Bound of a Set

Examples of upper and lower bound

1. $S_1 = (0,1) = \{x \mid 0 < x < 1\}$ (open interval 0 to 1), $S_2 = [0,1] = \{x \mid 0 \leq x \leq 1\}$.
Both set are bounded above and below.
$\max S_2 = \sup S_2 = 1$; $\min S_2 = \inf S_2 = 0$ but S_1 does not have maximum and minimum. However, $\sup S_1 = 1, \inf S_1 = 0$.

2. $S_3 = \left\{\frac{1}{n} \mid n \in \mathcal{Q}\right\}$. $\mathcal{Q}$ is the set of positive integers. S_3 is bounded above and below. $\max S_3 = \sup S_3 = 1$; $\inf S_3 = 0$. S_3 does not have minimum.

Illustration of various results

1. Result (iv): $A = (0,1)$ and $B = [1.5,2]$. Then $C = (1.5,3)$. Note that $\sup C = 3 = 2+1 = \sup A + \sup B$.
2. Result (vi): $A = (0,1)$ and $B = (1,2)$. $\sup A = 1 = \inf B$.
3. Nested interval Theorem: $I_n = [-\frac{1}{n}, \frac{1}{n}]$. Then $\cap_{n=1}^{\infty} I_n = \{0\}$

Sequence

Examples: Converging Sequence

1. $x_n = \frac{1}{n}$. This sequence converges to 0.
Check: Take any $\epsilon > 0$. Choose N such that $\frac{1}{N} < \epsilon$ or $N > \frac{1}{\epsilon}$. Then for all $n \geq N$, $a_n \in V_\epsilon(0)$
2. $x_n = \frac{n+1}{n}$. This sequence converges to 1.
Check: Take any $\epsilon > 0$. Choose N such that $\frac{1}{N} < \epsilon$ or $N > \frac{1}{\epsilon}$. Then for all $n \geq N$, $a_n \in V_\epsilon(1)$
3. $x_n = n^{\frac{1}{n}}$. This sequence converges to 1.
Check: First note that x_n lies in the interval $[1,2]$ for all n. Also note that x_n is a decreasing sequence after $n = 3$.
Now, take any $\epsilon > 0$. Choose N such that $N = \frac{1}{1-\epsilon}$. Then $N = 1 + N\epsilon$. By Taylor series expansion, $(1+\epsilon)^N = 1 + N\epsilon + \frac{N(N-1)}{2}\epsilon^2 + \ldots$, implying $(1+N\epsilon) < (1+\epsilon)^N$ because ϵ is positive.
Hence, $N < (1+\epsilon)^N$. That is $N^{\frac{1}{N}} < 1+\epsilon$. Since it is a decreasing sequence, for all $n \geq N$, $x_n \in V_\epsilon(1)$

Examples: Diverging Sequence

4. $x_n = n$. Suppose there is a such that $\lim x_n = a$. Choose $\epsilon = 1$. There is no N such that $x_n \in V_1(a)\, \forall n \geq N$. Thus $\lim x_n \neq a$ for any a. Hence the

sequence diverges.
5. $x_n = 1 + (-1)^n$. Lets take 2 as a potential convergence point (because every even term is 2). However it is not because we can not find any N such that $x_n \in V_1(2)\,\forall n \geq N$ (0 comes back in every odd term). Similarly 0 is not a limit. Check that everything else is easily ruled out.
6. $x_n = \sin\left(\frac{n\pi}{2}\right)$. Argument is similar to (ii). Potential limit points are $0, 1, -1$ but none of them can satisfy the neighbourhood test.

<u>Bounded sequence</u>
All examples above except 4 is bounded sequence. For instance,
Sequence 1 is bounded by $M = 1$, Sequence 2 is bounded by $M = 2$, Sequence 6 is bounded by $M = 1$.

<u>Monotone sequence</u>
Sequence 1, 2 are decreasing sequence; Sequence 4 is increasing sequence, Sequence 5 and 6 are not Monotone sequence.

<u>Subsequence</u>
$x_n = \frac{1}{n}$, that is the sequence is $1, \frac{1}{2}, \frac{1}{3}, \ldots$.
The followings are subsequences of $\{x_n\}_{n=0}^{\infty}$:
(i) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$; (ii) $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$; (iii) $1, \frac{1}{3}, \frac{1}{37}, \ldots$
The followings are not valid subsequence of $\{x_n\}_{n=0}^{\infty}$:
(i) $\frac{1}{2}, \frac{1}{20}, \frac{1}{5}, \frac{1}{50}, \ldots$; (ii) $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \ldots$;

<u>Divergence identification using subsequence</u>
Sequence 5 is divergent because we can find two subsequences $2, 2, 2 \ldots$ (even entries) ; $0, 0, 0 \ldots$ (odd entries); which converge to two different limits.
Similarly Sequence 6 is divergent because we can find two subsequences $0, 0, 0 \ldots$ (even entries); $1, 1, 1 \ldots$ (entries $1, 5, 9, \ldots$); which converge to two different limits.