except that the expression on the right-hand side makes no sense at the moment.

It is the pathologies that give rise to the need for rigor. A satisfying resolution to the questions raised will require that we be absolutely precise about what we mean as we manipulate these infinite objects. It may seem that progress is slow at first, but that is because we do not want to fall into the trap of letting the biases of our intuition corrupt our arguments. Rigorous proofs are meant to be a check on intuition, and in the end we will see that they actually improve our mental picture of the mathematical infinite. As a final example, consider something as intuitively fundamental as the associative property of addition applied to the series $\sum_{n=1}^{\infty}(-1)^n$. Grouping the terms one way gives

$$(-1+1)+(-1+1)+(-1+1)+(-1+1)+\cdots = 0+0+0+0+\cdots = 0,$$

whereas grouping in another yields

$$-1+(1-1)+(1-1)+(1-1)+(1-1)+\cdots = -1+0+0+0+0+\cdots = -1.$$

Manipulations that are legitimate in finite settings do not always extend to infinite settings. Deciding when they do and why they do not is one of the central themes of analysis.

2.2 The Limit of a Sequence

An understanding of infinite series depends heavily on a clear understanding of the theory of sequences. In fact, most of the concepts in analysis can be reduced to statements about the behavior of sequences. Thus, we will spend a significant amount of time investigating sequences before taking on infinite series.

Definition 2.2.1. A *sequence* is a function whose domain is $\mathbf{N}$.

This formal definition leads immediately to the familiar depiction of a sequence as an ordered list of real numbers. Given a function $f : \mathbf{N} \to \mathbf{R}$, $f(n)$ is just the nth term on the list. The notation for sequences reinforces this familiar understanding.

Example 2.2.2. Each of the following are common ways to describe a sequence.

(i) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$,

(ii) $(\frac{1+n}{n})_{n=1}^{\infty} = (\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \cdots)$,

(iii) (a_n), where $a_n = 2^n$ for each $n \in \mathbf{N}$,

(iv) (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$.

On occasion, it will be more convenient to index a sequence beginning with $n = 0$ or $n = n_0$ for some natural number n_0 different from 1. These minor variations should cause no confusion. What is essential is that a sequence be an *infinite* list of real numbers. What happens at the beginning of such a list is of little importance in most cases. The business of analysis is concerned with the behavior of the infinite "tail" of a given sequence.

We now present what is arguably the most important definition in the book.

Definition 2.2.3 (Convergence of a Sequence). A sequence (a_n) *converges* to a real number a if, for every positive number ϵ, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

To indicate that (a_n) converges to a, we write either $\lim a_n = a$ or $(a_n) \to a$.

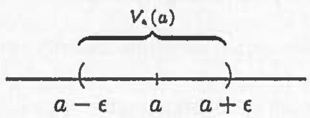
In an effort to decipher this complicated definition, it helps first to consider the ending phrase "$|a_n - a| < \epsilon$," and think about the points that satisfy an inequality of this type.

Definition 2.2.4. Given a real number $a \in \mathbf{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbf{R} : |x - a| < \epsilon\}$$

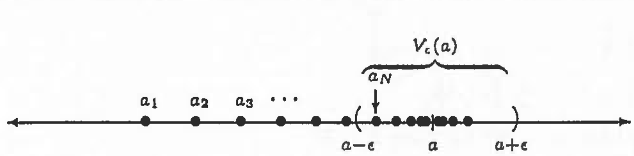
is called the *ϵ-neighborhood of a*.

Notice that $V_\epsilon(a)$ consists of all of those points whose distance from a is less than ϵ. Said another way, $V_\epsilon(a)$ is an interval, centered at a, with radius ϵ.



Recasting the definition of convergence in terms of ϵ-neighborhoods gives a more geometric impression of what is being described.

Definition 2.2.3B (Convergence of a Sequence: Topological Version). A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).



Definition 2.2.3 and Definition 2.2.3B say precisely the same thing; the natural number N in the original version of the definition is the point where the sequence (a_n) enters $V_\epsilon(a)$, never to leave. It should be apparent that *the value of N depends on the choice of ϵ*. The smaller the ϵ-neighborhood, the larger N may have to be.

Example 2.2.5. Consider the sequence (a_n), where $a_n = 1/\sqrt{n}$.

Our intuitive understanding of limits points confidently to the conclusion that

$$\lim\left(\frac{1}{\sqrt{n}}\right) = 0.$$

Before trying to prove this not too impressive fact, let's first explore the relationship between ϵ and N in the definition of convergence. For the moment, take ϵ to be $1/10$. This defines a sort of "target zone" for the terms in the sequence. By claiming that the limit of (a_n) is 0, we are saying that the terms in this sequence eventually get arbitrarily close to 0. How close? What do we mean by "eventually"? We have set $\epsilon = 1/10$ as our standard for closeness, which leads to the ϵ-neighborhood $(-1/10, 1/10)$ centered around the limit 0. How far out into the sequence must we look before the terms fall into this interval? The 100th term $a_{100} = 1/10$ puts us right on the boundary, and a little thought reveals that

$$\text{if} \quad n > 100, \quad \text{then} \quad a_n \in \left(-\frac{1}{10}, \frac{1}{10}\right).$$

Thus, for $\epsilon = 1/10$ we choose $N = 101$ (or anything larger) as our response.

Now, our choice of $\epsilon = 1/10$ was rather whimsical, and we can do this again, letting $\epsilon = 1/50$. In this case, our target neighborhood shrinks to $(-1/50, 1/50)$, and it is apparent that we must travel farther out into the sequence before a_n falls into this interval. How far? Essentially, we require that

$$\frac{1}{\sqrt{n}} < \frac{1}{50} \quad \text{which occurs as long as} \quad n > 50^2 = 2500.$$

Thus, $N = 2501$ is a suitable response to the challenge of $\epsilon = 1/50$.

It may seem as though this duel could continue forever, with different ϵ challenges being handed to us one after another, each one requiring a suitable value of N in response. In a sense, this is correct, except that the game is effectively over the instant we recognize a *rule* for how to choose N given an *arbitrary* $\epsilon > 0$. For this problem, the desired algorithm is implicit in the algebra carried out to compute the previous response of $N = 2501$. Whatever ϵ happens to be, we want

$$\frac{1}{\sqrt{n}} < \epsilon \quad \text{which is equivalent to insisting that} \quad n > \frac{1}{\epsilon^2}.$$

With this observation, we are ready to write the formal argument.

We claim that

$$\lim\left(\frac{1}{\sqrt{n}}\right) = 0.$$

Proof. Let $\epsilon > 0$ be an arbitrary positive number. Choose a natural number N satisfying

$$N > \frac{1}{\epsilon^2}.$$

We now verify that this choice of N has the desired property. Let $n \geq N$. Then,

$$n > \frac{1}{\epsilon^2} \quad \text{implies} \quad \frac{1}{\sqrt{n}} < \epsilon \quad \text{and hence} \quad |a_n - 0| < \epsilon.$$

□

Quantifiers

The definition of convergence given ealier is the result of hundreds of years of refining the intuitive notion of limit into a mathematically rigorous statement. The logic involved is complicated and is intimately tied to the use of the quantifiers "for all" and "there exists." Learning to write a grammatically correct convergence proof goes hand in hand with a deep understanding of why the quantifiers appear in the order that they do.

The definition begins with the phrase,

"*For all* ϵ, *there exists* $N \in \mathbf{N}$ such that ..."

Looking back at our first example, we see that our formal proof begins with, "Let $\epsilon > 0$ be an arbitrary positive number." This is followed by a construction of N and then a demonstration that this choice of N has the desired property. This, in fact, is a basic outline for how every convergence proof should be presented.

TEMPLATE FOR A PROOF THAT $(x_n) \to x$:

- "Let $\epsilon > 0$ be arbitrary."
- Demonstrate a choice for $N \in \mathbf{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N actually works.
- "Assume $n \geq N$."
- With N well chosen, it should be possible to derive the inequality $|x_n - x| < \epsilon$.

Example 2.2.6. Show

$$\lim \left(\frac{n+1}{n}\right) = 1.$$

As mentioned, before attempting a formal proof, we first need to do some preliminary scratch work. In the first example, we experimented by assigning specific values to ϵ (and it is not a bad idea to do this again), but let us skip straight to the algebraic punch line. The last line of our proof should be that for suitably large values of n,

$$\left|\frac{n+1}{n} - 1\right| < \epsilon.$$

Because
$$\left|\frac{n+1}{n}-1\right|=\frac{1}{n},$$
this is equivalent to the inequality $1/n<\epsilon$ or $n>1/\epsilon$. Thus, choosing N to be an integer greater than $1/\epsilon$ will suffice.

With the work of the proof done, all that remains is the formal writeup.

Proof. Let $\epsilon>0$ be arbitrary. Choose $N\in\mathbf{N}$ with $N>1/\epsilon$. To verify that this choice of N is appropriate, let $n\in\mathbf{N}$ satisfy $n\geq N$. Then, $n\geq N$ implies $n>1/\epsilon$, which is the same as saying $1/n<\epsilon$. Finally, this means
$$\left|\frac{n+1}{n}-1\right|<\epsilon,$$
as desired. □

Divergence

Significant insight into the role of the quantifiers in the definition of convergence can be gained by studying an example of a sequence that does not have a limit.

Example 2.2.7. Consider the sequence
$$\left(1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\frac{1}{5},-\frac{1}{5},\frac{1}{5},-\frac{1}{5},\frac{1}{5},-\frac{1}{5},\frac{1}{5},-\frac{1}{5},\frac{1}{5},-\frac{1}{5},\dots\right).$$

How can we argue that this sequence does not converge to zero? Looking at the first few terms, it seems the initial evidence actually supports such a conclusion. Given a challenge of $\epsilon=1/2$, a little reflection reveals that after $N=3$ all the terms fall into the neighborhood $(-1/2,1/2)$. We could also handle $\epsilon=1/4$. (What is the smallest possible N in this case?)

But the definition of convergence says "*For all* $\epsilon>0$...," and it should be apparent that there is no response to a choice of $\epsilon=1/10$, for instance. This leads us to an important observation about the logical negation of the definition of convergence of a sequence. To prove that a particular number x is *not* the limit of a sequence (x_n), we must produce a single value of ϵ for which no $N\in\mathbf{N}$ works. More generally speaking, the negation of a statement that begins "For all P, there exists Q..." is the statement, "For at least one P, no Q is possible..." For instance, how could we disprove the spurious claim that "At every college in the United States, there is a student who is at least seven feet tall"?

We have argued that the preceding sequence does not converge to 0. Let's argue against the claim that it converges to 1/5. Choosing $\epsilon=1/10$ produces the neighborhood $(1/10,3/10)$. Although the sequence continually revisits this neighborhood, there is no point at which it enters and never leaves as the definition requires. Thus, no N exists for $\epsilon=1/10$, so the sequence does not converge to 1/5.

Of course, this sequence does not converge to any other real number, and it would be more satisfying to simply say that this sequence does not converge.

Exercise 2.2.7. Informally speaking, the sequence $\sqrt{n}$ "converges to infinity."

(a) Imitate the logical structure of Definition 2.2.3 to create a rigorous definition for the mathematical statement $\lim x_n = \infty$. Use this definition to prove $\lim \sqrt{n} = \infty$.

(b) What does your definition in (a) say about the particular sequence $(1, 0, 2, 0, 3, 0, 4, 0, 5, 0, \ldots)$?

Exercise 2.2.8. Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

2.3 The Algebraic and Order Limit Theorems

The real purpose of creating a rigorous definition for convergence of a sequence is *not* to have a tool to verify computational statements such as $\lim 2n/(n+2) = 2$. Historically, a definition of the limit like Definition 2.2.3 came 150 years after the founders of calculus began working with intuitive notions of convergence. The point of having such a logically tight description of convergence is so that we can confidently state and prove statements *about convergence sequences in general.* We are ultimately trying to resolve arguments about what is and is not true regarding the behavior of limits with respect to the mathematical manipulations we intend to inflict on them.

As a first example, let us prove that convergent sequences are bounded. The term "bounded" has a rather familiar connotation but, like everything else, we need to be explicit about what it means in this context.

Definition 2.3.1. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbf{N}$.

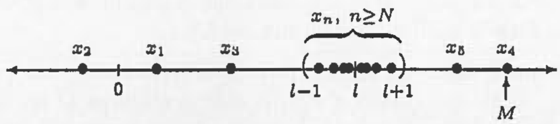
Geometrically, this means that we can find an interval $[-M, M]$ that contains every term in the sequence (x_n).

Theorem 2.3.2. *Every convergent sequence is bounded.*

Proof. Assume (x_n) converges to a limit l. This means that given a particular value of ϵ, say $\epsilon = 1$, we know there must exist an $N \in \mathbf{N}$ such that if $n \geq N$, then x_n is in the interval $(l-1, l+1)$. Not knowing whether l is positive or negative, we can certainly conclude that

$$|x_n| < |l| + 1$$

for all $n \geq N$.



We still need to worry (slightly) about the the terms in the sequence that come before the Nth term. Because there are only a finite number of these, we let

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |l| + 1\}.$$

It follows that $|x_n| \leq M$ for all $n \in \mathbf{N}$, as desired. □

This chapter began with a demonstration of how applying familiar algebraic properties (commutativity of addition) to infinite objects (series) can lead to paradoxical results. These examples are meant to instill in us a sense of caution and justify the extreme care we are taking in drawing our conclusions. The following theorems illustrate that sequences behave extremely well with respect to the operations of addition, multiplication, division, and order.

Theorem 2.3.3 (Algebraic Limit Theorem). *Let* $\lim a_n = a$, *and* $\lim b_n = b$. *Then,*

(i) $\lim(ca_n) = ca$, *for all* $c \in \mathbf{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$, *provided* $b \neq 0$.

Proof. (i) Consider the case where $c \neq 0$. We want to show that the sequence (ca_n) converges to ca, so the structure of the proof follows the template we described in Section 2.2. First, we let ϵ be some arbitrary positive number. Our goal is to find some point in the sequence (ca_n) after which we have

$$|ca_n - ca| < \epsilon.$$

Now,

$$|ca_n - ca| = |c||a_n - a|.$$

We are given that $(a_n) \to a$, so we know we can make $|a_n - a|$ as small as we like. In particular, we can choose an N such that

$$|a_n - a| < \frac{\epsilon}{|c|}$$

whenever $n \geq N$. To see that this N indeed works, observe that, for all $n \geq N$,

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\epsilon}{|c|} = \epsilon.$$

The case $c = 0$ reduces to showing that the constant sequence $(0, 0, 0, \ldots)$ converges to 0. This is addressed in Exercise 2.3.1.

Before continuing with parts (ii), (iii), and (iv), we should point out that the proof of (i), while somewhat short, is extremely typical for a convergence proof. Before embarking on a formal argument, it is a good idea to take an inventory of what we *want* to make less than ϵ, and what we are *given* can be made small for suitable choices of n. For the previous proof, we wanted to make $|ca_n - ca| < \epsilon$, and we were given $|a_n - a| <$ *anything we like* (for large values of n). Notice that in (i), and all of the ensuing arguments, the strategy each time is to bound the quantity we want to be less than ϵ, which in each case is

$$|(\text{terms of sequence}) - (\text{proposed limit})|,$$

with some algebraic combination of quantities over which we have control.

(ii) To prove this statement, we need to argue that the quantity

$$|(a_n + b_n) - (a + b)|$$

can be made less than an arbitrary ϵ using the assumptions that $|a_n - a|$ and $|b_n - b|$ can be made as small as we like for large n. The first step is to use the triangle inequality (Example 1.2.5) to say

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|.$$

Again, we let $\epsilon > 0$ be arbitrary. The technique this time is to divide the ϵ between the two expressions on the right-hand side in the preceding inequality. Using the hypothesis that $(a_n) \to a$, we know there exists an N_1 such that

$$|a_n - a| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_1.$$

Likewise, the assumption that $(b_n) \to b$ means that we can choose an N_2 so that

$$|b_n - b| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_2.$$

The question now arises as to which of N_1 or N_2 we should take to be our choice of N. By choosing $N = \max\{N_1, N_2\}$, we ensure that if $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. This allows us to conclude that

$$\begin{aligned} |(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n \geq N$, as desired.

(iii) To show that $(a_n b_n) \to ab$, we begin by observing that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b|. \end{aligned}$$

In the initial step, we subtracted and then added ab_n, which created an opportunity to use the triangle inequality. Essentially, we have broken up the distance from $a_n b_n$ to ab with a midway point and are using the sum of the two distances to overestimate the original distance. This clever trick will become a familiar technique in arguments to come.

Letting $\epsilon > 0$ be arbitrary, we again proceed with the strategy of making each piece in the preceding inequality less than $\epsilon/2$. For the piece on the right-hand side ($|a||b_n - b|$), if $a \neq 0$ we can choose N_1 so that

$$n \geq N_1 \quad \text{implies} \quad |b_n - b| < \frac{1}{|a|}\frac{\epsilon}{2}.$$

(The case when $a = 0$ is handled in Exercise 2.3.7.) Getting the term on the left-hand side ($|b_n||a_n - a|$) to be less than $\epsilon/2$ is complicated by the fact that we have a variable quantity $|b_n|$ to contend with as opposed to the constant $|a|$ we encountered in the right-hand term. The idea is to replace $|b_n|$ with a worst-case estimate. Using the fact that convergent sequences are bounded (Theorem 2.3.2), we know there exists a bound $M > 0$ satisfying $|b_n| \leq M$ for all $n \in \mathbf{N}$. Now, we can choose N_2 so that

$$|a_n - a| < \frac{1}{M}\frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_2.$$

To finish the argument, pick $N = \max\{N_1, N_2\}$, and observe that if $n \geq N$, then

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b| \\ &\leq M|a_n - a| + |a||b_n - b| \\ &< M\left(\frac{\epsilon}{M2}\right) + |a|\left(\frac{\epsilon}{|a|2}\right) = \epsilon. \end{aligned}$$

(iv) This final statement will follow from (iii) if we can prove that

$$(b_n) \to b \quad \text{implies} \quad \left(\frac{1}{b_n}\right) \to \frac{1}{b}$$

whenever $b \neq 0$. We begin by observing that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b||b_n|}.$$

Because $(b_n) \to b$, we can make the preceding numerator as small as we like by choosing n large. The problem comes in that we need a worst-case estimate on the size of $1/(|b||b_n|)$. Because the b_n terms are in the denominator, we are no longer interested in an upper bound on $|b_n|$ but rather in an inequality of the form $|b_n| \geq \delta > 0$. This will then lead to a bound on the size of $1/(|b||b_n|)$.

The trick is to look far enough out into the sequence (b_n) so that the terms are closer to b than they are to 0. Consider the particular value $\epsilon_0 = |b|/2$. Because $(b_n) \to b$, there exists an N_1 such that $|b_n - b| < |b|/2$ for all $n \geq N_1$. This implies $|b_n| > |b|/2$.

Next, choose N_2 so that $n \geq N_2$ implies

$$|b_n - b| < \frac{\epsilon|b|^2}{2}.$$

Finally, if we let $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = |b - b_n|\frac{1}{|b||b_n|} < \frac{\epsilon|b|^2}{2}\frac{1}{|b|\frac{|b|}{2}} = \epsilon.$$

□

Limits and Order

Although there are a few dangers to avoid (see Exercise 2.3.8), the Algebraic Limit Theorem verifies that the relationship between algebraic combinations of sequences and the limiting process is as trouble-free as we could hope for. Limits can be computed from the individual component sequences provided that each component limit exists. The limiting process is also well-behaved with respect to the order operation.

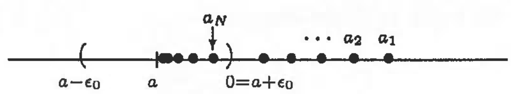
Theorem 2.3.4 (Order Limit Theorem). *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(i) *If* $a_n \geq 0$ *for all* $n \in \mathbf{N}$, *then* $a \geq 0$.

(ii) *If* $a_n \leq b_n$ *for all* $n \in \mathbf{N}$, *then* $a \leq b$.

(iii) *If there exists* $c \in \mathbf{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbf{N}$, *then* $c \leq b$. *Similarly, if* $a_n \leq c$ *for all* $n \in \mathbf{N}$, *then* $a \leq c$.

Proof. (i) We will prove this by contradiction; thus, let's assume $a < 0$. The idea is to produce a term in the sequence (a_n) that is also less than zero. To do this, we consider the particular value $\epsilon_0 = |a|$. The definition of convergence guarantees that we can find an N such that $|a_n - a| < |a|$ for all $n \geq N$. In particular, this would mean that $|a_N - a| < |a|$, which implies $a_N < 0$. This contradicts our hypothesis that $a_N \geq 0$. We therefore conclude that $a \geq 0$.



(ii) The Algebraic Limit Theorem ensures that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$, we can apply part (i) to get that $b - a \geq 0$.

(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbf{N}$, and apply (ii). □

A word about the idea of "tails" is in order. Loosely speaking, limits and their properties do not depend at all on what happens at the beginning of the sequence but are strictly determined by what happens when n gets large. Changing the value of the first ten—or ten thousand—terms in a particular sequence has no effect on the limit. Theorem 2.3.4, part (i), for instance, assumes that $a_n \geq 0$ *for all* $n \in \mathbf{N}$. However, the hypothesis could be weakened by assuming only that there exists some point N_1 where $a_n \geq 0$ for all $n \geq N_1$. The theorem remains true, and in fact the same proof is valid with the provision that when N is chosen it be at least as large as N_1.

In the language of analysis, when a property (such as non-negativity) is not necessarily true about some finite number of initial terms but is true for all terms in the sequence after some point N, we say that the sequence *eventually* has this property. (See Exercise 2.2.8.) Theorem 2.3.4, part (i), could be restated, "Convergent sequences that are eventually nonnegative converge to nonnegative limits." Parts (ii) and (iii) have similar modifications, as will many other upcoming results.

Exercises

Exercise 2.3.1. Show that the constant sequence $(a, a, a, a, \dots)$ converges to a.

Exercise 2.3.2. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Exercise 2.3.3 (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Exercise 2.3.4. Show that limits, if they exist, must be unique. In other words, assume $\lim a_n = l_1$ and $\lim a_n = l_2$, and prove that $l_1 = l_2$.

Exercise 2.3.5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \dots, x_n, y_n, \dots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Exercise 2.3.6. (a) Show that if $(b_n) \to b$, then the sequence of absolute values $|b_n|$ converges to $|b|$.

(b) Is the converse of part (a) true? If we know that $|b_n| \to |b|$, can we deduce that $(b_n) \to b$?

Exercise 2.3.7. (a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

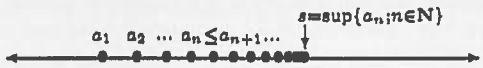
Definition 2.4.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.2 (Monotone Convergence Theorem). *If a sequence is monotone and bounded, then it converges.*

Proof. Let (a_n) be monotone and bounded. To prove (a_n) converges using the definition of convergence, we are going to need a candidate for the limit. Let's assume the sequence is increasing (the decreasing case is handled similarly), and consider the *set* of points $\{a_n : n \in \mathbf{N}\}$. By assumption, this set is bounded, so we can let

$$s = \sup\{a_n : n \in \mathbf{N}\}.$$

It seems reasonable to claim that $\lim(a_n) = s$.



To prove this, let $\epsilon > 0$. Because s is the least upper bound of $\{a_n : n \in \mathbf{N}\}$, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now, the fact that (a_n) is increasing implies that if $n \geq N$, then $a_N \leq a_n$. Hence,

$$s - \epsilon < a_N \leq a_n \leq s < s + \epsilon,$$

which implies $|a_n - s| < \epsilon$, as desired. □

The Monotone Convergence Theorem is extremely useful for the study of infinite series, largely because it asserts the convergence of a sequence without explicit mention of the actual limit. This is a good moment to do some preliminary investigations, so it is time to formalize the relationship between sequences and series.

Definition 2.4.3. Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots.$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges to* B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

2.5 Subsequences and the Bolzano–Weierstrass Theorem

In Example 2.4.5, we showed that the sequence of partial sums (s_m) of the harmonic series does not converge by focusing our attention on a particular *subsequence* (s_{2^k}) of the original sequence. For the moment, we will put the topic of infinite series aside and more fully develop the important concept of subsequences.

Definition 2.5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \cdots$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_j}), where $j \in \mathbf{N}$ indexes the subsequence.

Notice that the order of the terms in a subsequence is the same as in the original sequence, and repetitions are not allowed. Thus if

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots\right),$$

then

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\right) \quad \text{and} \quad \left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \cdots\right)$$

are examples of legitimate subsequences, whereas

$$\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \frac{1}{1000}, \frac{1}{500}, \cdots\right) \quad \text{and} \quad \left(1, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots\right)$$

are not.

Theorem 2.5.2. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. Exercise 2.5.1 □

This not too surprising result has several somewhat surprising applications. It is the key ingredient for understanding when infinite sums are associative (Exercise 2.5.2). We can also use it in the following clever way to compute values of some familiar limits.

Example 2.5.3. Let $0 < b < 1$. Because

$$b > b^2 > b^3 > b^4 > \cdots > 0,$$

the sequence (b^n) is decreasing and bounded below. The Monotone Convergence Theorem allows us to conclude that (b^n) converges to some l satisfying $b > l \geq 0$. To compute l, notice that (b^{2n}) is a subsequence, so $(b^{2n}) \to l$ by Theorem 2.5.2. But $(b^{2n}) = (b^n)(b^n)$, so by the Algebraic Limit Theorem, $(b^{2n}) \to l \cdot l = l^2$. Because limits are unique, $l^2 = l$, and thus $l = 0$.

Without much trouble (Exercise 2.5.5), we can generalize this example to conclude $(b^n) \to 0$ whenever $-1 < b < 1$.

Example 2.5.4 (Divergence Criterion). Theorem 2.5.2 is also useful for providing economical proofs for divergence. In Example 2.2.7, we were quite sure that

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \cdots\right)$$

did not converge to any proposed limit. Notice that

$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \cdots\right)$$

is a subsequence that converges to $1/5$. Also,

$$\left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \cdots\right)$$

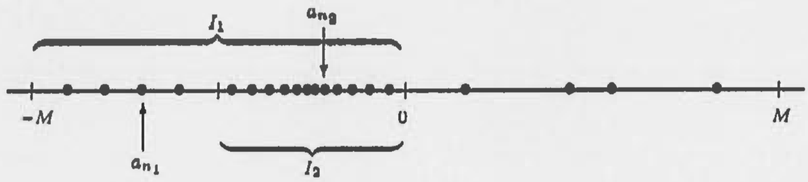
is a different subsequence of the original sequence that converges to $-1/5$. Because we have two subsequences converging to two different limits, we can rigorously conclude that the original sequence diverges.

The Bolzano–Weierstrass Theorem

In the previous example, it was rather easy to spot a convergent subsequence (or two) hiding in the original sequence. For *bounded* sequences, it turns out that it is always possible to find at least one such convergent subsequence.

Theorem 2.5.5 (Bolzano–Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all $n \in \mathbf{N}$. Bisect the closed interval $[-M, M]$ into the two closed intervals $[-M, 0]$ and $[0, M]$. (The midpoint is included in both halves.) Now, it must be that at least one of these closed intervals contains an infinite number of the points in the sequence (a_n). Select a half for which this is the case and label that interval as I_1. Then, let a_{n_1} be some point in the sequence (a_n) satisfying $a_{n_1} \in I_1$.



Next, we bisect I_1 into closed intervals of equal length, and let I_2 be a half that again contains an infinite number of points of the original sequence. Because there are an infinite number of points from (a_n) to choose from, we can select an a_{n_2} from the original sequence with $n_2 > n_1$ and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of points of (a_n) and then select $n_k > n_{k-1} > \cdots > n_2 > n_1$ so that $a_{n_k} \in I_k$.

We want to argue that (a_{n_k}) is a convergent subsequence, but we need a candidate for the limit. The sets

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

form a nested sequence of closed intervals, and by the Nested Interval Property there exists at least one point $x \in \mathbf{R}$ contained in every I_k. This provides us with the candidate we were looking for. It just remains to show that $(a_{n_k}) \to x$.

Let $\epsilon > 0$. By construction, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. (This follows from Example 2.5.3 and the Algebraic Limit Theorem.) Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. Because x and a_{n_k} are both in I_k, it follows that $|a_{n_k} - x| < \epsilon$. □

Exercises

Exercise 2.5.1. Prove Theorem 2.5.2.

Exercise 2.5.2. (a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$