Metric Space

Definition: Metric Space

Take a set X. A function $d : X \times X \to \mathcal{R}$ is called a **Metric** or **distance function** on E if d satisfies the following properties:
(a) $d(p, q) \geq 0$ for all $p, q \in X$. Moreover $d(p, q) = 0$ iff $p = q$.
(b) For all $p, q \in X$, $d(p, q) = d(q, p)$
(c) (Triangle Inequality) $d(p, q) + d(q, r) \geq d(p, r)$ for all $p, q, r \in X$.
$d(p, q)$ is called the distance between p and q. The set X along with d, is called a Metric space.

Observation: Take $X' \subset X$. Then X' along with d is also a metric space.

Examples: Distance Function

1. $X=$ Set of integers; $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$
2. $X = \mathcal{R}$; $d(x, y) = |x - y|$
3. $X = \mathcal{R}^l$; $d(x, y) = \sqrt{\sum_{k=1}^{l} (x_k - y_k)^2}$
4. $X = \mathcal{R}^l$; $d(x, y) = \max_k |x_k - y_k|$
5. $X = \{f \mid f : [a, b] \to \mathcal{R} \text{ and } f \text{ is continuous}\}$; $d(f, g) = max_{t\in[a,b]}|f(t) - g(t)|$

Metric Space

On $\mathcal{R}$, we used the distance function $d(p, q) = |p - q|$. We can now generalize many concepts, introduced earlier on $\mathcal{R}$, in terms of distance function.

Let (X, d) be a metric space.

• $A \subseteq X$ is **bounded** if there exists $q \in A$ and $M > 0$ such that $d(p, q) \leq M$ for all $p \in A$.
Examples: Bounded metric space
1. $X = \mathcal{R}^2$ and $A = \{x \in X \mid x_1^2 + x_2^2 < 1\}$. Here A is bounded.
2. $X = \mathcal{R}^2$ and $A = \{x \in X \mid x_1 = x_2\}$. A is not bounded.

• For any $a \in X$ and $\epsilon > 0$, the ϵ-**neighbourhood** of a is the set of all points in X whose distance from a is strictly less than ϵ. Formally, $B_\epsilon(a) = \{x \in X \mid d(x, a) < \epsilon\}$.

• A sequence in X, $\{a_n\}_{n=1}^{\infty}$, **converges** to a: If for every $\epsilon > 0$, there exists $N \in \mathcal{N}$ such that $\forall n \geq N, \ a_n \in B_\epsilon(a)$.

Examples: Convergent Sequences

1. $X = \mathcal{R}^2$; $d(x, y) = \sqrt{\sum_{k=1}^{2} (x_k - y_k)^2}$; $a_n = (\frac{1}{n}, 1)$. This sequence converges to $(0, 1)$.
2. $X = \{f \mid f : [0, 1] \to \mathcal{R}\}$; $d(f, g) = max_{t \in [a,b]} |f(t) - g(t)|$; $f_n(t) = \frac{t}{n}$. This sequence converges to $f(t) = 0 \ \forall t$.

• A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded** if there exists $M > 0$ such that $d(a_1, a_n) \leq M \ \forall n$

Vector Space

Some of our results are easily generalized with minor modifications. Some results will require a generalized notion of addition and multiplication.

Definition: Vector Space

A linear space or vector space V over scalar $\mathcal{R}$ is a set of elements (called vectors), along with two binary operations $\bigoplus$ (vector addition) and $\bigodot$ (scalar multiplication) that satisfies following properties. Let $v, w, z \in V$ and $c, d \in \mathcal{R}$

1. V close under $\bigoplus$: $v \bigoplus w \in V$,
2. Commutativity of $\bigoplus$: $v \bigoplus w = w \bigoplus v$,
3. Associativity of $\bigoplus$: $(v \bigoplus w) \bigoplus z = v \bigoplus (w \bigoplus z)$
4. Existence of additive identity: $\exists$ a vector called $\mathbf{0}$ such that $v \bigoplus \mathbf{0} = v \; \forall v$
5. Existence of additive inverse: For each v, $\exists \; w \in V$ such that $v \bigoplus w = \mathbf{0}$
6. V close under $\bigodot$: $c \bigodot v \in V$,
7. Associativity of $\bigodot$: $(cd) \bigodot x = c \bigodot (d \bigodot x)$
8. Multiplicative identity: $1 \bigodot v = v$
9. Distributive property: $c \bigodot (v \bigoplus w) = (c \bigodot v) \bigoplus (c \bigodot w)$,
$(c + d) \bigodot v = (c \bigodot v) \bigoplus (d \bigodot v)$

Normed Vector Space

From now on, we shall use the usual addition and multiplication sign for vector space as well. Additive inverse shall be denoted by usual '$-$' sign. That is inverse of vector $z \in V$ is denoted by $-z$. Moreover for $y, z \in V$, $y + (-z)$ will be simply denoted by $y - z$.

Check: Additive inverse is unique.

Definition: Normed Vector Space

A normed vector space is a vector space V with a norm function $|| \ || : V \to \mathcal{R}$. The norm function has to satisfy properties similar to distance function.

(a) $||x|| \geq 0$ for all $x \in V$. Moreover $||x|| = 0$ iff $x = \mathbf{0}$.

(b) For all $x, y \in V$, $||x - y|| = ||y - x||$

(c) For all $c \in \mathcal{R}$ and $x \in V$, $||cx|| = |c| \, ||x||$

(d) (Triangle Inequality) For all $x, y \in V$, $||x + y|| \leq ||x|| + ||y||$

A norm can be easily extended to a distance function in the following manner, $d(x, \mathbf{0}) = ||x||$ and $d(x, y) = d(x - y, \mathbf{0}) = ||x - y||$.

Check: d satisfies all properties of distance function.

Observation: A normed vector space is a metric space with generalized addition and multiplication.

Examples: Normed Vector Spaces

1. $X = \mathcal{R}^l$; $||x|| = \sqrt{\sum_{k=1}^{l} x_k^2}$
2. $X = \{f \mid f : [a, b] \to \mathcal{R}$ and f is continuous$\}$; $||f|| = max_{t \in [a,b]} |f(t)|$

Sequence in Normed Vector Space

Lets check whether our results remain the same in normed vector space.

Result 5 - **Sequence**: Every convergent sequence is bounded.
(**Proof**): Take any sequence $\{a_n\}_{n=1}^{\infty}$ which converges to a. We can find N such that for all $n \geq N$, $d(a_n, a) < 1$. Hence for all $n \geq N$,
$d(a_n, a_1) \leq d(a_1, a) + d(a, a_n) < d(a_1, a) + 1$ (Using Triangle inequality).
Now choose $M = \max\{d(a_1, a_2), d(a_1, a_3), \ldots, d(a_1, a_{N-1}), (d(a_1, 1) + 1)\}$.

Check (**Result 7-Sequence**): Subsequence of a convergent sequence converges to the same limit.

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences in a normed vector space V. $\lim a_n = a$ and $\lim b_n = b$.
(**Result 1-Sequence**) Take the sequence $\{ca_n\}_{n=1}^{\infty}$, where $c \in \mathcal{R}$. $\lim(ca_n) = ca$.
(**Result 2-Sequence**) Take the sequence $\{a_n + b_n\}_{n=1}^{\infty}$. $\lim(a_n + b_n) = a + b$.

Open and Closed Set

The following discussion is applicable for metric spaces and hence for normed vector spaces as well.
Let (X, d) be a metric space and $E \subseteq X$.

Definitions:
$p \in E$ is an **interior point of** E: There exists $\epsilon > 0$ such that $B_\epsilon(p) \subseteq E$.

E is an **open set**: All $p \in E$ are interior points of E.

$p \in X$ is a **limit point** of E if for all $\epsilon > 0$ $B_\epsilon(p)$ contains an element of E other than p. Note, that p may or may be an element of E.

Alternative definition of limit point: $p \in X$ is a **limit point** of E if there is a non trivial sequence in E that converges to p. That is we can find a sequence $\{a_n\}_{n=1}^{\infty}$ such that (i) $a_n \in E\, \forall n$, (ii) $a_n \neq p\, \forall n$, (iii) $\lim a_n = p$.

E is **closed** if all limit points of E belong to E.

Observations:
1. Finite sets are closed set.
2. A set can be both closed and open.
Example: $X = \mathcal{N}$, $E = \{1, 2, 3\}$, $d(x, y) = |x - y|$.
3. A set can be neither open nor closed.
Example: $X = \mathcal{R}^2$, $E = \{x \mid 0 < x_1 < 1, x_2 = 0\}$, $d(x, y) = \sqrt{\sum_{k=1}^{2}(x_k - y_k)^2}$
4. A set can be open but not closed.Example: $X = \mathcal{R}$, $E = (0, 1)$, $d(x, y) = |x - y|$.
5. A set can be closed but not open. Example: $X = \mathcal{R}$, $E = [0, 1]$, $d(x, y) = |x - y|$.

Open and Closed Set

Results

1. (X, d) is a metric space and $E \subseteq X$. E is open $\Leftrightarrow$ $(X \setminus E)$ is closed.

2. If $E_1, E_2, E_3, \ldots$ are open sets in (X, d). Then
(i) $\cup_{k=1}^{\infty} E_k$ (infinite union) is an open set.
(ii) $\cup_{k=1}^{n} E_k$ (finite union) is an open set.
(iii) $\cap_{k=1}^{n} E_k$ (finite intersection) is an open set.
(iv) However $\cap_{k=1}^{\infty} E_k$ (infinite intersection) may not be an open set.

3. If $E_1, E_2, E_3, \ldots$ are closed sets in (X, d). Then
(i) $\cap_{k=1}^{\infty} E_k$ (infinite intersection) is a closed set.
(ii) $\cap_{k=1}^{m} E_k$ (finite intersection) is a closed set.
(iii) $\cup_{k=1}^{m} E_k$ (finite union) is a closed set.
(iv) However $\cup_{k=1}^{\infty} E_k$ (infinite union) may not be a closed set.

Proof of 2: (i) Take any $p \in \cup_{k=1}^{\infty} E_k$, which implies $p \in E_k$ for some k. Since E_k is open there exists $\epsilon > 0$ such that $B_\epsilon(p) \subseteq E_k$. This implies $B_\epsilon(p) \subseteq \cup_{k=1}^{\infty} E_k$. Hence p is an interior point of $\cup_{k=1}^{\infty} E_k$.
(ii) Same as (i).
(iii) Take any $p \in \cap_{k=1}^{m} E_k$, which implies $p \in E_k$ for all k. Since E_k is open there exists $\epsilon_k > 0$ such that $B_{\epsilon_k}(p) \subseteq E_k$. Take $\epsilon = \min_{k=1}^{m} \epsilon_k$. Then $B_\epsilon(p) \subseteq E_k$ for all k. Hence $B_\epsilon(p) \subseteq \cap_{k=1}^{m} E_k$ and p is an interior point of $\cap_{k=1}^{m} E_k$.
(iv) Take $X = \mathcal{R}$, $d(x, y) = |x - y|$, $E_k = (-\frac{1}{k}, \frac{1}{k})$. $\cap_{k=1}^{\infty} E_k = \{0\}$, which is not an open set.

Open and Closed Set

Proof of 3: (i) Take any convergent sequence $\{a_n\}_{n=1}^{\infty}$ in $\cap_{k=1}^{\infty} E_k$, which implies $\{a_n\}_{n=1}^{\infty}$ is a sequence in E_k for all k. Since E_k is closed, $\lim a_n \in E_k$ for all k. Hence $\lim a_n \in \cap_{k=1}^{\infty} E_k$.
(ii) Same as (i).
(iii) Take any convergent sequence $\{a_n\}_{n=1}^{\infty}$ in $\cup_{k=1}^{m} E_k$, which implies there is a subsequence of $\{a_n\}_{n=1}^{\infty}$ in E_k, for some k. Since E_k is closed, the subsequence converges to an element $p \in E_k$. However $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence so it should also converge to $p \in E_k$. Thus $\{a_n\}_{n=1}^{\infty}$ converges to a point in $\cup_{k=1}^{m} E_k$.
(iv) $X = \mathcal{R}$, $d(x, y) = |x - y|$, $E_k = [\frac{1}{k}, 1]$. $\cup_{k=1}^{\infty} E_k = (0, 1]$, which is not closed.

Proof of 1: E is open $\Rightarrow$ $(X \setminus E)$ is closed:
Take a limit point p of $(X \setminus E)$. We want to show that $p \in (X \setminus E)$. We shall prove this by contradiction.
Suppose $p \in E$. Since E is an open set p must be an interior point of E. Then there exists $\epsilon > 0$, such that $B_\epsilon(p) \subseteq E$. So for that ϵ there is no point of $(X \setminus E)$, which is in $B_\epsilon(p)$. This contradicts the fact that p is a limit point of $(X \setminus E)$.
$(X \setminus E)$ is closed $\Rightarrow$ E is open:
Take any $p \in E$. We want to show that p is an interior point of E. Once again we prove by contradiction.
Suppose the contrary is true and for every $\epsilon > 0$, $B_\epsilon(p)$ is not a subset of E. Hence each $B_\epsilon(p)$ contains some element of $(X \setminus E)$ other than p. Hence p is a limit point of $(X \setminus E)$. Since $(X \setminus E)$ is closed, p must be in $(X \setminus E)$, which contradicts with the fact that $p \in E$.

Compact Set

Definition: Compact Set:
Let (X, d) be a metric space and $E \subseteq X$. E is a **compact set** if every sequence has a subsequence that converges to a point in E.

Heine-Borel Theorem: Take $S \subseteq \mathcal{R}^n$ with Euclidian norm. S is compact iff S is closed and bounded.
Proof: We shall prove just one part of this result, S compact $\Rightarrow$ S is closed and bounded.
Take a converging sequence $\{a_n\}_{n=1}^{\infty}$ in S. Since S is compact, $\{a_n\}_{n=1}^{\infty}$ has a converging subsequence that converges to a point in S. But $\{a_n\}_{n=1}^{\infty}$ converges, hence it must converge to the same point in S. Thus S is closed.
Now suppose that S is not bounded then for all $M > 0$, there is a $a_M \in S$ such that $||a_M|| > M$. Take the sequence $\{a_M\}_{M=1}^{\infty}$. There is no subsequence of $\{a_M\}_{M=1}^{\infty}$ which is bounded and hence there is no subsequence which converges. We have reached a contradiction.
The converse follows a proof similar to Bolzano-Weierstrass Theorem.

Observation: A sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathcal{R}^n$ converges to $a \Leftrightarrow$ for every $k = 1, 2, \ldots n$; $\{a_n^k\}_{n=1}^{\infty}$ converges to a^k.