lie to the right of 0 and negative numbers to the left of 0. If $a < b$, a point $x$ satisfies the inequalities $a < x < b$ if and only if $x$ is *between* $a$ and $b$.

This device for representing real numbers geometrically is a very worthwhile aid that helps us to discover and understand better certain properties of real numbers. However, the reader should realize that all properties of real numbers that are to be accepted as theorems must be deducible from the axioms without any reference to geometry. This does not mean that one should not make use of geometry in studying properties of real numbers. On the contrary, the geometry often suggests the method of proof of a particular theorem, and sometimes a geometric argument is more illuminating than a purely *analytic* proof (one depending entirely on the axioms for the real numbers). In this book, geometric


FIGURE I.7 Real numbers represented geometrically on a line.

arguments are used to a large extent to help motivate or clarify a particular discussion. Nevertheless, the proofs of all the important theorems are presented in analytic form.

### I 3.8 Upper bound of a set, maximum element, least upper bound (supremum)

The nine axioms listed above contain all the properties of real numbers usually discussed in elementary algebra. There is another axiom of fundamental importance in calculus that is ordinarily not discussed in elementary algebra courses. This axiom (or some property equivalent to it) is used to establish the existence of irrational numbers.

Irrational numbers arise in elementary algebra when we try to solve certain quadratic equations. For example, it is desirable to have a real number $x$ such that $x^2 = 2$. From the nine axioms above, we cannot prove that such an $x$ exists in **R**, because these nine axioms are also satisfied by **Q**, and there is no rational number $x$ whose square is 2. (A proof of this statement is outlined in Exercise 11 of Section I 3.12.) Axiom 10 allows us to introduce irrational numbers in the real-number system, and it gives the real-number system a property of continuity that is a keystone in the logical structure of calculus.

Before we describe Axiom 10, it is convenient to introduce some more terminology and notation. Suppose $S$ is a nonempty set of real numbers and suppose there is a number $B$ such that

$$x \leq B$$

for every $x$ in $S$. Then $S$ is said to be *bounded above* by $B$. The number $B$ is called an *upper bound* for $S$. We say *an* upper bound because every number greater than $B$ will also be an upper bound. If an upper bound $B$ is also a member of $S$, then $B$ is called the *largest member* or the *maximum element* of $S$. There can be at most one such $B$. If it exists, we write

$$B = \max S .$$

Thus, $B = \max S$ if $B \in S$ and $x \leq B$ for all $x$ in $S$. A set with no upper bound is said to be *unbounded above*.

The following examples serve to illustrate the meaning of these terms.

EXAMPLE 1. Let $S$ be the set of all positive real numbers. This set is unbounded above. It has no upper bounds and it has no maximum element.

EXAMPLE 2. Let $S$ be the set of all real $x$ satisfying $0 \leq x \leq 1$. This set is bounded above by 1. In fact, 1 is its maximum element.

EXAMPLE 3. Let $T$ be the set of all real $x$ satisfying $0 \leq x < 1$. This is like the set in Example 2 except that the point 1 is not included. This set is bounded above by 1 but it has no maximum element.

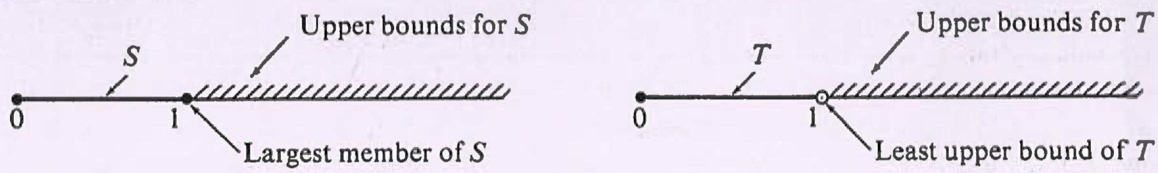
Some sets, like the one in Example 3, are bounded above but have no maximum element. For these sets there is a concept which takes the place of the maximum element. This is called the *least upper bound* of the set and it is defined as follows:

DEFINITION OF LEAST UPPER BOUND. *A number $B$ is called a least upper bound of a nonempty set $S$ if $B$ has the following two properties:*

(a) *$B$ is an upper bound for $S$.*

(b) *No number less than $B$ is an upper bound for $S$.*

If $S$ has a maximum element, this maximum is also a least upper bound for $S$. But if $S$ does not have a maximum element, it may still have a least upper bound. In Example 3 above, the number 1 is a least upper bound for $T$ although $T$ has no maximum element. (See Figure I.8.)




(a) $S$ has a largest member: $\max S = 1$

(b) $T$ has no largest member, but it has a least upper bound: $\sup T = 1$

FIGURE I.8 Upper bounds, maximum element, supremum.

THEOREM I.26. *Two different numbers cannot be least upper bounds for the same set.*

*Proof.* Suppose that $B$ and $C$ are two least upper bounds for a set $S$. Property (b) implies that $C \geq B$ since $B$ is a least upper bound; similarly, $B \geq C$ since $C$ is a least upper bound. Hence, we have $B = C$.

This theorem tells us that if there is a least upper bound for a set $S$, there is *only* one and we may speak of *the* least upper bound.

It is common practice to refer to the least upper bound of a set by the more concise term *supremum*, abbreviated *sup*. We shall adopt this convention and write

$$B = \sup S$$

to express the fact that $B$ is the least upper bound, or supremum, of $S$.

## I 3.9 The least-upper-bound axiom (completeness axiom)

Now we are ready to state the least-upper-bound axiom for the real-number system.

AXIOM 10. *Every nonempty set $S$ of real numbers which is bounded above has a supremum; that is, there is a real number $B$ such that $B = \sup S$.*

We emphasize once more that the supremum of $S$ need not be a member of $S$. In fact, $\sup S$ belongs to $S$ if and only if $S$ has a maximum element, in which case $\max S = \sup S$.

Definitions of the terms *lower bound, bounded below, smallest member* (or *minimum element*) may be similarly formulated. The reader should formulate these for himself. If $S$ has a minimum element, we denote it by $\min S$.

A number $L$ is called a *greatest lower bound* (or *infimum*) of $S$ if (a) $L$ is a lower bound for $S$, and (b) no number greater than $L$ is a lower bound for $S$. The infimum of $S$, when it exists, is uniquely determined and we denote it by $\inf S$. If $S$ has a minimum element, then $\min S = \inf S$.

Using Axiom 10, we can prove the following.

THEOREM I.27. *Every nonempty set $S$ that is bounded below has a greatest lower bound; that is, there is a real number $L$ such that $L = \inf S$.*

*Proof.* Let $-S$ denote the set of negatives of numbers in $S$. Then $-S$ is nonempty and bounded above. Axiom 10 tells us that there is a number $B$ which is a supremum for $-S$. It is easy to verify that $-B = \inf S$.

Let us refer once more to the examples in the foregoing section. In Example 1, the set of all positive real numbers, the number 0 is the infimum of $S$. This set has no minimum element. In Examples 2 and 3, the number 0 is the minimum element.

In all these examples it was easy to decide whether or not the set $S$ was bounded above or below, and it was also easy to determine the numbers $\sup S$ and $\inf S$. The next example shows that it may be difficult to determine whether upper or lower bounds exist.

EXAMPLE 4. Let $S$ be the set of all numbers of the form $(1 + 1/n)^n$, where $n = 1, 2, 3, \ldots$. For example, taking $n = 1$, 2, and 3, we find that the numbers 2, $\frac{9}{4}$, and $\frac{64}{27}$ are in $S$. Every number in the set is greater than 1, so the set is bounded below and hence has an infimum. With a little effort we can show that 2 is the smallest element of $S$ so $\inf S = \min S = 2$. The set $S$ is also bounded above, although this fact is not as easy to prove. (Try it!) Once we know that $S$ is bounded above, Axiom 10 tells us that there is a number which is the supremum of $S$. In this case it is not easy to determine the value of $\sup S$ from the description of $S$. In a later chapter we will learn that $\sup S$ is an irrational number approximately equal to 2.718. It is an important number in calculus called the Euler number $e$.

## I 3.10 The Archimedean property of the real-number system

This section contains a number of important properties of the real-number system which are consequences of the least-upper-bound axiom.

THEOREM I.28. *The set* **P** *of positive integers* 1, 2, 3, . . . *is unbounded above.*

*Proof.* Assume **P** is bounded above. We shall show that this leads to a contradiction. Since **P** is nonempty, Axiom 10 tells us that **P** has a least upper bound, say $b$. The number $b - 1$, being less than $b$, cannot be an upper bound for **P**. Hence, there is at least one positive integer $n$ such that $n > b - 1$. For this $n$ we have $n + 1 > b$. Since $n + 1$ is in **P**, this contradicts the fact that $b$ is an upper bound for **P**.

As corollaries of Theorem I.28, we immediately obtain the following consequences:

THEOREM I.29. *For every real $x$ there exists a positive integer $n$ such that $n > x$.*

*Proof.* If this were not so, some $x$ would be an upper bound for **P**, contradicting Theorem I.28.

THEOREM I.30. *If $x > 0$ and if $y$ is an arbitrary real number, there exists a positive integer $n$ such that $nx > y$.*

*Proof.* Apply Theorem I.29 with $x$ replaced by $y/x$.

The property described in Theorem I.30 is called the *Archimedean property* of the real-number system. Geometrically it means that any line segment, no matter how long, may be covered by a finite number of line segments of a given positive length, no matter how small. In other words, a small ruler used often enough can measure arbitrarily large distances. Archimedes realized that this was a fundamental property of the straight line and stated it explicitly as one of the axioms of geometry. In the 19th and 20th centuries, non-Archimedean geometries have been constructed in which this axiom is rejected.

From the Archimedean property, we can prove the following theorem, which will be useful in our discussion of integral calculus.

THEOREM I.31. *If three real numbers $a$, $x$, and $y$ satisfy the inequalities*

$$a \leq x \leq a + \frac{y}{n} \tag{I.14}$$

*for every integer $n \geq 1$, then $x = a$.*

*Proof.* If $x > a$, Theorem I.30 tells us that there is a positive integer $n$ satisfying $n(x - a) > y$, contradicting (I.14). Hence we cannot have $x > a$, so we must have $x = a$.

### I 3.11 Fundamental properties of the supremum and infimum

This section discusses three fundamental properties of the supremum and infimum that we shall use in our development of calculus. The first property states that any set of numbers with a supremum contains points arbitrarily close to its supremum; similarly, a set with an infimum contains points arbitrarily close to its infimum.

THEOREM I.32. *Let $h$ be a given positive number and let $S$ be a set of real numbers.*

(a) *If $S$ has a supremum, then for some $x$ in $S$ we have*

$$x > \sup S - h .$$

(b) *If $S$ has an infimum, then for some $x$ in $S$ we have*

$$x < \inf S + h .$$

*Proof of* (a). If we had $x \leq \sup S - h$ for *all* $x$ in $S$, then $\sup S - h$ would be an upper bound for $S$ smaller than its least upper bound. Therefore we must have $x > \sup S - h$ for at least one $x$ in $S$. This proves (a). The proof of (b) is similar.

THEOREM I.33. ADDITIVE PROPERTY. *Given nonempty subsets $A$ and $B$ of* $\mathbf{R}$, *let $C$ denote the set*

$$C = \{a + b \mid a \in A, b \in B\} .$$

(a) *If each of $A$ and $B$ has a supremum, then $C$ has a supremum, and*

$$\sup C = \sup A + \sup B .$$

(b) *If each of $A$ and $B$ has an infimum, then $C$ has an infimum, and*

$$\inf C = \inf A + \inf B .$$

*Proof.* Assume each of $A$ and $B$ has a supremum. If $c \in C$, then $c = a + b$, where $a \in A$ and $b \in B$. Therefore $c \leq \sup A + \sup B$; so $\sup A + \sup B$ is an upper bound for $C$. This shows that $C$ has a supremum and that

$$\sup C \leq \sup A + \sup B .$$

Now let $n$ be any positive integer. By Theorem I.32 (with $h = 1/n$) there is an $a$ in $A$ and a $b$ in $B$ such that

$$a > \sup A - \frac{1}{n}, \qquad b > \sup B - \frac{1}{n}.$$

Adding these inequalities, we obtain

$$a + b > \sup A + \sup B - \frac{2}{n}, \qquad \text{or} \qquad \sup A + \sup B < a + b + \frac{2}{n} \leq \sup C + \frac{2}{n},$$

since $a + b \leq \sup C$. Therefore we have shown that

$$\sup C \leq \sup A + \sup B < \sup C + \frac{2}{n}$$

for every integer $n \geq 1$. By Theorem I.31, we must have $\sup C = \sup A + \sup B$. This proves (a), and the proof of (b) is similar.

THEOREM I.34. *Given two nonempty subsets S and T of* **R** *such that*

$$s \leq t$$

*for every s in S and every t in T. Then S has a supremum, and T has an infimum, and they satisfy the inequality*

$$\sup S \leq \inf T .$$

*Proof.* Each $t$ in $T$ is an upper bound for $S$. Therefore $S$ has a supremum which satisfies the inequality $\sup S \leq t$ for all $t$ in $T$. Hence $\sup S$ is a lower bound for $T$, so $T$ has an infimum which cannot be less than $\sup S$. In other words, we have $\sup S \leq \inf T$, as asserted.

### *I 3.12 Exercises

1. If $x$ and $y$ are arbitrary real numbers with $x < y$, prove that there is at least one real $z$ satisfying $x < z < y$.
2. If $x$ is an arbitrary real number, prove that there are integers $m$ and $n$ such that $m < x < n$.
3. If $x > 0$, prove that there is a positive integer $n$ such that $1/n < x$.
4. If $x$ is an arbitrary real number, prove that there is exactly one integer $n$ which satisfies the inequalities $n \leq x < n + 1$. This $n$ is called the greatest integer in $x$ and is denoted by $[x]$. For example, $[5] = 5$, $[\frac{5}{2}] = 2$, $[-\frac{8}{3}] = -3$.
5. If $x$ is an arbitrary real number, prove that there is exactly one integer $n$ which satisfies $x \leq n < x + 1$.
6. If $x$ and $y$ are arbitrary real numbers, $x < y$, prove that there exists at least one rational number $r$ satisfying $x < r < y$, and hence infinitely many. This property is often described by saying that the rational numbers are *dense* in the real-number system.
7. If $x$ is rational, $x \neq 0$, and $y$ irrational, prove that $x + y$, $x - y$, $xy$, $x/y$, and $y/x$ are all irrational.
8. Is the sum or product of two irrational numbers always irrational?
9. If $x$ and $y$ are arbitrary real numbers, $x < y$, prove that there exists at least one irrational number $z$ satisfying $x < z < y$, and hence infinitely many.
10. An integer $n$ is called *even* if $n = 2m$ for some integer $m$, and *odd* if $n + 1$ is even. Prove the following statements:
    (a) An integer cannot be both even and odd.
    (b) Every integer is either even or odd.
    (c) The sum or product of two even integers is even. What can you say about the sum or product of two odd integers?
    (d) If $n^2$ is even, so is $n$. If $a^2 = 2b^2$, where $a$ and $b$ are integers, then both $a$ and $b$ are even.
    (e) Every rational number can be expressed in the form $a/b$, where $a$ and $b$ are integers, at least one of which is odd.
11. Prove that there is no rational number whose square is 2.

    [*Hint:* Argue by contradiction. Assume $(a/b)^2 = 2$, where $a$ and $b$ are integers, at least one of which is odd. Use parts of Exercise 10 to deduce a contradiction.]