1 Comparative Static results for ratio Form

Suppose that the players participating in the contest face strictly convex cost function $\psi_i(e_i)$ for $i = 1, 2$. Our objective functions are given by:

$$\max_{e_1} \left\{ \frac{\theta e_1^m}{\theta e_1^m + e_2^n} v - \psi_1(e_1) \right\}$$

and

$$\max_{e_2} \left\{ \frac{e_2^n}{\theta e_1^m + e_2^n} v - \psi_2(e_2) \right\}$$

The above optimization can be redefined in terms of the cost of input provided by the parties. Let $x_i = \psi_i(e_i)$. Since $\psi_i(e_i)$ is strictly monotonic, it has a well behaved inverse function; $g_i(x_i) = \psi_i^{-1}(x_i)$. Therefore, $e_i = g_i(\psi_i(e_i)) = g_i(x_i)$. With this transformation we can rewrite our optimization problem as

$$\max_{x_1} \left\{ \frac{\theta (g_1(x_1))^m}{\theta (g_1(x_1))^m + (g_2(x_2))^n} v - x_1 \right\}$$

and

$$\max_{x_2} \left\{ \frac{(g_2(x_2))^n}{\theta (g_1(x_1))^m + (g_2(x_2))^n} v - x_2 \right\}$$

where x_i is cost of effort/input.

Let us restrict our attention to the convex cost functions of the form $\psi_i(e_i) = e_i^k,\ i = 1, 2$ where $k > 1$ (note that we are only considering $e_i > 0$ and hence such a function is strictly increasing in e_i given k). Then, from the above definitions we get $g_i(x_i) = x_i^{\frac{1}{k}} = e_i,\ i = 1, 2$. Using these expressions we can rewrite our optimization problem as

$$\max_{x_1} \left\{ \frac{\theta x_1^{m'}}{\theta x_1^{m'} + x_2^{n'}} v - x_1 \right\}$$

and

$$\max_{x_2} \left\{ \frac{x_2^{n'}}{\theta x_1^{m'} + x_2^{n'}} v - x_2 \right\}$$

where $m' = \frac{m}{k}$ and $n' = \frac{n}{k}$. (If $m, n < 1$ then $m', n' < 1$ since $k > 1$)

Note that we have transformed a problem with convex cost of efforts to a standard ratio form problem with linear cost function extensively studied in the literature. However, efforts (e_i) have been replaced by costs (x_i).
We can solve the entire problem for cost of effort (x_i) and then deduce efforts using

the relationship $e_i = g_i(\psi_i(e_i)) = g_i(x_i)$ which in our specific case is $e_i = g_i(x_i) = x_i^{\frac{1}{k}}$. Note that for given value of k, e_i and x_i are monotonically related i.e. increase in $x_i \iff$ increase in e_i. The magnitude of change will differ but the direction of change would be the same. With this information we now look at the comparative static results for x_i and deduce the comparative static results for e_i.
The FOC's for the above problem are given by

$$FOC1 - \quad g_1^r = \frac{m'p(1-p)v}{x_1} - 1 = 0$$

$$FOC2 - \quad g_2^r = \frac{n'p(1-p)v}{x_2} - 1 = 0$$

which can be rearranged and written as:

$$m'p(1-p)v - x_1 = 0$$
$$n'p(1-p)v - x_2 = 0$$

After rearranging, the FOC's are similar to the case with difference form CSF and convex costs (with efforts replaced by costs).

Following the arguments used for proving the existence of a solution to the FOC's in the difference form CSF with quadratic costs (using Intermediate value theorem), here also it can be shown that at least one solution to the FOC's is guaranteed to exist. Further, for $m, n \leq 1$ SOC will also be satisfied and hence a Nash equilibrium exists. Alternatively, using the results of Szidarovszky and Okuguchi (1997), we can claim that for the given ratio form CSF with $m, n \leq 1$ there exists a unique pure Nash equilibrium. In fact We can relax this condition and say that we need $m', n' \leq 1 \implies m, n \leq k$ where $k > 1$

We know from the FOC's that if a Nash equilibrium (x_1^{r*}, x_2^{r*}) exists, then $x_1^{r*} = \frac{m'}{n'} x_2^{r*} \implies x_1^{r*} = \frac{m}{n} x_2^{r*}$ i.e. $x_1^{r*} \gtreqless x_2^{r*}$ as $m \gtreqless n$ independent of the 'natural advantage' i.e. $\theta's$.

As seen above, $g_i^r = 0 \;\; i = 1, 2$ gives us the FOCs. We can also see that

$$\frac{\partial g_1^r}{\partial x_1} = \frac{m'^2 v(1-2p^{r*})p^{r*}(1-p^{r*})}{(x_1^{r*})^2} - \frac{m'p^{r*}(1-p^{r*})v}{(x_1^{r*})^2}$$

$$\frac{\partial g_2^r}{\partial x_2} = \frac{n'^2 v(2p^{r*}-1)p^{r*}(1-p^{r*})}{(x_2^{r*})^2} - \frac{n'p^{r*}(1-p^{r*})v}{(x_2^{r*})^2}$$

$$\frac{\partial g_1^r}{\partial x_2} = \frac{m'v(1-2p)}{x_1}\frac{\partial p}{\partial x_2} = \frac{m'n'v(2p^{r*}-1)p^{r*}(1-p^{r*})}{x_1^{r*}x_2^{r*}}$$

$$\frac{\partial g_2^r}{\partial x_1} = \frac{n'v(1-2p)}{x_2}\frac{\partial p}{\partial x_1} = \frac{m'n'v(1-2p^{r*})p^{r*}(1-p^{r*})}{x_1^{r*}x_2^{r*}}$$

$\frac{\partial g_i^r}{\partial x_i} \leq 0$ for $i = 1, 2$ when the SOC is satisfied i.e. at any Nash equilibrium.
From the above expressions we know that

$$\frac{\partial g_1^r}{\partial x_2}\frac{\partial g_2^r}{\partial x_1} \leq 0$$

and at any Nash equilibrium we have

$$\frac{\partial g_1^r}{\partial x_1}\frac{\partial g_2^r}{\partial x_2} \geq 0$$

2 Changing value of prize (v)

Let us see the impact of expected prize value on the equilibrium cost of efforts and probability of winning of the first player.
On derivating the FOC's w.r.t v we get

$$\frac{\partial g_1^r}{\partial v} + \frac{\partial g_1^r}{\partial x_1}\frac{\partial x_1^{r*}}{\partial v} + \frac{\partial g_1^r}{\partial x_2}\frac{\partial x_2^{r*}}{\partial v} = 0$$

$$\frac{\partial g_2^r}{\partial v} + \frac{\partial g_2^r}{\partial x_1}\frac{\partial x_1^{r*}}{\partial v} + \frac{\partial g_2^r}{\partial x_2}\frac{\partial x_2^{r*}}{\partial v} = 0$$

where

$$\frac{\partial g_1^r}{\partial v} = \frac{m'p^{r*}(1-p^{r*})}{x_1^{r*}} > 0$$

$$\frac{\partial g_2^r}{\partial v} = \frac{n'p^{r*}(1-p^{r*})}{x_2^{r*}} > 0$$

The other partials are the same as given earlier.
On solving the system of linear equations, we get

$$\frac{\partial x_1^{r*}}{\partial v} = \frac{\frac{\partial g_1^r}{\partial v}\frac{\partial g_2^r}{\partial x_2} - \frac{\partial g_2^r}{\partial v}\frac{\partial g_1^r}{\partial x_2}}{\frac{\partial g_1^r}{\partial x_2}\frac{\partial g_2^r}{\partial x_1} - \frac{\partial g_2^r}{\partial x_2}\frac{\partial g_1^r}{\partial x_1}} \tag{1}$$

$$\frac{\partial x_2^{r*}}{\partial v} = \frac{\frac{\partial g_2^r}{\partial v}\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial v}\frac{\partial g_2^r}{\partial x_1}}{\frac{\partial g_1^r}{\partial x_2}\frac{\partial g_2^r}{\partial x_1} - \frac{\partial g_2^r}{\partial x_2}\frac{\partial g_1^r}{\partial x_1}} \tag{2}$$

Let $A = \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} - \frac{\partial g_2}{\partial x_2}\frac{\partial g_1}{\partial x_1}$. Using arguments given in the previous section, it is easy to see that at any NE, $A < 0$. Here we can use the partials which were calculated earlier to obtain,

$$\frac{\partial x_1^{r*}}{\partial v} = \frac{\frac{-m'n'(p^{r*}(1-p)^{r*})^2 v}{x_1^{r*}(x_2^{r*})^2}}{A} > 0$$

$$\frac{\partial x_2^{*}}{\partial v} = \frac{\frac{-m'n'(p^{r*}(1-p)^{r*})^2 v}{(x_1^{r*})^2 x_2^{r*}}}{A} > 0$$

Also, since p^{r*} doesn't depend on v directly, so we get

$$\frac{dp^{r*}}{dv} = \frac{\partial p^{r*}}{\partial x_1^{r*}}\frac{\partial x_1^{r*}}{\partial v} + \frac{\partial p^{r*}}{\partial x_2^{r*}}\frac{\partial x_2^{r*}}{\partial v}$$

$$\frac{dp^{r*}}{dv} = \frac{m'p^{r*}(1-p^{r*})}{x_1^{r*}}\frac{\partial x_1^{r*}}{\partial v} - \frac{n'p^{r*}(1-p^{r*})}{x_2^{r*}}\frac{\partial x_2^{r*}}{\partial v}$$

Since in any equilibrium, we have $\frac{x_1^{r*}}{m'} = \frac{x_2^{r*}}{n'}$, thus we can simplify the above equation to get

$$\frac{dp^{r*}}{dv} = \frac{m'}{x_1^{r*}} p^{r*}(1-p^{r*})\big(\frac{\partial x_1^{r*}}{\partial v} - \frac{\partial x_2^{r*}}{\partial v}\big)$$

The dynamics of the above expression depend on $\frac{\partial x_1^{r*}}{\partial v} - \frac{\partial x_2^{*}}{\partial v}$.

$$\frac{\partial x_1^{r*}}{\partial v} - \frac{\partial x_2^{*}}{\partial v} = \frac{\frac{-m'n'(p^{r*}(1-p)^{r*})^2 v}{x_1^{r*}(x_2^{r*})^2} - \frac{-m'n'(p^{r*}(1-p)^{r*})^2 v}{(x_1^{r*})^2 x_2^{r*}}}{A}$$

$$= \frac{\frac{-m'n'(p^{r*}(1-p)^{r*})^2 v}{x_1^{r*}x_2^{r*}}\big(\frac{1}{x_2^{r*}} - \frac{1}{x_1^{r*}}\big)}{A}$$

Given that the denominator in the above expression is negative at any Nash equilibrium, $\frac{\partial x_1^{r*}}{\partial v} - \frac{\partial x_2^{*}}{\partial v} \gtreqless 0 \quad as \quad \frac{1}{x_2^{r*}} - \frac{1}{x_1^{r*}} \gtreqless 0$

Note that in any equilibrium, $\frac{x_1^{r*}}{m'} = \frac{x_2^{r*}}{n'}$ is true i.e. $\frac{x_1^{r*}}{x_2^{r*}} = \frac{m}{n}$. So, $m \gtreqless n \implies x_1^{r*} \gtreqless x_2^{r*} \implies \frac{1}{x_2^{r*}} \gtreqless \frac{1}{x_1^{r*}} \implies \frac{1}{x_2^{r*}} - \frac{1}{x_1^{r*}} \gtreqless 0$.

Thus, $\frac{\partial x_1^{r*}}{\partial v} - \frac{\partial x_2^{r*}}{\partial v} \gtreqless 0 \quad as \quad m \gtreqless n \implies \frac{dp^{r*}}{dv} \gtreqless 0 \quad as \quad m \gtreqless n$

2.1 Impact of change in natural advantage, (θ)

Here we have $\frac{\partial g_1^r}{\partial \theta} = \frac{m'v(1-2p^{r*})}{x_1}\frac{\partial p}{\partial \theta}\Big|_{x_1^{r*},x_2^{r*}}$; $\frac{\partial g_2^r}{\partial \theta} = \frac{n'v(1-2p^{r*})}{x_2}\frac{\partial p}{\partial \theta}\Big|_{x_1^{r*},x_2^{r*}}$, where

$$\frac{\partial p}{\partial \theta}\Big|_{x_1^{r*},x_2^{r*}} = \frac{x_1^{m'}x_2^{n'}}{(\theta x_1^{m'} + x_2^{n'})^2}\Big|_{x_1^{r*},x_2^{r*}} = \frac{p^{r*}(1-p^{r*})}{\theta} > 0.$$

For this case, since $\frac{x_1^{r*}}{m'} = \frac{x_2^{r*}}{n'}$ in an equilibrium, $\frac{\partial g_1^r}{\partial \theta} = \frac{\partial g_2^r}{\partial \theta}$. The other partials are the same as in the previous case. The effect of θ depends on the value of the equilibrium value of p, i.e., p^{r*}.

$$\frac{\partial x_1^{r*}}{\partial \theta} = \frac{\frac{\partial g_1^r}{\partial \theta}\left(\frac{\partial g_2^r}{\partial x_2} - \frac{\partial g_1^r}{\partial x_2}\right)}{A} = \frac{\frac{\partial g_1^r}{\partial \theta}\left(\frac{-n'vp^{r*}(1-p^{r*})}{(x_2^{r*})^2}\right)}{A} \tag{3}$$

$$\frac{\partial x_2^{r*}}{\partial \theta} = \frac{\frac{\partial g_1^r}{\partial \theta}\left(\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_2^r}{\partial x_1}\right)}{A} = \frac{\frac{\partial g_1^r}{\partial \theta}\left(\frac{-m'vp^{r*}(1-p^{r*})}{(x_1^{r*})^2}\right)}{A} \tag{4}$$

It is easy to see that $\frac{\partial x_1^{r*}}{\partial \theta} \gtreqless 0$ *as* $p^{r*} \lesseqgtr 1/2$. Same is true for $\frac{\partial x_2^{r*}}{\partial \theta}$, i.e. $\frac{\partial x_2^{r*}}{\partial \theta} \gtreqless 0$ *as* $p^{r*} \lesseqgtr 1/2$. x_1^{r*} and x_1^{r*} are maximum when $p^{r*} = 1/2$ i.e. when the contest is symmetric.

For instance, when $m' = n'$, an equilibrium say $(x_1^{r*}, x_2^{r*}, p^{r*}) = (\frac{mv\theta}{(\theta+1)^2}, \frac{mv\theta}{(\theta+1)^2}, \frac{\theta}{\theta+1})$. In this case, if $\theta < 1$ then $p^{r*} < 1/2$ and hence $\frac{\partial x_1^{r*}}{\partial \theta} > 0$ and $\frac{\partial x_1^{r*}}{\partial \theta} > 0$. The opposite holds true for $\theta > 1$. The closer θ is to 1 higher is the value.

Let's look at how equilibrium probability of win, p^{r*}, changes with θ, i.e., $\frac{dp^{r*}}{d\theta} > 0$ holds.

$$\begin{aligned}
\frac{dp^{r*}}{d\theta} &= \frac{\partial p^{r*}}{\partial \theta} + \frac{\partial p^{r*}}{\partial x_1^{r*}}\frac{\partial x_1^{r*}}{\partial \theta} + \frac{\partial p^{r*}}{\partial x_2^{r*}}\frac{\partial x_2^{r*}}{\partial \theta} \\
&= \frac{\partial p^{r*}}{\partial \theta} + \frac{m'p^{r*}(1-p^{r*})}{x_1^{r*}}\left(\frac{\partial x_1^{r*}}{\partial \theta} - \frac{\partial x_2^{r*}}{\partial \theta}\right) \\
&= \frac{\partial p^{r*}}{\partial \theta}\left[\frac{1}{1-(m'-n')(1-2p^{r*})}\right]
\end{aligned}$$

In the literature, mostly the case $m = n$ is considered in which case probability increases with natural advantage.