= I_t + (1-\delta)K_t = sY_t + (1-\delta)K_t \\ \text{i.e., } K_{t+1} &= sF(N_t, K_t) + (1-\delta)K_t, \end{aligned} \tag{1}$$

- While labour stock increases due to population growth (at a constant rate n):

$$N_{t+1} = (1+n)N_t. \tag{2}$$

Dynamics of Capital and Labour (Contd):

- Equations (1) and (2) represent a 2×2 system of difference equations, which we can directly analyse to determine the time paths of N_t and K_t, and therefore the corresponding dynamics of Y_t.
- However, given the properties of the production function, we can transform the 2×2 system into a single-variable difference equation - which is easier to analyse.
- We shall follow the latter method here.

Capital-Labour Ratio & Per Capita Production Function:

- Using the CRS property, we can write:

$$y_t \equiv \frac{Y_t}{N_t} = \frac{F(N_t, K_t)}{N_t} = F\left(1, \frac{K_t}{N_t}\right) \equiv f(k_t),$$

  where y_t represents per capita output, and k_t represents the capital-labour ratio (or the per capita capital stock) in the economy at time t.

- The function $f(k_t)$ is often referred to as the *per capita* production function.

- Notice that using the relationship that $F(N_t, K_t) = N_t f(k_t)$, we can easily show that:

$$\begin{aligned} F_N(N_t, K_t) &= f(k_t) - k_t f'(k_t); \\ F_K(N_t, K_t) &= f'(k_t). \end{aligned}$$

  [Derive these two expressions yourselves].

Properties of Per Capita Production Function:

- Given the properties of the aggregate production function, one can derive the following properties of the per capita production function:

$$
\begin{aligned}
\text{(i) } f(0) &= 0; \\
\text{(ii) } f'(k) &> 0; \;\; f''(k) < 0; \\
\text{(iii) } \underset{k \to 0}{Lim} f'(k) &= \infty; \;\; \underset{k \to \infty}{Lim} f'(k) = 0.
\end{aligned}
$$

- Condition (i) indicates that capital is an essential input of production;
- Condition (ii) indicates diminishing marginal product of capital;
- Condition (iii) indicates the Inada conditions with respect to capital .
- Finally, using the definition that $k_t \equiv \frac{K_t}{N_t}$, we can write

$$
\begin{aligned}
k_{t+1} &\equiv \frac{K_{t+1}}{N_{t+1}} = \frac{sF(N_t, K_t) + (1-\delta)K_t}{(1+n)N_t} \\
&\Rightarrow \;\; k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{(1+n)} \equiv g(k_t). \qquad (3)
\end{aligned}
$$

Dynamics of Capital-Labour Ratio:

- Equation (3) represents the basic dynamic equation in the discrete time Solow model. (**Interpretation?**)
- Notice that Equation (3) represents a single non-linear difference equation in k_t. Once again we use the phase diagram technique to analyse the dynamic behaviour of k_t.
- Recall that to draw the phase diagram of a single difference equation, we first plot the $g(k_t)$ function with respect to k_t. Then we identify its possible points of intersection with the 45° line - which denote the steady state points of the system.
- In plotting the $g(k_t)$ function, note:

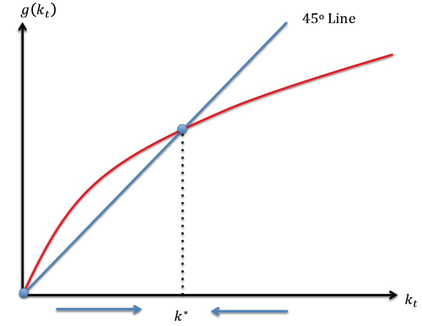
$$
\begin{aligned}
g(0) &= \frac{sf(0) + (1-\delta).0}{(1+n)} = 0; \\
g'(k) &= \frac{1}{(1+n)}\left[sf'(k) + (1-\delta)\right] > 0; \\
g''(k) &= \frac{1}{(1+n)} sf''(k) < 0.
\end{aligned}
$$

Dynamics of Capital-Labour Ratio (Contd.):

- Moreover,

$$
\begin{aligned}
\underset{k\to 0}{Lim} g'(k) &= \frac{1}{(1+n)}\left[s \underset{k\to 0}{Lim} f'(k) + (1-\delta)\right] = \infty; \\
\underset{k\to \infty}{Lim} g'(k) &= \frac{1}{(1+n)}\left[s \underset{k\to \infty}{Lim} f'(k) + (1-\delta)\right] = \frac{(1-\delta)}{(1+n)} < 1.
\end{aligned}
$$

- We can now draw the phase diagram for k_t :



Dynamics of Capital-Labour Ratio (Contd.):

- From the phase diagram we can identify two possible steady states:

$$\begin{aligned} \text{(i) } k &= 0 \text{ (Trivial Steady State);} \\ \text{(ii) } k &= k^* > 0 \text{ (Non-trivial Steady State).} \end{aligned}$$

- Since an economy is always assumed to start with some positive capital-labour ratio (however small), we shall ignore the non-trivial steady state.
- The economy has a unique non-trivial steady state, given by k^*, which is globally asymptotically stable: starting from any initial capital-labour ratio $k_0 > 0$, the economy would always move to k^* in the long run.
- Implications:
  - In the long run, per capita output: $y_t \equiv f(k_t)$ will be constant at $f(k^*)$ while aggregate output $Y_t \equiv N_t f(k_t)$ will be growing at the same rate as N_t (namely at the exogenously given rate n).

Verification of Stability Property: Linearization

- Let us now veify the stability property of the non-trivial steady state (k^*) via the linearization technique. (Notice though that the lineraization technique will only tell us about the local stability; not global stability).
- Recall that the difference equation is given by:

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{(1+n)} \equiv g(k_t)$$

- Linearizing around the non-trivial steady state:

$$\begin{aligned} k_{t+1} &= g(k^*) + g'(k^*)(k_t - k^*) \\ &= g'(k^*)k_t + [g(k^*) - g'(k^*)k^*] \end{aligned}$$

- The non-trivial steady state would be locally stable iff $g'(k^*) = \left[\frac{sf'(k^*) + (1-\delta)}{(1+n)}\right] < 1.$

Verification of Stability Property: Linearization (Contd.)

- To see how the linearization works, let us take a specific production function of the Cobb-Douglas variety:

$$f(k) = k^{\alpha};\ 0 < \alpha < 1.$$

- Then

$$g(k_t) = \frac{s(k_t)^{\alpha} + (1-\delta)k_t}{(1+n)}$$

- And the nontrivial steady state is:

$$k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- Verify that at $k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$, $g'(k^*) = \left[\frac{s\alpha(k^*)^{\alpha-1} + (1-\delta)}{(1+n)}\right] < 1.$

Some Long Run Implictions of Solow Growth Model:

- Notice that the non-trivial steady state k^* can be written as:

$$\begin{aligned} k^* &= \frac{sf(k^*) + (1-\delta)k^*}{(1+n)} \\ \Rightarrow & \quad (1+n)\,k^* = sf(k^*) + (1-\delta)k^* \\ \Rightarrow & \quad \frac{f(k^*)}{k^*} = \frac{n+\delta}{s}. \end{aligned}$$

- Total differentiating and using the properties of the $f(k)$ function, it is easy to show that,

$$\frac{dk^*}{ds} > 0; \frac{dk^*}{dn} < 0; \frac{dk^*}{d\delta} < 0.$$

  - A higher savings ratio generates a *higher* **level** of per capita output in the long run;
  - A higher rate of grwoth of population generates a *lower* **level** of per capita output in the long run;
  - A higher rate of depreciation generates a *lower* **level** of per capita output in the long run.

Some Long Run Implictions of Solow Growth Model (Contd.):

- But these are all **level** effects. What would be the impact on the long run **growth**?
- It is easy to see that in this Solovian economy the per capita income does not grow in the long run ($f(k^*)$ is a constant).
- On the other hand, **the long run growth rate of aggregate income in the Solovian economy is always equal to** n (and is independent of other parameters, e.g., s or δ)
- Notice there is no role for the government in influencing the long run growth rate here. In particular, if the government tries to manipulate the savings ratio (by imposing an appropriate tax on households' income and investing the tax revenue in capital formation), then such a policy will have no long run **growth** effect.

Solow Model: Long Run vis-a-vis Short/Medium Run

- To summarise:
  - The per capita income does not grow in the long run; it remains constatnt at $f(k^*)$ - the exact level being determined by various parameters (s, n, δ).
  - The aggregate income grows at a constant rate - given by the exogenous rate of growth of population (n).
- But all these happen only in the long run, i.e., as $t \longrightarrow \infty$.
- Starting from a given initial capital-labour ratio k_0 $(\neq k^*)$, it will obviously take the economy some time before it reaches k^*.
- What happens during these transitional periods?
- In particular, what would be the rate of growth of per capita income and that of aggregate income in the short run - when the economy is yet to reach its steady state?

Transitional Dynamics in Solow Growth Model:

- When the economy is out of steady state, the rate of growth of capital-labour ratio is given by:

$$\gamma_k \equiv \frac{k_{t+1}-k_t}{k_t} = \frac{\frac{sf(k_t)+(1-\delta)k_t}{(1+n)} - k_t}{k_t}$$
$$= \frac{sf(k_t)-(n+\delta)k_t}{(1+n)k_t} \gtreqless 0 \text{ according as } k_t \lesseqgtr k^*.$$

- Moreover,

$$\frac{d\gamma_k}{dk} = \frac{-s(1+n)\left[f(k)-kf'(k)\right]}{\left[(1+n)k\right]^2} < 0.$$

- In other words, during transition, the higher is the capital-labour ratio of the economy, the lower is its (short run) growth rate.
- This last result has important implications for cross country growth comparisons.

Further Implictions of Solow Growth Model: Absolute vis-a-vis Conditional Convergence

- The above result implies that the transitional growth rate of per capita income in the poorer countries (with low k_0) will be higher than that of the rich countries (with high k_0); and eventually they will converge to the same level of per capita income (**Absolute Convergence**).
- This proposition however has been strongly rejected by data. In fact empirical studies show the opposite: richers countries have remained rich and poorer countries have remained poor and there is no significant tendencies towards convergence - even when one looks at long run time series data.
- The proposition of absolute convergence of course pre-supposes that the underlying parameters for all economies (rich and poor alike) are the same.

Absolute vis-a-vis Conditional Convergence (Contd.)

- If we allow rich and poor countries to have different values of s, δ, n etc. (which is plausible), then the Solow model generates a much weaker prediction of **Conditional Convergence.**
- Conditional Convergence states that a country grows faster - the further away it is from its own steady state.
- An alternative (and more useful) statement of Conditional Convergence runs as follows: Among a group of countries which are similar (similar values of s, δ, n etc.), the relatively poorer ones will grow faster and eventually the per capita income of all these countries will converge.
- This weaker hypothesis in generally supported by data.
- However, Conditional Convergence Hypothesis is not very helpful in explaining the persistent differences in per capita income amongst the rich and the poor countries.

Solow Model: Golden Rule & Dynamic Inefficiency

- Let us now get back from short run to long run (steady state).
- Recall that for given values of δ and n, the savings rate in the economy uniquely pins down the corresponding steady state capital-labour ratio:

$$k^*(s) : \frac{f(k^*)}{k^*} = \frac{n+\delta}{s}.$$

- We have already seen that a higher value of s is associated with a higher k^*, and therefore, a higher **level** of steady state per capita income $(f(k^*))$.
- So it seems that a higher savings ratio - though has no growth effect - may still be welcome because it generates higher standard of living (as reflected by a higher per capita income) at the steady state.
- But welfare of agents do not depend on just income; it depends on the level of consumption. So what is the corresponding level of consumption associated with the steady state $k^*(s)$?

Solow Model: Golden Rule & Dynamic Inefficiency (Contd.)

- Notice that in this model, per capita consumption is defined as:

$$\begin{aligned} \frac{C_t}{N_t} &\equiv \frac{Y_t - S_t}{N_t} \\ &\Rightarrow \quad c_t = f(k_t) - sf(k_t) \end{aligned}$$

- Accordingly, for given values of δ and n, steady state level of per capita consumption is related to the savings ratio of the economy in the following way:

$$\begin{aligned} c^*(s) &= f(k^*(s)) - sf(k^*(s)) \\ &= f(k^*(s)) - (n+\delta)\, k^*(s). \quad \text{[Using the definition of } k^*\text{]} \end{aligned}$$

- We have already noted that if the government tries to manipulate the savings ratio (by imposing an appropriate tax on household's income and using the tax proceeds for capital formation), then such a policy would have no long run **growth** effect.

Golden Rule & Dynamic Inefficiency in Solow Model (Contd.)

- But can such a policy still generate a higher **level** of steady state per capita consumption at least?
- If yes, then such a policy would still be desirable, even if it does not impact on growth.
- Taking derivative of $c^*(s)$ with respect to s :

$$\frac{dc^*(s)}{ds} = \left[f'(k^*(s)) - (n + \delta) \right] \frac{dk^*(s)}{ds}.$$

- Since $\dfrac{dk^*}{ds} > 0$,

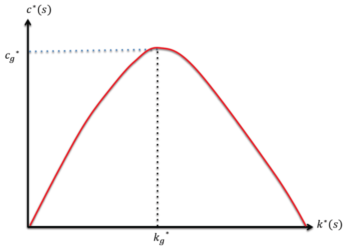
$$\frac{dc^*(s)}{ds} \gtreqless 0 \ \text{ according as } \ f'(k^*(s)) \gtreqless (n + \delta).$$

- In other words, steady state value of per capita consumption, $c^*(s)$, is maximised at that level of savings ratio and associated $k^*(s)$ where

$$f'(k^*) = (n + \delta).$$

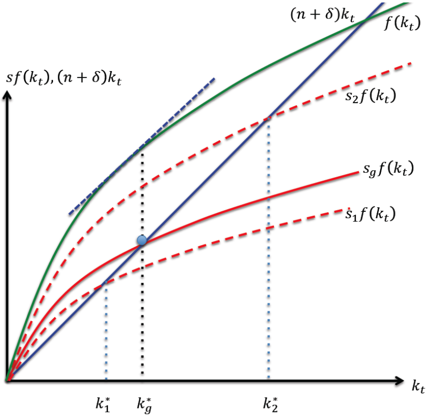
Digrammatic Representation of the Golden Rule Steady State:

- We shall denote this savings ratio as s_g and the corresponding **steady state** capital-labour ratio as k_g^* - where the subscript 'g' stand for **golden rule.**



- The point (k_g^*, c_g^*) in some sense represents the 'best' or the 'most desirable' steady state point (although in the absence of an explicit utility function, such qualifications remain somewhat vague).

Alternative Digrammatic Representation of the Golden Rule Steady State:



- There are many possible steady states to the left and to the right of k_g^* - associated with various other savings ratios.

Golden Rule & the Concept of 'Dynamic Inefficiency'

- Importantly, all the steady states to the **right** of k_g^* are called '**dynamically inefficient**' steady states.
- From any such point one can '**costlessly**' move to the left - to a lower steady state point - and in the process enjoy a higher level of current consumption as well as higher levels of future consumption at all subsequent dates. (**How?**)
- Notice however that the steady states to the **left** of k_g^* are **not** 'dynamically inefficient'. (**Why not?**)

Cause of 'Dynamic Inefficiency' in Solow Model

- Dynamic inefficiency occurs because people oversave.
- This possibility arises in the Solow model because **the savings ratio is exogenously given - it is not chosen through households' optimization process.**
- Note that if indeed the steady state of the economy happens to lie in the dynamically inefficient region, then that in itself would justify a pro-active, interventionist role of the government in the Solow model - even though government cannot influence the long run rate of growth of the economy.

Limitations of the Solow Growth Model:

- There are three major criticisms of the Solow model.

1. It does not take into account the demand side of the economy. In fact the model *assumes* that demand is always equal to supply. (In fact there exists some post-Keynesian demand-led growth models which address this issue. However, due to paucity of time, we shall not consider those 'Keynesian' growth models in this course).
2. The steady state in the Solow model might be dynamically inefficient, because people may oversave. If one allows households to choose their savings ratio optimally, then this inefficinency should disappear. But this latter possibility is simply not allowed in the Solow model.
3. Even though the Solow model is supposed to be a growth model - it cannot really explain long run growth:

- - The per capita income does not grow at all in the long run;
  - The aggregate income grows at an exogenously given rate n, which the model does not attempt to explain.

Extensions of the Solow Growth Model:

- The basic Solow growth model has subsequently been extended to counter some of these critisisms.
- We shall look at two such extensions:
  1. Neoclassical Growth Model with Optimizing Households: This extension allows the households to choose their consumption/savings behaviour optimally. There are two different characterization of the optimizing household model:
     - Households optimize over infinite horizon (The Ramsey-Cass-Koopmans Framework)
     - Households optimize over finite horizon in an overlapping generations set up (The Samuelson-Diamond Framework).
  2. Growth Models with non-Neoclassical Technology: This extension allows the per capita income to grow in the long run.In this context we shall also discuss why the aggregate production technology may **not** exhibit all the Neoclassical properties.

Neoclassical Growth with Optimizing Agents:

- Let us start with first extension which allows for optimizing consumption/savings behaviour by households over infinite horizon: **The Ramsey-Cass-Koopmans Inifinite Horizon Framework** (henceforth **R-C-K**)
- This framework retains all the production side assuption of the Solovian economy; but households now choose their consumption and savings decision optimally be maximising their utility defined over an infinite horizon.
- This latter statement should immediately tell you that the underlying macro structure would be very similar to the DGE framework that we have constructed earlier.
- Let us revisit the underlying macro framework.

Neoclassical Growth with Optimizing Agents: The R-C-K Model

- The R-C-K model is considered Neoclassical - because it retains **all** the assumptions of the Neoclassical production function (including the diminishing returns property and the Inada conditions.)
- In fact the production side story is exactly identical to Solow.
- As before, the economy starts with a given stock of capital (K_t) and a given level of population (L_t) at time t.
- These factors are supplied **inelastically** to the market in every period. This implies that households do not care for leisure.
- Population grows at a constant rate n.
- Capital stocks grows due to optimal savings (and investment) decisions by the households.
- Notice that once again savings and investment are always identical. So just like Solow, this is a supply driven growth model.

The R-C-K Model: The Household Side Story

- There are H **identical** households indexed by h.
- Each household consists of a single **infinitely lived** member to begin with (at $t = 0$). However **population within a household increases over time at a constant rate** n. (And each newly born member is infinitely lived too!)
- At any point of time t, the total capital stock and the total labour force in the economy are equally distributed across all the households, which they offer inelastically to the market at the market wage rate w_t and the market rental rate r_t.
- Thus total earning of a household at time t: $w_t N_t^h + r_t K_t^h$.
- Corresponding **per member** earning: $y_t^h = w_t + r_t k_t^h$,

  where k_t^h is the **per member capital stock** in household h, which is also the **per capita capital stock** (or the capital-labour ratio, k_t) in the economy.

The Household Side Story (Contd.):

- In every time period, the instantaneous utility of the household depends on its **per member** consumption:

$$u_t = u\left(c_t^h\right); \ u' > 0; \ u'' < 0; \ \lim_{c^h \to 0} u'(c^h) = \infty; \ \lim_{c^h \to \infty} u'(c^h) = 0.$$

- The household at time 0 chooses its entire consumption profile $\left\{c_t^h\right\}_{t=0}^{\infty}$ so as to maximise the discounted sum of its life-time utility:

$$U_0^h = \sum_{t=0}^{\infty} \beta^t u\left(c_t^h\right)$$

  subject to its period by period budget constraint.

- Notice once again that identical households implied that **per member** consumption (c_t^h) of any household is also equal to the per capita consumption (c_t) in the economy at time t.

The R-C-K Model: Centralized Version (Optimal Growth)

- There are two version of the R-C-K model:
  - A centralized version - which analyses the problem from the perspective of a social planner (who is **omniscient**, **omnipotent** and **benevolent**).
  - A decentralized version - which analyses the problem from the perspective of a perfectly competitive market economy where 'atomistic' households and firms take optimal decisions in their respective individual spheres.
- The centralized version was developed by Ramsey (way back in 1928) and is often referred to as the 'optimal growth' problem.
- In the DGE framework (discussed in Module 1) we have solved the problem both from the perspective of the social planner as well as for the perfectly competitive market economy.
- We have shown that **under rational expectations and perfect information on the part of the households**, the solution paths for the economy under the two institutional structures are identical.
- We shall therefore characterise the solution paths for only one of them, namely that of the social planner.

- From our analysis of the DGE frameowrk, we know that the dynamic optimization problem of the social planner is:

$$\underset{\{c_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}}{Max.} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

subject to

$$\begin{aligned} \text{(i) } c_t &\leqq f(k_t) + (1-\delta)k_t \text{ for all } t \geqq 0; \\ \text{(ii) } k_{t+1} &= \frac{f(k_t) + (1-\delta)k_t - c_t}{1+n};\ k_t \geqq 0 \text{ for all } t \geqq 0;\ k_0 \text{ given.} \end{aligned}$$

R-C-K Model: Centralized Version (Contd.)

- In Module 1, we have seen how to solve a dynamic programming problem (using the Bellman equation) to arrive at the dynamic equations characterizing the optimal paths for the control and state variable.
- Using this method, we can derive the following two dynamic equations characterizing the solution to the social planner's optimization problem as:

$$u'(c_t)[1+n] = \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right]. \tag{4}$$

$$k_{t+1} = \frac{f(k_t) + (1-\delta)k_t - c_t}{1+n}; \;\; k_0 \text{ given.} \tag{5}$$

- These two equations represent a $2X2$ system of difference equations which implicitly defines the 'optimal' trajectories of c_t and k_t.
- Of course, we still need two boundary conditons to precisely characterise the solution paths for this $2X2$ system.

R-C-K Model: Centralized Version (Contd.)

- One boundary condition is given by the initial condition: k_0.
- The other boundary condition is provided by the Transversality condition:
$$\lim_{t\to\infty} \beta^t \left[f'(k_t) + (1-\delta) \right] .u'(c_t) k_t = 0$$
- However, the dynamic equations given above are very involved and chracterizing the optimal path is not very easy.
- To simplify, let us begin by assuming specific functional forms for $u(c)$ and $f(k)$.
- Let
$$u(c_t) = \log c_t;$$
$$f(k_t) = (k_t)^\alpha;\ 0 < \alpha < 1.$$

R-C-K Model (Centralized Version): Optimal Paths (Contd.)

- Given these specific functional forms, the dynamic equations for the centralized R-C-K model are represented by the following two equations:

$$
\begin{aligned}
c_{t+1} &= \frac{\beta\left[\alpha(k_{t+1})^{\alpha-1} + (1-\delta)\right]}{1+n} c_t; \qquad (6) \\
k_{t+1} &= \frac{(k_t)^{\alpha} + (1-\delta)k_t - c_t}{1+n}. \qquad (7)
\end{aligned}
$$

- The associated boundary conditions are:

$$
k_0 \text{ given; } \lim_{t\to\infty} \beta^t \left[\alpha(k_t)^{\alpha-1} + (1-\delta)\right] . \frac{k_t}{c_t} = 0
$$

Characterization of the Optimal Paths:

- We are now all set to characterise the dynamic paths of c_t and k_t as charted out by the above dynamic system.
- Before that let us quickly characterize the steady state.
- Using the steady state condition that $c_t = c_{t+1} = c^*$ and $k_t = k_{t+1} = k^*$ in equations (6) and (7), the steady state values are given by:

$$
\begin{aligned}
k^* &= \left[\frac{\alpha\beta}{1+n-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}; \\
c^* &= (k^*)^{\alpha} - (n+\delta)k^*.
\end{aligned}
$$

- Now let us chart out the optimal paths of c_t and k_t - starting from any given initial value of k_0. (Note that the initial value of c_0 is **not** given - it is to be optimally determined).
- We now construct the phase diagram to qualitatively characterise the solution paths for these specific functional forms.

R-C-K Model (Centralized Version): Phase Diagram

- To construct the phase diagram for this 2×2 system we have to plot the two level curves $\triangle c = 0$ and $\triangle k = 0$ in the (k_t, c_t) plane.
- The equations of these two level curves are as follows:

$$c_{t+1} - c_t \equiv \triangle c = \left[\frac{\beta \left[\alpha (k_{t+1})^{\alpha - 1} + (1 - \delta) \right]}{1 + n} - 1 \right] c_t = 0; \quad (8)$$

$$k_{t+1} - k_t \equiv \triangle k = \frac{(k_t)^{\alpha} - (n + \delta) k_t - c_t}{1 + n} = 0. \quad (9)$$

- From (8):

$$\triangle c = 0 \Rightarrow \text{either } \left[\frac{\beta \left[\alpha (k_{t+1})^{\alpha - 1} + (1 - \delta) \right]}{1 + n} - 1 \right] = 0;$$

$$\text{i.e., either } k_{t+1} = \left(\frac{\alpha \beta}{(1 + n) - \beta (1 - \delta)} \right)^{\frac{1}{1-\alpha}} = k^* = k_t; \quad \text{or } c_t = 0$$

R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- From (9):

$$\triangle k = 0 \Rightarrow c_t = (k_t)^{\alpha} - (n+\delta)k_t$$

- Thus along the $\triangle k = 0$ locus:

$$\left.\frac{dc_t}{dk_t}\right|_{\triangle k=0} \gtreqless 0 \text{ according as } \alpha(k_t)^{\alpha-1} \gtreqless (n+\delta)$$

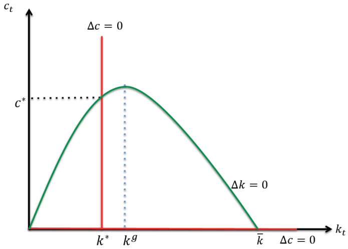
$$\text{i.e., } k_t \lesseqgtr \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}} \equiv k_g$$

$$\text{Also, when } k_t = 0,\ c_t = 0;\ \&\ \text{when } k_t \geqq \left(\frac{1}{n+\delta}\right)^{\frac{1}{1-\alpha}} \equiv \bar{k},\ c_t = 0.$$

- A Comment: Earlier (in the context of the Solow model), we had defined the 'golden rule' capital-labour ratio as $k_g : f'(k) = n + \delta$. Now with the Cobb-Douglas production function, $f'(k) = \alpha(k_t)^{\alpha-1}$. Therefore 'golden rule' is suitably defined as: $k_g = \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}$.

R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- All these information can be put together in constructing the following phase diagram:



- This phase diagram is not complete yet. We have to put in the direction of movements of c and k, which we do next.

R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- In order to determine the direction of movements of k and c, we proceed in the following way:
  - First consider the movement of c which is captured by the $\triangle c$ function, as given by

$$\triangle c = \left[\frac{\beta \left[\alpha (k_{t+1})^{\alpha - 1} + (1 - \delta) \right]}{1 + n} - 1 \right] c_t$$

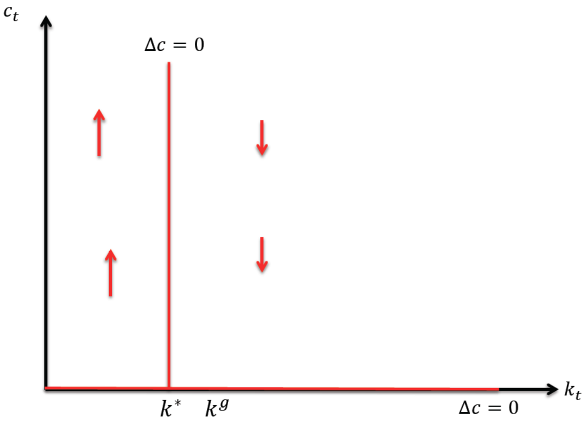
  - It is easy to see that for any $c_t > 0$,

$$\begin{aligned} \triangle c &\gtreqless 0 \\ \text{according as } \frac{\beta \left[\alpha (k_{t+1})^{\alpha - 1} + (1 - \delta) \right]}{1 + n} - 1 &\gtreqless 0 \end{aligned}$$

$$\Rightarrow k_{t+1} \lesseqgtr \left(\frac{\alpha \beta}{(1 + n) - \beta (1 - \delta)} \right)^{\frac{1}{1 - \alpha}} = k^*$$

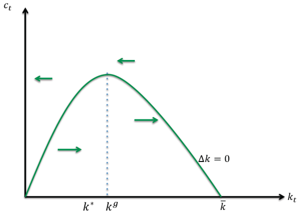
R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- Accordingly, the movement of c is shown in the diagram below:



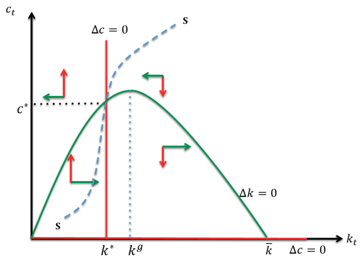
R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- Next consider the movement of k.
  - The movement of k is captured by the $\triangle k$ function, as given by
$$\triangle k = \frac{(k_t)^{\alpha} - (n+\delta)k_t - c_t}{1+n}$$
  - It is easy to see that
$$\begin{aligned} \triangle k &\gtreqqless 0 \\ \text{according as } c_t &\lesseqqgtr (k_t)^{\alpha} - (n+\delta)k_t \end{aligned}$$
  - Accordingly, the movement of c is shown in the diagram below:



R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- Combining the two, we get the complete phase diagram for this dynamic system as follows:



- There exists a unique path (denoted by SS) which will take us to the non-trivial steady state. This is indeed the optimal path. All other paths violate the TVC.

R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- Although we have drawn the phase diagram here for a specific example where the instantaneous utility function is logarithmic, the diagrammatic analysis will go through for any other utility function which belongs to the CRRA family, i.e.,

$$u(c_t) = \frac{(c_t)^{1-\sigma}}{1-\sigma}; \quad \sigma \neq 1.$$

(Verify this yourself)

R-C-K Model (Centralized Version): Growth Conclusions

- In the R-C-K model, for any given k_0, the optimal c_0 will be chosen such that the economy is on the SS path.
- Along this path, the average capital stock and average consumption approach their steady state values $(k^*.c^*)$ in the long run (i.e, as $t \longrightarrow \infty$).
- Thus the long run growth conclusions of the Solow model prevails:
    - The per capita income does not grow in the long run; it remains constatnt at $f(k^*)$ - the exact level being determined by various parameters (s, n, δ).
    - The aggregate income grows at a constant rate - given by the exogenous rate of growth of population (n).
- BUT, is the steady state now dynamically efficient? For that we have to compare this steady state with the 'golden rule'.

R-C-K Model (Centralized Version): Dynamic Efficiency

- It is easy to verify that for our specific example (with a Cobb-Douglas production function):

$$k^* = \left(\frac{\alpha\beta}{(1+n) - \beta(1-\delta)} \right)^{\frac{1}{1-\alpha}} < \left(\frac{\alpha}{n+\delta} \right)^{\frac{1}{1-\alpha}} = k_g$$

- But is it true in general, or is this an artifact of the Cobb-Douglas production function?
- To answer this question, let us go back to the generalized R-C-K model and examine whether its steady state is dynamically efficient or not.

R-C-K Model (Centralized Version): Dynamic Efficiency (Contd.)

- Recall that the dynamic equations characterizing the optimal paths of the generalized R-C-K model were given by:

$$u'(c_t)[1+n] = \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right] \quad (10)$$

$$k_{t+1} = \frac{f(k_t) + (1-\delta)k_t - c_t}{1+n} \quad (11)$$

- We know that at steady state, by definition:

$$c_t = c_{t+1} = c^*;$$

$$k_t = k_{t+1} = k^*.$$

- Using this steady state definition in the dyamic equations, the steady state for this system is given by:

$$k^* \quad : \quad f'(k^*) = \frac{(1+n) - \beta(1-\delta)}{\beta}; \quad (12)$$

$$c^* \quad = \quad f(k^*) - (n+\delta)k^* \quad (13)$$

R-C-K Model (Centralized Version): Golden Rule & Dynamic Efficiency

- Recall that the 'golden rule' capital-labour ratio is defined as:

$$k_g : f'(k_g) = (n + \delta)$$

- Given that $0 < \delta, \beta < 1$, one can again easily verify that

$$\begin{aligned} f'(k^*) &= \frac{(1+n) - \beta(1-\delta)}{\beta} > (n+\delta) = f'(k_g) \\ &\Rightarrow \quad k^* < k_g. \end{aligned}$$

- In other words, the steady state of the R-C-K model necessarily lies in the dynamically efficient region.
- This is true for any n and δ, as long as $0 < \beta < 1$ (i.e., households prefer curren consumption to future consumption). Indeed it is only when $\beta = 1$ that household will optimally reach the golden rule such that:

$$k^* = k_g.$$

R-C-K Model (Centralized Version): Golden Rule & Dynamic Efficiency (Contd.)

- Thus we see that in the centralized version of the R-C-K model, when the social planner decides on how much to save and how much to leave for households' consumption in order to maximise households' utility over infinite horizon, it ensures that the corresponding steady state is **always dynamically efficient**.
- Finally, will all these results hold for a decentralized market economy as well?
- The answer is: "yes" - because (under rational expectation, complete information and no externality) the solution path for the social planner is exactly identical to the solution path for the competitive market economy.
- Thus the growth path of the decentralized market economy will be exactly identical to the growth path of the planned economy; moreover the steady state of the market economy will be dynamically efficient too.

Samuelson-Diamond Overlapping Generations Model

- We have so far analysed optimizing savings behaviour by the households in the context of the inifinite horizon R-C-K framework.
- We now turn to an alternative framework, where agents optimize over a finite time horizon.
- This framework was first developed by Samuelson (1958) in the context of an exchange economy, which was later extended to a production economy by Peter Diamond (1965).
- Each agent now lives exactly for T periods. For convenience, we shall assume that $T = 2$.
- We shall denote these two periods of an agent's life time by 'youth' and 'old-age' respectively.
- The agent works only in the first period of his life (when young) and is retired in the second period (when old).
- Thus he has to make provisions for his old-age consumption from his first period wage income itself (through savings).
- The agent optimally decides about his consumption profile by maximizing his life-time utility (to be specified later).

OLG Model: Production Side Story

- Once again, the production side story in the OLG model is identical to that of Solow.
- Thus the economy starts with a given stock of capital (K_t) and a given stock of labour force (N_t) at time t.
- Notice that since people do not work during their oldage, the current labour force consist only of the current youth.
- We also assume that all firms have access to an identical production technology - **which satisfies all standard neoclassical properties**.
- The firm-specific production functions can be aggregated to generate an aggregate production function such that :

$$Y_t = F(K_t, N_t).$$

- At every point of time the market clearing wage rate and the rental rate of capital are given by:

$$w_t = F_N(K_t, N_t); \quad r_t = F_K(K_t, N_t).$$

OLG Model: Household Side Story

- There are H households or dynasties in the economy. In every household, at any point of time t, there are two cohorts of agents - those who are born in period t ('generation t' - who are currently young) and those who were born in the previous period ('generation $t-1$' - who are currently old): hence the name 'overlapping generations'.
- Thus at any point of time t, total population consists of two successive generations of people:

$$L_t = N_t + N_{t-1}.$$

(Notice that although total population is L_t, total labour force at time t in only N_t.)

- We shall assume that population in successive generations grows at a constant rate n :

$$N_{t+1} = (1+n)N_t.$$

- Hence total population in the economy (L_t) also grows at the same constant rate n.

Life Cycle of a Representative Member of Generation t:

- All agents **within** a generation are identical. So we can talk in terms of a representative agent belonging to 'generation t'.
- The agent is born at the beginning of period t with an endowment of one unit of labour.
- **All agents in this model are selfish - they care only about their own consumption/utility and not about their childrens' utility.** Hence they do not leave any bequest.
- No bequest implies that the young agent has only labour endowment and no capital endowment.
- The young agent in period t supplies his labour inelastically to the labour market in period t to earn a wage income w_t.
- Out of this wage income, the agent consumes a part and saves the rest - which becomes his capital stock **in the next period** and allows him to earn a rental income **in the next period**.

Representative Agent's Utility Function:

- Notice that since the agent would not be working in the next period, the savings and the consequent ownership of capital is the only source of income for him in the next period.
- Thus an agent is a worker in the first period of his life and becomes a capitalist (capital-owner) in the second period of his life.
- The young agent in period t optimally decides on his current consumption (c_t^1) and current savings (s_t) (or equivalently, his current consumption (c_t^1) and future consumption (c_{t+1}^2)) so as to maximise his lifetime utility:

$$U(c_t^1, c_{t+1}^2) \equiv u(c_t^1) + \beta u(c_{t+1}^2); \; 0 < \beta < 1, \tag{14}$$

where $u' > 0$; $u'' < 0$; $\lim_{c \to 0} u'(c) = \infty$; $\lim_{c \to \infty} u'(c) = 0$.

- β is the standard discount factor which tells us that agents prefer current consumption more than future consumption.

Representative Agent's Budget Constraints:

- The first period budget constraint of the agent:
$$c_t^1 + s_t = w_t.$$
- The second period budget constraint of the agent:
$$c_{t+1}^2 = (1 + r_{t+1}^e - \delta)s_t.$$
- Combining, we get the life-time budget constraint of the agent as:
$$c_t^1 + \frac{c_{t+1}^2}{(1 + r_{t+1}^e - \delta)} = w_t. \tag{15}$$
- The agent decides on his optimal consumption in the two periods by maximising (1) subject to (2).
- Since the agent is taking his savings decision in the first period, when second period's market interest rate is not yet known, he must optimize on the basis of some expected value of the r_{t+1}.
- As before, we shall assume that **agents have perfect foresight/rational expectations** such that $r_{t+1}^e = r_{t+1}$ for all t.

Representative Agent's Optimal Consumption & Savings:

- From the FONC of the optimization exercise:

$$\frac{u'(c_t^1)}{u'(c_{t+1}^2)} = \beta(1 + r_{t+1} - \delta). \tag{16}$$

- From the FONC and the life-time budget constraint, we can derive the optimal solutions as:

$$\begin{aligned} c_t^1 &= \psi(w_t, r_{t+1}); \\ c_{t+1}^2 &= \eta(w_t, r_{t+1}). \end{aligned}$$

- Corresponding optimal savings:

$$s_t = w_t - \psi(w_t, r_{t+1}) \equiv \phi(w_t, r_{t+1}).$$

- Before we proceed further, it is useful to note the signs of the partial derivatives s_w and s_r.

Representative Agent's Optimal Consumption & Savings (Contd.):

- Notice that

$$s_w \equiv \frac{\partial \phi(w_t, r_{t+1})}{\partial w_t} = 1 - \frac{\partial \psi(w_t, r_{t+1})}{\partial w_t} = 1 - \frac{\partial c_t^1}{\partial w_t}.$$

- From the life-time budget constraint of the agent:

$$\frac{\partial c_t^1}{\partial w_t} + \frac{1}{(1 + r_{t+1} - \delta)} \frac{\partial c_{t+1}^2}{\partial w_t} = 1.$$

- Under the assumption that both c_t^1 and $c_{t\_1}^2$ are normal goods, a unit increase in the wage rate w_t **ceteris paribus** must increase both c_t^1 and c_{t+1}^2. This implies that $0 < \dfrac{\partial c_t^1}{\partial w_t} < 1$ (since both $\frac{\partial c_{t+1}^2}{\partial w_t}$ and $\frac{1}{(1+r_{t+1}-\delta)}$ are positive).
- This in turn implies that

$$0 < s_w < 1.$$

Representative Agent's Optimal Consumption & Savings (Contd.):

- The sign of s_r however is ambiguous.
- Notice that

$$s_r \equiv \frac{\partial \phi(w_t, r_{t+1})}{\partial r_{t+1}} = -\frac{\partial \psi(w_t, r_{t+1})}{\partial r_{t+1}} = -\frac{\partial c_t^1}{\partial r_{t+1}}.$$

- So the sign of s_r depends on how first period consumption responds to a unit change in r_{t+1}.
- Recall however that $\dfrac{1}{1 + r_{t+1} - \delta}$ is the relative price of c_{t+1}^2 in terms of c_t^1.
- Thus a unit increase in r_{t+1} *ceteris paribus* **reduces** the relative price of future consumption in terms of current consumption.

Representative Agent's Optimal Consumption & Savings (Contd.):

- Any such price change will be associated with two effects:
  - a substitution effect ($\Rightarrow$ consumption should move in favour of the relatively cheaper good);
  - an income effect ($\Rightarrow$ the budget set of the consumer expands, which increases consumption of both goods)
- Thus due to an increase in r_{t+1} **ceteris paribus**
  - c_t^1 should decrease due to the substitution effect, while
  - c_t^1 should increase due to the income effect of a price change.
- The sign of $\dfrac{\partial c_t^1}{\partial r_{t+1}}$ depends on which effect dominates.
- This in turn implies that

$$\begin{aligned} s_r &\gtreqless 0 \\ \text{according as, substitution effect} &\gtreqless \text{income effect.} \end{aligned}$$

Representative Agent's Optimal Consumption & Savings (Contd.):

- Notice that $s_r < 0$ implies that savings responds negatively to a change in the future rate of interest: a rise in the (expected) rate of interest leades to lower savings.
- Apriori we have no reason to rule out this case, even though it is counter-intutive. It depends on the characteristics of the utility function.
- However, henceforth we shall assume that the utility function of the agents is such that the substitution effect of a price change dominates the income effect so that

$$s_r \geqq 0.$$

Aggregate Consumption & Savings:

- Let us now turn our attention to the aggregate economy.
- Recall that in every period the total output is distributed as wage income and capital income:

$$Y_t = w_t N_t + r_t K_t. \tag{17}$$

- The entire wage income goes to the current young generation. Each of them saves a part of the wage income (s_t) and consume the rest. Thus,

$$w_t N_t = c_t^1 N_t + s_t N_t.$$

- On the other hand, the entire interest income goes to the current old generation. Each of them consume not only the interest earnings but the left over capital stock as well. Thus,

$$c_t^2 N_{t-1} = r_t K_t + (1 - \delta) K_t.$$

Aggregate Consumption & Savings (Contd.):

- Thus aggregate consumption in this economy at time t :

$$\begin{aligned} C_t &= c_t^1 N_t + c_t^2 N_{t-1} \\ &= [w_t N_t - s_t N_t] + [r_t K_t + (1-\delta) K_t] \\ &= w_t N_t + r_t K_t - \underbrace{[s_t N_t - (1-\delta) K_t]}_{S_t} \end{aligned} \quad (18)$$

- Notice that aggregate savings S_t has two components:
  1. The positive savings by the young $(s_t N_t)$;
  2. The negative savings by the old $(-(1-\delta) K_t)$.
- As before, from the Savings-Investment equality for the aggregate economy:

$$\begin{aligned} I_t &= S_t, \text{ where } I_t \equiv K_{t+1} - (1-\delta) K_t \\ &\Rightarrow K_{t+1} = s_t N_t. \end{aligned}$$

Dynamics of Capital-Labour Ratio:

- Since we know that labour force in this economy is growing at the rate n, i.e., $N_{t+1} = (1+n)N_t$, we can derive the dynamic of capital-labour ratio (k_t) as:

$$k_{t+1} = \frac{s_t}{(1+n)} = \frac{\phi(w_t, r_{t+1})}{(1+n)}. \tag{19}$$

- Again, from the production side of the story, we already know that

$$\begin{aligned} w_t &= f(k_t) - k_t f'(k_t); \\ r_{t+1} &= f'(k_{t+1}). \end{aligned}$$

- Thus we can write (4) as:

$$k_{t+1} = \frac{\phi(w_t(k_t), r_{t+1}(k_{t+1}))}{(1+n)} = \Phi(k_t, k_{t+1}). \tag{20}$$

- Equation (20) is the basic dynamic equation of the OLG model, which implicitly defines k_{t+1} as a function of k_t. Given k_0, we should be able to trace the evolution of the capital-labour ratio over time.

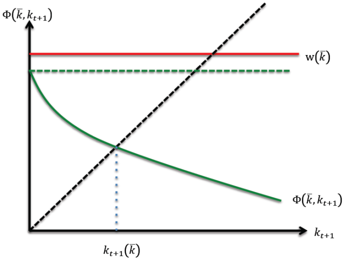
Existence of a Unique Perfect Foresight path:

- Let us look at dynamic equation (20). Notice that it is an 'implicit' difference equation with k_{t+1} entering on both sides.
- In fact this implicit nature of the function arises precisely due to assumption of perfect foresight. For any given k_t, the LHS denotes the k_{t+1} that will acutally emerge in the economy while the $r_{t+1}(k_{t+1})$ in the RHS captures the k_{t+1} that people expect to hold. Perfect foresight will hold if and only if there two match with each other.
- This then raises the following question: for every k_t do we neceassrily get a **unique** k_{t+1} that satisfy the dynamic equation (20)?
- In other words, for any given initial value of k_0, **does a unique perfect foresight path exist**?
- Notice that uniqueness is important because otherwise the future trajectory of the economy will become indeterminate.

Existence of a Unique Perfect Foresight Path (Contd.):

- Notice that for any given value of k_t,say $\bar{k}$, from (20) we shall have a unique solution for k_{t+1} if and only if the curve representing $\Phi(\bar{k}, k_{t+1})$ has a unique point of intersection with the 45° line in the positive quadrant.
- A *sufficient* condition for this to happen is $\Phi(\bar{k}, k_{t+1})$ is either a flat line or is downward sloping with respect to k_{t+1}, i.e.,

$$\frac{\partial \Phi(\bar{k}, k_{t+1})}{\partial k_{t+1}} \leqq 0 \ \text{ for all } \bar{k} \in (0, \infty).$$



Existence of a Unique Perfect Foresight Path (Contd.):

- Notice that

$$\begin{aligned}
\frac{\partial \Phi(\bar{k}, k_{t+1})}{\partial k_{t+1}} &= \frac{1}{(1+n)} \frac{\partial \phi(w_t, r_{t+1})}{\partial k_{t+1}} \\
&= \frac{1}{(1+n)} \frac{\partial \phi(w_t, r_{t+1})}{\partial r_{t+1}} \frac{dr_{t+1}}{dk_{t+1}} \\
&= \frac{1}{(1+n)} \, s_r \; f''(k_{t+1}).
\end{aligned}$$

- Thus a *sufficient* condition for the existence of a unique perfect foresight path is: $s_r \geqq 0$.

Dynamics of Capital-Labour Ratio (Contd.):

- Let us now characterise the evolution of k_t over time.
- Since $\Phi(k_t, k_{t+1})$ is a nonlinear function of k_t and k_{t+1}, we shall have to use the phase diagram technique to qualitatively characterise the dynamics.
- In drawing the phase diagram, first note that the slope of the phase line can be determined by total diffentiating (20):

$$\begin{aligned}(1+n)dk_{t+1} &= s_w \frac{dw_t}{dk_t} dk_t + s_r \frac{dr_{t+1}}{dk_{t+1}} dk_{t+1} \\ \text{i.e., } \frac{dk_{t+1}}{dk_t} &= \frac{s_w\left[-k_t f''(k_t)\right]}{(1+n) - s_r f''(k_{t+1})}.\end{aligned}$$

- Under the assumption that $s_r \geqq 0$, the slope of the phase line is necessarily positive.

Dynamics of Capital-Labour Ratio (Contd.):

- But even when the slope is positive, the curvature is not necessarily concave - since it would involve the third derivative of the utility function and the $f(k)$ function - whose signs are not known.
- Hence anything is possible: we may have situations of no steady state; unique stable steady state; unique unstable steady state; multiple steady states (some stable, some unstable):

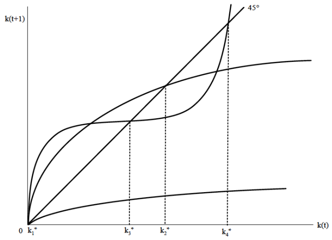


FIGURE 9.1. Various types of steady-state equilibria in the baseline overlapping generations model.

- In other words, the nice result of the Solow model of a unique and globally stable steady state is no longer gauranteed - **despite the production function satifying all the standard neoclassical properties** - **including diminishing returns and the Inada conditions!**
- What about dynamic efficiency?
- Even that is not gauranteed any more!
- Below we provide an example - with specific functional forms - to show that dynamic efficiency is not necessarily gauranteed under the OLG model - despite optimizing savings behaviour by the agents.

Golden Rule & Dynamic Efficiency in the OLG Model:

- Before we turn to the specific functional forms, let us first define the golden rule in the context of the OLG model.
- As before let us define the 'golden rule' as that particular steady state which maximises the steady state level of per capita (average) consumption.
- Notice however that now there are two sets of people at any point of time t - current young (N_t) and current old (N_{t-1}).
- Hence per capita (average) consumption at any point of time t would be defined as:

$$c_t = \frac{c_t^1 N_t + c_t^2 N_{t-1}}{L_t} = \frac{c_t^1 N_t + c_t^2 N_{t-1}}{N_t + N_{t-1}} = \frac{(1+n)c_t^1 + c_t^2}{1+(1+n)}.$$

- In other words, the per capita consumption in period t is the weighted average of the consumption of the current young and that of the current old.

Characterization of the Steady State in OLG Model:

- So what would be the steady state value of per capita consumption?
- Recall that the basic dynamic equation is the OLG model is given by:

$$k_{t+1} = \frac{s_t}{(1+n)} = \frac{\phi(w(k_t), r(k_{t+1}))}{(1+n)}$$

- Accordingly, the steady state(s) of the OLG model is defined as:

$$\begin{aligned} k^* &= \frac{s^*}{(1+n)} = \frac{\phi(w(k^*), r(k^*))}{(1+n)} \\ &\Rightarrow s^* = (1+n)k^* \end{aligned} \qquad (21)$$

- On the other hand, from the optimal solutions of c^1 and c^2, we know that at steady state:

$$\left(c^1\right)^* = \psi(w(k^*), r(k^*)) = w(k^*) - s^*; \qquad (22)$$

$$\left(c^2\right)^* = \eta(w(k^*), r(k^*)) = \left[1 + r(k^*) - \delta\right] s^*. \qquad (23)$$

Steady State Per Capita Consumption in the OLG Model:

- Hence from (1), (2) & (3) steady state per capita consumption:

$$
\begin{aligned}
c^* &= \frac{(1+n)\left(c^1\right)^* + \left(c^2\right)^*}{1+(1+n)} \\
&= \frac{(1+n)\left[w(k^*) - s^*\right] + \left[(1+r(k^*) - \delta)\right] s^*}{1+(1+n)} \\
&= \frac{(1+n)w(k^*) + \left[r(k^*) - \delta - n)\right] s^*}{1+(1+n)} \\
&= \frac{(1+n)\left[\{w(k^*) + r(k^*)k^*\} - (n+\delta)k^*\right]}{1+(1+n)} \\
&= \frac{1+n}{1+(1+n)}\left[f(k^*) - (n+\delta)k^*\right].
\end{aligned}
$$

- Maximizing c^* with respect to k^*, we would still get the golden rule condition as:

$$k_g : f'(k^*) = (n+\delta)$$

OLG Model with Specific Functional Forms:

- Let us now assume specific functional forms.
- Let

$$U(c_t^1, c_{t+1}^2) \equiv \log c_t^1 + \beta \log c_{t+1}^2;$$

$$f(k_t) = (k_t)^{\alpha};\ 0 < \alpha < 1.$$

- Also let the rate of depreciation be 100%, i.e., $\delta = 1$.
- Thus the life-time budget constraint of the representative agent of generation t is given by:

$$c_t^1 + \frac{c_{t+1}^2}{r_{t+1}} = w_t.$$

- The corresponding FONC:

$$\frac{c_{t+1}^2}{c_t^1} = \beta r_{t+1}. \tag{24}$$

OLG Model: Specific Functional Forms (Contd.)

- From the FONC and the life-time budget constraint, we can derive the optimal solutions as:

$$\begin{aligned} c_t^1 &= \frac{1}{1+\beta} w_t; \\ s_t &= \frac{\beta}{1+\beta} w_t; \\ c_{t+1}^2 &= \beta r_{t+1} \left[\frac{1}{1+\beta} w_t \right]. \end{aligned}$$

- Corresponding dynamic equation:

$$k_{t+1} = \frac{s_t}{(1+n)} = \left(\frac{1}{1+n} \right) \left[\frac{\beta}{1+\beta} w_t \right].$$

- Finally, given the production function,

$$w_t = (1-\alpha)\,(k_t)^{\alpha}$$

OLG Model: Dynamics with Specific Functional Forms

- Thus the dynamics of this specific case is simple:

$$k_{t+1} = \left(\frac{1}{1+n}\right)\left(\frac{\beta}{1+\beta}\right)(1-\alpha)(k_t)^{\alpha}.$$

- It is easy to see that the above phase line will generate a unique non-trivial steady state which is globally stable.
- The corresponding steady state solution is defined as:

$$\begin{aligned} k^* &= \left(\frac{1}{1+n}\right)\left(\frac{\beta}{1+\beta}\right)(1-\alpha)(k^*)^{\alpha} \\ \text{i.e., } k^* &= \left[\left(\frac{1}{1+n}\right)\left(\frac{\beta}{1+\beta}\right)(1-\alpha)\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

- Is this steady state dynamically efficient? Not necessarily!

OLG Model: Dynamic Inefficiency

- Notice that given the specific functional form, the 'golden rule' value of the capital-labour ratio can be derived as:

$$\begin{aligned} k_g &: \quad f'(k^*) = (n+\delta) \\ \text{i.e., } k_g &: \quad \alpha(k^*)^{\alpha-1} = 1+n \\ \text{i.e., } k_g &= \quad \left[\frac{\alpha}{1+n}\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

- It is easy to verify that the steady state under this specific example will be dynamically inefficient whenever

$$\frac{\beta}{1+\beta} > \frac{\alpha}{1-\alpha}.$$

- Example: $\beta = \frac{1}{2}; \alpha = \frac{1}{5}$.

Reason for Dynamic Inefficiency in the OLG Model:

- Thus we see that dynamic inefficiency may arise in the OLG model - despite optimizing savings behaviour by households.
- This apparently paradoxical result stems from the fact that agents are 'selfish' in the OLG model; they do not care for their children's utility/consumption.
- Thus when they optimise they equate the MRS with (the discounted value of) the actual future return $(r_{t+1} + 1 - \delta)$:

$$\frac{u'(c_t^1)}{u'(c_{t+1}^2)} = \beta(r_{t+1} + 1 - \delta).$$

- Now notice that

$$\frac{u'(c_t)}{u'(c_{t+1})} \gtreqless 1 \Rightarrow u'(c_t) \gtreqless u'(c_{t+1}) \Rightarrow c_t \lesseqgtr c_{t+1}.$$

Reason for Dynamic Inefficiency in OLG Model (Contd.)

- This implies that in the OLG framework, $c_t \lesseqqgtr c_{t+1}$ i.e, consumption would rise, fall or remain unchanged over time if the corresponding gross (future) return, measured by $(r_{t+1} + 1 - \delta)$ is greater, equal to or less than the subjective cost $\frac{1}{\beta}$, ie. according as

$$r_{t+1} + 1 - \delta \gtreqqless \frac{1}{\beta}.$$

Dynamic Inefficiency: OLG vis-a-vis R-C-K Model

- Recall that in the R-C-K model the corresponding equation was given by:

$$u'(c_t)[1+n] = \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right].$$

$$\text{i.e., } \frac{u'(c_t)}{u'(c_{t+1})} \gtreqqless 1 \text{ according as } \frac{[f'(k_{t+1}) + (1-\delta)]}{1+n} \gtreqqless \frac{1}{\beta}$$

- Thus in the R-C-K framework, $c_t \lesseqqgtr c_{t+1}$ i.e, consumption would rise, fall or remain unchanged over time if the corresponding **population-adjusted** (future) return, measured by $\frac{[r_{t+1}+(1-\delta)]}{1+n}$ is greater, equal to or less than the subjective cost measured by $\frac{1}{\beta}$.

Reason for Dynamic Inefficiency in OLG Model (Contd.)

- Notice that in both the OLG model and the R-C-K model, households will stop saving whenever their percieved return is less than the subjective cost $\frac{1}{\beta}$.
- But due to presence of intergenerational altrauism, in the R-C-K model the percieved return is adjusted for popultion growth and is given by $\frac{[r_{t+1}+(1-\delta)]}{1+n}$, whereas in the OLG model this return is just $\left[r_{t+1}+(1-\delta)\right]$.
- This implies that in the R-C-K model households will *necessarily* stop saving when the (net) marginal product has fallen below the population growth rate $\frac{[r_{t+1}+(1-\delta)]}{1+n} < 1$.
- But there is no reason why in the OLG model, households would stop saving in this scenario, because even though the gross marginal product $\left[r_{t+1}+(1-\delta)\right]$ has fallen below $1+n$ - it might still greater than the subjective cost $\frac{1}{\beta}$.Thus there is a tendency to oversave (compared to the R-C-K model), which persists even when the economy has moved into a dynamically inefficient region.

Dynamic Inefficiency & Scope for Government Intervention in the OLG Model:

- Since under the OLG framework, the steady state of the decentralized market economy may be dynamically inefficient (despite rational expectations on part of the agents), this again justifies a role of government in improving efficiency.
- Thus the conclusions of the OLG model are diametrically opposite of that of the R-C-K model - even though both are based on strictly neoclassical production function and optimizing agents.
- The difference arises primarily due to the absence of parental altruism in the OLG framework.
- It can be shown that if we introduce parental altruism in the OLG model (by incorporating a bequest term that each parent leaves to his child at the end of his life time), then the OLG framework will be very similar to the R-C-K framework.

OLG Model: Summary

- Even though the production function is Neoclassical in the OLG model, the strong stability result of Solow (as well as R-C-K model) does not hold:
  - a steady state may not exists;
  - even if it exists it may not be unique;
  - even if it is unique, it may not be stable.
- So the growth conclusions of the Solow model do not hold either.
- Moreover the dynamic inefficiency problem may reappear here- even though each agent is optimally determining his savings behaviour.
- Thus in this model there is scope for government intervention in terms of improving efficiency.
- Since the OLG model does not obey the growth properties of the Solow/R-C-K model, we do not identify this structure with the "Neoclassical Growth Model" (even though the production structure here is identical to Solow/R-C-K).

Long Run Growth of Per Capita?

- Recall that in all the growth models discussed so far, the capital-labour ratio (k_t) in the long run becomes a constant. Hence none of them can explain the steady rise in per capita income that is historically observed in almost all economies since the industrial revolution in Europe in the late 18th century.
- Such staedy growth is often explained by appealing to exogenous technological shocks.
- Can we have long run growth of per capita income in a Solow-type economy even without such exogenous shocks?
- The answer is: yes, but only if you allow some of the Neoclassical properties of the production function to be relaxed.
- The long run constancy of the per capita income in the Solow model arises due to the strong uniqueness and stability property of the steady state - which in turn depends on two key assumptions: the property of diminishing returns and the Inada Conditions. One can generate long run growth of per capita income in the Solow model if we relax one of these conditions.

AK Production Technology: First Step Towards Endogenous Growth

- In particular, steady growth along a balanced growth path is possible if the production function exhibits **non-diminishing** returns.
- A specific example of such non-diminishing returns technology is the AK technology where output is a linear function of capital:

$$Y_t = AK_t.$$

- The AK model is the precursor to the latter-day 'endogenous growth theory' which (unlike Solow) attempts to explain long run growth in per capita income in terms endogenous factors within the economy.
- When technology interacts with the factor accumulation, there is no reason why the production function would necessarily exhibit diminishing returns.

AK Technology: Dynamics of Capital and Labour

- Let us now replace the Neoclassical production function in the Solow model by an AK production function.
- As before, capital stock over time gets augmented by the savings/investment made by the households.
- Thus the capital accumulation equation in this economy is given by:

$$\begin{aligned} K_{t+1} &= I_t + (1-\delta)K_t = sY_t + (1-\delta)K_t \\ \text{i.e., } K_{t+1} &= sAK_t + (1-\delta)K_t, \end{aligned} \tag{25}$$

- Labour stock, on the other hand, increases due to population growth (at a constant rate n):

$$N_{t+1} = (1+n)N_t. \tag{26}$$

AK Technogy: Dynamics of Capital-Labour Ratio

- Using the definition that $k_t \equiv \frac{K_t}{N_t}$, we can write

$$\begin{aligned} k_{t+1} &\equiv \frac{K_{t+1}}{N_{t+1}} = \frac{sAK_t + (1-\delta)K_t}{(1+n)N_t} \\ &\Rightarrow k_{t+1} = \frac{sA + (1-\delta)}{(1+n)} k_t. \end{aligned} \tag{27}$$

- From above, the rate of growth of capital-labour ratio, and hence that of per capita output, is immediately given by:

$$\frac{k_{t+1} - k_t}{k_t} = \frac{sA - (\delta + n)}{(1+n)}$$

- As long as $sA > (\delta + n)$, per capita output grows at a constant rate - both in short run and long run.

Justifications for AK Technology:

- There are many justifications as to why the ‘aggregate’ production technology might be linear. Here we look at two examples - each providing a different justification as to why diminishing returns might not work:
    1. Fixed Coefficient Technogy - exhibiting complemetarity between factors of production;
    2. Production Technology with Learning-by-Doing and Knowledge Spillover (Frankel-Romer).

Justification for AK Technology: Leontief Production Function

- The first example of an AK technology is of course the Fixed Coefficient (Leontief) Production Function:

$$Y_t = \min\left[aK_t, bN_t\right],$$

where a and b are the constant coefficients representing the productivity of capital and labour respectively.

- Suppose the economy starts with a (historically) given stock of capital and a given amount of labour force at time t.
- Then fixed coefficient technology implies:

$$\begin{aligned} Y_t &= aK_t \text{ if } aK_t < bN_t \text{ (capital constrained economy);} \\ Y_t &= bN_t \text{ if } bN_t < aK_t \text{ (labour constrained economy).} \end{aligned}$$

- In the first case, the production technology would in effect be AK - although notice that labour is still used in the production process.

Justification for AK Technology: Learning by Doing & Knowledge Spillover

- A more nuanced justification was provided by Frankel (1962), which was later exploited by Romer (1986) in developing the first model of endogenous growth.
- This was already discussed in the context of the Micro-foundations. Let us quickly recapitulate.
- Consider an economy with S identical firms - each having access to an identical firm-specific technology:

$$Y_i = \bar{A}_t F(K_{it}, N_{it}) \equiv \bar{A}_t \left(K_{it}\right)^{\alpha} \left(N_{it}\right)^{1-\alpha};\ 0 < \alpha < 1.$$

- The firm-specific production function exhibits all the neoclassical properties.
- $\bar{A}_t$ represents the current state of the technology in the economy, which is treated as exogenous by each firm.

AK Technology: Frankel-Romer Explanation

- The $\bar{A}_t$ depends on the **aggregate capital labour-ratio** in the economy - due to '**learning by doing**'. In particular, let us assume

$$\bar{A}_t = \left(\frac{K_t}{N_t}\right)^{\beta}; \ \beta > 0; \ \text{where } K_t = SK_{it}; \ N = SN_{it}.$$

- Corresponding Aggregate Production Function:

$$\begin{aligned} Y_t &= \sum Y_{it} = S\left[\bar{A}_t (K_{it})^{\alpha} (N_{it})^{1-\alpha}\right] \\ &= \bar{A}_t (SK_{it})^{\alpha} (SN_{it})^{1-\alpha} \\ &= \bar{A}_t (K_t)^{\alpha} (N_t)^{1-\alpha}. \end{aligned}$$

- Replacing the value of $\bar{A}_t$ in the aggregate production technology:

$$Y_t = \bar{A}_t (K_t)^{\alpha} (N_t)^{1-\alpha} = (K_t)^{\alpha+\beta} (N_t)^{1-\alpha-\beta}.$$

- In the special case where $\alpha + \beta = 1$, the aggregate production technology is indeed AK.

AK Technology: General Insight

- Even though the AK models hint at some explanation for the technology parameter (e.g., learning by doing in the Frankel-Romer model), they do not model technological progress explicitly.
- The subsequent endogenous growth models explicitly specify a process of technological change in terms of R&D, education, improvement in infrastructure etc. They also move away from the CRS technology to allow for IRS - which necessitates a movement away from competitive market structure.
- Even more recent developments in growth theory go beyond technology and explain growth in terms of other deeper factors like institutions, political economy, income and wealth distribution etc.
- These are however beyond the purview of the current course.

References:

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  - D. Acemoglu: Introduction to Modern Economic Growth, Chapter 2, Sections 2.1,2.2, 2.3 & Chapter 5, Sections 5.1,5.2, 5.3, 5.4, 5.5, 5.9.
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