Problem Set 2

1. A is a non-empty set. Suppose $\bar{a} = \sup A$ and $\underline{a} = \inf A$.
(i) Take $\alpha > 0$ and define $C = \{\alpha x \mid x \in A\}$. Show that, $\sup C = \alpha\bar{a}$ and $\inf C = \alpha\underline{a}$.
(ii) Take $\alpha < 0$ and define $C = \{\alpha x \mid x \in A\}$. Show that, $\inf C = \alpha\bar{a}$ and $\sup C = \alpha\underline{a}$.

2. A and B are non-empty sets such that $a \leq b$ for all $a \in A$ and $b \in B$. Show that, $\sup A \leq \inf B$.

3. Let A and B be non-empty bounded sets with $A \subseteq B$. Show that $\sup A \leq \sup B$ and $\inf A \geq \inf B$.

4. Prove or provide counterexample
(i) A finite non-empty set always has maximum and minimum.
(ii) A and B are non-empty sets such that $a < b$ for all $a \in A$ and $b \in B$. Then $\sup A < \inf B$.
(iii) If $a = \sup A$ then we can find a sequence $\{a_n\}_{n=0}^{\infty}$ such that $a_n \in A$ for all n and $\lim a_n = a$.

5. Let $\lim a_n = a$ and $\lim b_n = b$
(i) Take the sequence $\{a_n b_n\}_{n=0}^{\infty}$. Show that $\lim a_n b_n = ab$.
(ii) Suppose $b_n \neq 0$ for all n and $b \neq 0$. Take the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=0}^{\infty}$. Show that $\lim \left(\frac{a_n}{b_n}\right) = \dfrac{a}{b}$.

6. Find the limit of following sequence:
(i) $a_n = \frac{n+3}{2n-3}$, (ii) $a_n = \frac{n}{2^n}$, (iii) $a_n = \left(\frac{1}{n}\right)^{(-1)^n}$
(iv) $a_1 = 0$ and $a_{n+1} = \frac{10-a_n}{2}$
(v) $1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, (5 \text{ zeros}), 1, \ldots$

7. (Squeeze Theorem) If $x_n \leq y_n \leq z_n$ for all n and $\lim x_n = \lim z_n = L$ then $\lim y_n = L$.

8. Prove or provide counterexample
(i) If a sequence is not bounded then it does not have converging subsequence.
(ii) If a monotone sequence diverges then it does not have converging subsequence.
(iii) If a sequence $\{a_n\}_{n=0}^{\infty}$ converges then for every $\epsilon > 0$, there exists N such that $|a_m - a_n| < \epsilon \ \forall m, n \geq N$.
(iv) If $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ both diverge then $\{(a_n + b_n)\}_{n=0}^{\infty}$ also diverge.
(v) If $\sum_{i=1}^{n} a_i = 1$ then $\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}$.