# Bound of set

Upper bound of a set

Let $S$ be a nonempty set of real numbers. Suppose there is a real number $B$ such that $B \geq x \, \forall x \in S$. Then $B$ is an **upper bound** for $S$. $S$ is said to be **bounded above** by $B$. A set which has no upper bound is said to be **unbounded above**.
Observation 1: There can be many upper bounds for a set.
Observation 2: $B$ need not be a member of $S$

Maximum of a set

If an upper bound $B$ for set $S$ (set of real numbers) is also a member of $S$, then $B$ is the **maximum** of $S$. Thus $B = \max S$ if $B \in S$ and $x \leq B \, \forall x \in S$.
Observation 1: A set may not have maximum (Formally: There exists at least one set which does not have maximum).
Observation 2: Can there be multiple maximum? NO
Observation 3: If $B = \max S$ then $B$ is the smallest upper bound of $S$.

Qn: Is there a smallest upper bound for sets without maximum? Let us first define our intended object formally.

Least upper bound or Supremum of a set

A real number $B$ is called a **least upper bound** (or **supremum**) of a nonempty set $S$, if $B$ has the following two properties:
$(i)$ $B$ is an upper bound for $S$. $(ii)$ No number less than $B$ is an upper bound for $S$.
We denote supremum by $B = \sup S$.
Observation 1: Can there be multiple supremums? NO

# Supremum and Infimum

Completeness Axiom

Every **nonempty** set $S$ of real numbers which is **bounded above** has a supremum; that is, there is a real number $B$ such that $B = \sup S$.

Observation 1: $\sup S$ need not be in $S$.
Observation 2: $\sup S \in S \Leftrightarrow \sup S = \max S$

We can define lower bound, minimum and greatest lower bound (or infimum) in a similar fashion.
A real number $L$ is called a **greatest lower bound** (or **infimum**) of a nonempty set $S$, if $L$ has the following two properties:
$(i)$ $L$ is a lower bound for $S$, that is $L \leq x \, \forall x \in S$. $(ii)$ No number greater than $L$ is a lower bound for $S$. We denote infimum by $L = \inf S$.
If $\inf S \in S$ then $\inf S = \min S$.

Useful results

$(i)$ (Reflection) Let $S$ be a non-empty set. Define $-S = \{-x \mid x \in S\}$.
$B = \sup S \Leftrightarrow -B = \inf -S$.
$(ii)$ Every nonempty set $S$ of real numbers which is bounded below has an infimum.
$(iii)$ (How to find a supremum?) Suppose that $z \in R$ is an upper bound of $S$.
Moreover for every choice of $\epsilon > 0$, there exists an element $a \in S$ such that $a > z - \epsilon$.
$\Leftrightarrow z = \sup S$.
Does this result hold for finite set?
Write a similar result for infimum and prove.

# Properties of Sup and Inf

More results

Let $A$ and $B$ are non-empty and bounded sets (that is bounded above and below) of $\mathcal{R}$.

($iv$) (Addition) Suppose $C = \{a + b \mid a \in A,\ b \in B\}$. Then $\sup C = \sup A + \sup B$.

($v$) (multiplication) Suppose $\alpha > 0$ and $C = \{\alpha a \mid a \in A\}$. Then $\sup C = \alpha \sup A$.

($vi$) (Order) Suppose $A$ and $B$ are such that for every pair $(a, b) \in A \times B$, $a \leq b$, then $\sup A \leq \inf B$.

Write similar properties for infimum and prove.

Result: Nested Interval Theorem

Assume we are given a closed interval $I_n = [a_n, b_n] = \{x \mid a_n \leq x \leq b_n\}$ for each positive integer $n$. Assume further that each $I_n$ contains $I_{n+1}$. That is we have a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \ldots$. Then their intersection is nonempty, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

(**Sketch of a Proof**): Define $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$.

Step 1: $A$ is bounded above (by $b_1$) and bounded below (by $a_1$). Similarly $B$ is bounded.

Step 2: For any pair $a_n, b_m$, show that $a_n \leq b_m$.

Step 3: Invoke Result ($vi$). We have $\sup A \leq \inf B$. Let $a = \sup A$ and $b = \sup B$. Take the interval $[a, b]$.

Step 4: Show that $[a, b] \subseteq \cap_{n=1}^{\infty} I_n$: Take $x \in [a, b] \Rightarrow x \geq a$. Since $\sup A = a$, we have $a \geq a_n \forall n$. Thus $x \geq a_n$. Similarly, $x \leq b_n$. Hence $x \in I_n$ for all $n$, implying $x \in \cap_{n=1}^{\infty} I_n$.

# Proof of some results

(**Proof of Result** (*iii*)):
$\Rightarrow$: $z$ is an upper bound for $S$. We want to show that there is no other upper bound for $S$ which is smaller than $z$. We prove **by contradiction**. That is we start with the negation that a smaller upper bound for $S$ exists and reach a logical contradiction. Suppose $z' < z$ and $z'$ is a upper bound for $S$. Choose $\epsilon = \frac{z-z'}{2}$, which is $> 0$. There is no $a \in S$ such that $a > z - \epsilon$. Because for all $a \in S$, $a \leq z'$ ($z'$ is an upper bound for $S$) and $z' < z - \epsilon$. This contradicts LHS; hence done.
$\Leftarrow$: Let $z = \sup S$. By definition $z$ is an upper bound for $S$. We want to show that for each $\epsilon > 0$, there exists $a \in S$ such that $a > z - \epsilon$. Again we prove by contradiction. We start with the negation. There exists $\epsilon > 0$ such that for all $a \in S$, $a \leq z - \epsilon$. That would mean $(z - \epsilon)$ is an upper bound for $S$. This contradicts with the fact that $z$ is the supremum.

(**Proof of Result** (*iv*)):
We prove the result in two steps - (1) $(\sup A + \sup B) \geq \sup C$ and (2) for every $\epsilon > 0$, there exists $c \in C$ such that $c > (\sup A + \sup B) - \epsilon$. (this is sufficient by Result (*iii*))
Step 1: Take $\sup A + \sup B$. $(\sup A + \sup B) \geq a + b \ \forall a \in A, b \in B$. Equivalently $(\sup A + \sup B) \geq c \ \forall c \in C$. Thus $(\sup A + \sup B)$ an upper bound for $C$. By completeness axiom $C$ has supremum. By definition of supremum, $(\sup A + \sup B) \geq \sup C$.
Step 2: Take any $\epsilon > 0$. $(\sup A + \sup B) - \epsilon = \left(\sup A - \frac{\epsilon}{2}\right) + \left(\sup B - \frac{\epsilon}{2}\right)$. From Result (*iii*), we know that there exists $\tilde{a} \in A$ such that $\tilde{a} > \left(\sup A - \frac{\epsilon}{2}\right)$ and there exists $\tilde{b} \in B$ such that $\tilde{b} > \left(\sup B - \frac{\epsilon}{2}\right)$. Note $\tilde{c} = \tilde{a} + \tilde{b}$ is in $C$ and $\tilde{c} > (\sup A + \sup B) - \epsilon$.

# Sequence

Definition: Sequence

A **sequence** is a function $f : \mathcal{N} \to \mathcal{R}$. $f(n)$ is the $n$-th term on the list. We shall denote a sequence by $\{a_n\}_1^\infty$. So $f(n) = a_n$.

We are interested about the 'tail' of a sequence, that is how it behaves for large $n$, or as $n \to \infty$.

Different ways of writing a sequence

$(i)$ $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$; $(ii)$ $\left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty}$; $(iii)$ $a_1 = 1$, $a_{n+1} = \frac{1}{2}a_n$

Before we proceed, we need one more definition.

Definition: Neighbourhood

For any $a \in \mathcal{R}$ and $\epsilon > 0$, the $\epsilon$-**neighbourhood** of $a$ is the set of all points whose distance from $a$ is strictly less than $\epsilon$. Formally, $B_\epsilon(a) = \{x \mid |x - a| < \epsilon\}$.

Convergence of a sequence

A sequence $\{a_n\}_{n=1}^\infty$ **converges** to $a$: If for every $\epsilon > 0$, there exists $N \in \mathcal{N}$ such that $\forall n \geq N,\ a_n \in B_\epsilon(a)$.

Observation 1: Choice of $N$ is dependent on $\epsilon$.

A sequence which does not converge to any $a \in \mathcal{R}$ is said to **diverge**.

# Important Results

Results: Let $\lim a_n = a$ and $\lim b_n = b$

1. Take the sequence $\{ca_n\}_{n=1}^{\infty}$, where $c \in \mathcal{R}$. $\lim(ca_n) = ca$.
2. Take the sequence $\{a_n + b_n\}_{n=1}^{\infty}$. $\lim(a_n + b_n) = a + b$.
3. If $a_n \leq c$ for all $n$, then $a \leq c$. Similarly, if $a_n \geq c$ then $a \geq c$.
4. If $a_n \leq b_n$ for all $n$, then $a \leq b$.

(**Proof of Result** 1.): If $c = 0$ then it is trivial. Take $c \neq 0$. We want to show that for $\epsilon > 0$, we can find $N$ such that for all $n \geq N$, $ca_n \in B_\epsilon(ca)$.
Lets choose that $N$ for which $a_n \in B_{\frac{\epsilon}{|c|}}(a)$ for all $n \geq N$. Lets check that this $N$ will indeed work. $|ca_n - ca| = |c||a_n - a| < |c|.\frac{\epsilon}{|c|} = \epsilon$.
(**Proof of Result** 2.): We want to show that for $\epsilon > 0$, we can find $N$ such that for all $n \geq N$, $a_n + b_n \in B_\epsilon(a + b)$. Rest of the proof is about construction of such $N$.
Pick $N_1$ such that $a_n \in B_{\frac{\epsilon}{2}}(a)$ for all $n \geq N_1$ (this is possible because $\lim a_n = a$).
Pick $N_2$ such that $b_n \in B_{\frac{\epsilon}{2}}(b)$ for all $n \geq N_2$ (this is possible because $\lim b_n = b$).
Take $N = \max\{N_1, N_2\}$. Lets check that this $N$ will work
$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

(**Proof of Result** 3.): We shall prove this by contradiction. Suppose $a_n \leq c$ for all $n$ but $a > c$. Let us choose $\epsilon = a - c/2$. By convergence, there must exist $N$ such that for all $n \geq N$ such that $a_n \in B_\epsilon(a)$. But then for all such $n$, $a_n > (a - \frac{a-c}{2}) = \frac{a+c}{2} > c$ (the last equality follows from $a > c$). We reach a contradiction.

(**Proof of Result** 4.): Construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n = (b_n - a_n)$ for all $n$. Using Result 1 and 2, $\lim x_n = (\lim b_n - \lim a_n) = b - a$. Since $x_n \geq 0$ for all $n$, by Result 3, $b - a \geq 0$.

# Bounded and Monotone Sequence

Definition: Bounded Sequence
A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded** by $M \in \mathcal{R}$, if for all $n$, $|a_n| \leq M$.

Definition: Monotone Sequence
A sequence $\{a_n\}_{n=1}^{\infty}$ is **monotone** if it is increasing or decreasing. A sequence is increasing if $a_{n+1} \geq a_n$ for all $n$. Similarly a sequence is decreasing if $a_{n+1} \leq a_n \, \forall n$.

Results
5. Every convergent sequence is bounded. However the converse (every bounded sequence is convergent) is not true.
(**Proof**): Take any sequence $\{a_n\}_{n=1}^{\infty}$ which converges to $a$. We can find $N$ such that for all $n \geq N$, $|a_n - a| < 1$. Hence $|a_n| < |a| + 1$ (Triangle inequality).
Now choose $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, (|a| + 1)\}$. Thus for all $n$, $|a_n| \leq M$.
Here is a bounded sequence which does not converge $1, -1, 1, -1, \ldots$

6. If a sequence is monotone and bounded then it converges (to its supremum).
(**Sketch of a proof**): Suppose $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded (do the other case yourself).
Define $S = \{a_1, a_2, \ldots\}$. $S$ is non-empty and bounded and hence has a supremum (completeness axiom). Let $s = \sup S$. We shall show that $\lim a_n = s$. That is for every $\epsilon > 0$, we can find $N$ such that for all $n \geq N$, $a_n \in B_\epsilon(s)$.
Take any $\epsilon > 0$. Since $s = \sup S$, there exist $a_m$ such that $s > a_m > s - \epsilon$. As $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence, for all $n > m$, $s > a_n \geq a_m > s - \epsilon$. Hence we are done.

# Subsequence

Definition: Subequence

A subsequence $\{b_n\}_{n=1}^{\infty}$ of a sequence $\{a_n\}_{n=1}^{\infty}$ is a selection from the original sequence. That is $b_1 = a_{n_1}, b_2 = a_{n_2}, b_3 = a_{n_3}, \ldots$, where $n_1 < n_2 < n_3 < \ldots$
$(i)$ Order of the terms in subsequence same as original sequence, $(ii)$ repetitions are not allowed.

Results

**7**. Subsequence of a convergent sequence converges to the same limit.
(**Sketch of a proof**): Suppose $\{b_n\}_{n=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ and $\lim a_n = a$. We want to show that $\lim b_n = a$. For every $\epsilon > 0$, we want to find $N$ such that for all $n \geq N$, $b_n \in B_\epsilon(a)$. Choose the $N$ such that $n \geq N$, $a_n \in B_\epsilon(a)$. This $N$ will do the job for $b_n$ as well.
**Observation**: In a sequence if we can find two subsequences which converge to different limits, then the original sequence diverges.

# Bolzano-Weierstrass Theorem

**8**. (**Bolzano-Weierstrass Theorem**) Every bounded sequence (convergent or not) of real numbers has a convergent subsequence.
(**Sketch of a proof**): Let $\{a_n\}_{n=1}^{\infty}$ be bounded by $M$. That is all $-M \leq a_n \leq M \, \forall n$.
Step 1: Divide the interval $[-M, M]$ into two equal parts $[-M, 0]$ and $[0, M]$. At least one of these two interval has infinite number of elements from $\{a_n\}_{n=1}^{\infty}$. Call that interval $I_1$ and pick one element of $\{a_n\}_{n=1}^{\infty}$, say $a_{n_1}$ such that $a_{n_1} \in I_1$.
Step 2: Divide $I_1$ in two equal intervals. Once again at least one of these two intervals has infinite number of elements from $\{a_n\}_{n=1}^{\infty}$. Call that interval $I_2$ and pick one element of $\{a_n\}_{n=1}^{\infty}$, say $a_{n_2}$ such that $a_{n_2} \in I_2$ and $n_2 > n_1$. This is possible because $I_2$ has infinite element from $\{a_n\}_{n=1}^{\infty}$.
Repeat step 2 of this algorithm to obtain a subsequence $a_{n_1}, a_{n_2}, \dots$ of $\{a_n\}_{n=1}^{\infty}$. We want to show that this sequence converges. We now need a candidate for limit. Note that the sets $I_1 \supseteq I_2 \supseteq \dots$ by construction. By the 'Nested Interval Theorem', $\cap_k I_k$ is non-empty. Pick any $x^* \in \cap_k I_k$. We shall show that $\lim a_{n_k} = x^*$.
Take any $\epsilon > 0$. Choose $N$ such that the length of $I_N$ is less than $\epsilon$. This is possible because the length of $I_N$ is $M.2^{-N}$, which converges to 0. Thus for all $k \geq N$, $|a_{n_k} - x^*| \leq M.2^{-k} \leq \epsilon$. Hence proved.

# Functional Limit

**Reading**: Simon and Blume Ch. 14

We shall restrict our attention to functions from $\mathcal{R}^l$ to $\mathcal{R}$. From now on, we shall use Eucledian distance.

Definition: Functional limit
If for all nontrivial sequence $\{x_n\}_{n=1}^{\infty}$ in $A$ which converge to $c$, the sequence of functional values $\{f(x_n)\}_{n=1}^{\infty}$ converges to $M$ then **functional limit** of $f$ at $c$ is $M$. This is denoted by $\lim_{x \to c} f(x) = M$.

**Observation**: Note that the value of $f$ at $c$ is not relevant for the limit.

Examples
1. $A = [0, 2)$; $f(x) = 2x$. What is $\lim_{x \to 2} f(x)$? Ans: 4
2. $A = \mathcal{R}$, $f(x) = |x|$. What is $\lim_{x \to 0} f(x)$? Ans: 0
3. $A = \mathcal{R}^2$, $f(x) = x_1 x_2$. What is $\lim_{x \to 0} f(x)$? Ans: 0
4. $A = \mathcal{R}$, $f(x) = \sin\left(\frac{1}{x}\right)$. What is $\lim_{x \to 0} f(x)$?

Take two sequence: $a_n = \frac{1}{2n\pi}$ and $b_n = \frac{1}{\left(2n\pi + \frac{\pi}{2}\right)}$. $\{f(a_n)\}_{n=1}^{\infty}$ converges to 0 but $\{f(b_n)\}_{n=1}^{\infty}$ converges to 1. Hence $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Alternative definition: Functional limit
If for every $\epsilon > 0$, there exists $\delta > 0$ such that $x \in V_\delta(c)$ and $x \neq c$ implies $f(x) \in V_\epsilon(M)$ then **functional limit** of $f$ at $c$ is $M$.

# Functional Limit

**Observation** Two definitions of functional limit are equivalent.

**Sketch of a proof**: First defn. $\Rightarrow$ Second defn.: Let $\lim_{x\to c} f(x) = M$ by first definition. We shall prove by contradiction. Suppose there exists an $\epsilon$, for which we can not find a $\delta$. That is for for all $\delta$, there exist a point $x$, which is in $\delta$-neighbourhood of $c$ but $f(x)$ is not in $\epsilon$-neighbourhood of $M$. We can choose $\delta = \frac{1}{k}$ for $k = 1, 2, \ldots$. For each we shall get $x_k$ ($x_k \neq c$) such that $d(x_k, c) < \frac{1}{k}$ but $d(f(x_k), M) \geq \epsilon$. This $\{x_k\}_{k=1}^{\infty}$ is a non-trivial sequence which converges to $c$ but $\{f(x_k)\}_{k=1}^{\infty}$ does not converge to $M$. We have reached a contradiction.
Second defn. $\Rightarrow$ First defn.: Let $\lim_{x\to c} f(x) = M$ by second definition. Take a non-trivial sequence $\{x_k\}_{k=1}^{\infty}$ which converges to $c$. We want to show that $\{f(x_k)\}_{k=1}^{\infty}$ converges to $M$. Take any $\epsilon > 0$. By second definition, we can find a $\delta > 0$ such that for all $x \in B_\delta(x)$ implies $f(x) \in B_\epsilon(M)$. Since $\lim x_k = c$, given $\delta$, I can find $N$ such that $x_k \in B_\delta(x)$ for all $k \geq N$. Hence $f(x_k) \in B_\epsilon(M)$ for all $k \geq N$. Therefore $\{f(x_k)\}_{k=1}^{\infty}$ converges to $M$.

# Continuity

Definition: Continuity

Let $f : A \rightarrow \mathcal{R}$, where $A \subseteq \mathcal{R}^l$. $f$ is **continuous** at $c \in A$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $x \in V_\delta(c)$ implies $f(x) \in V_\epsilon(f(c))$.

**Observation 1**: This is the same definition as functional limit with $\lim_{x \to c} = f(c)$, except that we have removed the restriction on $x \neq c$. Equivalently it removes the restriction on non-trivial sequence in first definition.
**Observation 2**: If $f$ is continuous at every point in the domain $A$, then we say that $f$ is continuous on $A$.

Examples
1. $A = \mathcal{R}$, $f(x) = |x|$ is continuous at 0. In fact $f$ is continuous on $A$.
2. $A = \mathcal{R}^2$, $f(x) = x_1 x_2$ is continuous function on $A$.
3. $A = \mathcal{R}^2$, $f(x) = \max\{x_1, x_2\}$ is continuous function on $A$.
4. $A = \mathcal{R}$, $f(x) = \lceil x \rceil$ = smallest integer $\geq x$. $f$ is not continuous at integers.
4. Any function defined on finite domain is continuous.

Useful Results
Let $f : A \rightarrow \mathcal{R}$ and $g : A \rightarrow \mathcal{R}$ are continuous at $c \in A$. Then
1. For $\alpha \in \mathcal{R}$, $\alpha f(x)$ is continuous at $c$.
2. $f(x) + g(x)$ is continuous at $c$.
3. $f(x)g(x)$ is continuous at $c$.
4. $f(x)/g(x)$ is continuous at $c$.
5. $f : A \rightarrow \mathcal{R}$ is continuous at $c \in A$. $g : f(A) \rightarrow \mathcal{R}$ is continuous at $f(c)$. Then $f \circ g$ is continuous at $c$.

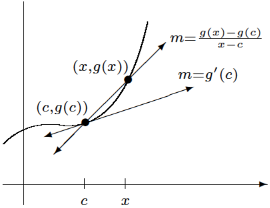
# Intermediate Value Theorem

Intermediate Value Theorem: $f : [a, b] \to \mathcal{R}$ is continuous function such that $f(a) \geq 0$ and $f(b) \leq 0$ (or the opposite). Then there exists $c \in [a, b]$ such that $f(c) = 0$.

**Sketch of a proof**: Lets take the case $f(a) > 0$ and $f(b) < 0$. Define $S = \{x \in [a, b] \mid f(x) > 0\}$.
$S$ is non-empty because $a \in S$. Thus $S$ has a supremum, denote it by $c$. $c$ must be smaller than $b$ because $b \notin S$. We want to show that $f(c) = 0$.
Suppose that $f(c) > 0$. Lets show that there is an interval $(c, c + \delta_1)$ where $f(x) > 0$. This will contradict the fact that $c$ is supremum of $S$.
By continuity, for an $\epsilon \in (0, f(c))$, we can find $\delta_1 > 0$ such that $x \in V_{\delta_1}(c)$ implies $f(x) \in V_\epsilon(f(c))$. Thus for all $x$ in $c < x < c + \delta_1$, $f(x) > f(c) - \epsilon > 0$.
Now suppose that $f(c) < 0$. Lets show that there is an interval $(c - \delta_2, c)$ such that $f(x) < 0$. Then $c$ can not be the supremum (second definition of supremum).
Once again by continuity, for an $\epsilon \in (-f(c), 0)$, we can find $\delta_2 > 0$ such that $x \in V_{\delta_2}(c)$ implies $f(x) \in V_\epsilon(f(c))$. Thus for all $x$ in $c - \delta_2 < x < c$, $f(x) < f(c) + \epsilon < 0$.
Hence $f(x) = 0$.

# Derivative of function from R to R

This section will deal with functions $f : \mathcal{R} \to \mathcal{R}$. Derivative of $f$ at $c$ is the slope of graph of $f$ at $c$. The difference quotient $(f(x) - f(c))/(x - c)$ represents the line through two points $(x, f(x)$ and $(c, f(c))$. We take the functional limit of this quotient as $x$ approaches $c$ to get the slope of tangent at $c$.




Definition: Derivative

**Derivative** of $f$ at $c$ is $f'(c) = \lim_{x \to c} \frac{(f(x)-f(c))}{(x-c)}$ provided this limit exists. Otherwise $f$ is not differentiable at $c$.

Examples:

1. $f(x) = x^2$. $f'(c) = \lim_{x \to c} \frac{(x^2-c^2)}{(x-c)} = \lim_{x \to c}(x + c) = 2c$
2. $f(x) = |x|$. $f'(0) = \lim_{x \to 0} \frac{(|x|-|c|)}{(x-c)}$ does not exist because the limit depends on whether we take positive or negative sequence.

# Derivative of function from R to R

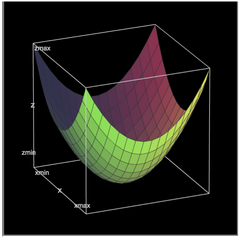
**Observation 1**: If $f$ is differentiable at $c$ then $f$ is continuous at $c$. The converse is not true.
Proof: $\lim_{x\to c} f(x) - f(c) = \lim_{x\to c}\left[\frac{(f(x)-f(c))}{(x-c)}(x-c)\right] =$
$\left[\lim_{x\to c}\frac{(f(x)-f(c))}{(x-c)}\right][\lim_{x\to c}(x-c)] = f'(c)\cdot 0 = 0$. Hence, $\lim_{x\to c} f(x) = f(c)$

**Observation 2**: Let $f$ be differentiable at $c$. If $h$ is small then $f(c+h)$ is approximated by $[f(c)+f'(c)h]$. This follows from the definition of derivative.

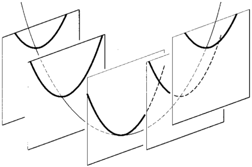
**Result 3**: Let $f$ and $g$ have derivative at $c$. Then
$(i)$ $(f+g)'(c) = f'(c)+g'(c)$
$(ii)$ $(fg)'(c) = f'(c)g(c)+g'(c)f(c)$
$(iii)$ $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c)-g'(c)f(c)}{(g(c))^2}$
$(iv)$ If $f$ is differentiable at $c$ and $g$ is differentiable at $f(c)$ then
$(g\circ f)'(c) = g'(f(c))f'(c)$

**Proof of** $(iv)$: Define $d(y) = \frac{g(y)-g(f(c))}{y-f(c)} - g'(f(c))$. Note that $d(y)$ is defined for all $y\neq f(c)$ and $\lim_{y\to f(c)} d(y) = 0$. To complete the definition, choose $d(f(c)) = 0$ so that $d$ is continuous at $f(c)$. we can rewrite the above equation as
$g(y)-g(f(c)) = [g'(f(c))+d(y)](y-f(c))$.
For all $t\neq c$, (using the above) we have $\frac{g(f(t))-g(f(c))}{t-c} = [g'(f(c))+d(f(t))]\frac{f(t)-f(c)}{t-c}$.
Taking $\lim_{t\to c}$, we obtain
$(g\circ f)'(c) = (\lim_{t\to c}[g'(f(c))+d(f(t))])\left(\lim_{t\to c}\frac{f(t)-f(c)}{t-c}\right) = g'(f(c))f'(c)$

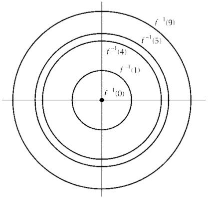
# Visualizing multivariable functions




(a) $f(x, y) = x^2 + y^2$



(b) cross section along $x = c$




*Level curves of $z = x^2 + y^2$.*

Figure: Level curves

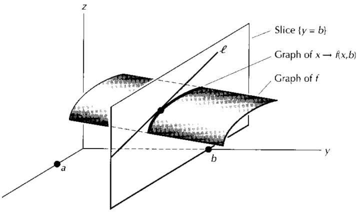
# Derivative of function from $\mathcal{R}^l$ to $\mathcal{R}$

Take $f : A \to \mathcal{R}$, where $A \subseteq \mathcal{R}^l$. First we shall study partial derivative of $f$.

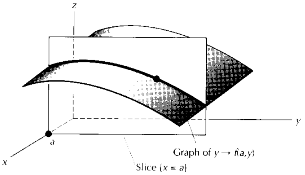
Partial Derivative

**Partial derivative** of $f(x_1, x_2, \ldots, x_l)$ at $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_l)$, with respect to $x_i$ is defined as

$$\frac{\partial f}{\partial x_i} = \lim_{t \to 0} \frac{f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_i + t, \ldots, \bar{x}_l) - f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_i, \ldots, \bar{x}_l)}{t} = \lim_{t \to 0} \frac{f(\bar{x} + te_i) - f(\bar{x})}{t}$$




*The graph of $x \mapsto f(x, b)$ on the slice $\{y = b\}$.*




*The graph of $y \mapsto f(a, y)$ on the slice $\{x = a\}$.*

In effect while calculating partial derivative, we treat $f$ as a function of one variable at a time.

Example: $f(x_1, x_2) = x_1 x_2$. Partial of $f$ at $\bar{x}$ is $\frac{\partial f}{\partial x_1} = \bar{x}_2$, $\frac{\partial f}{\partial x_2} = \bar{x}_1$

# Derivative of function from $\mathcal{R}^l$ to $\mathcal{R}$

Notation:
1. $Df(\bar{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_l}\right)$ calculated at $\bar{x}$.
2. If the partial derivative of $f$ exists for all $i = 1, 2 \ldots, l$ for all $x \in A$ and these partial derivatives are continuous functions in $A$ then we say that $f \in \mathcal{C}^1(A)$ (called $f$ is continuously differentiable on $A$).

Derivative

$f$ is differentiable at $\bar{x} \in A$ if there is a $1 \times l$ vector $\gamma$ such that

$$\lim_{y \to \bar{x}} \frac{[f(y) - f(\bar{x}) - \gamma \cdot (y - \bar{x})]}{||y - \bar{x}||} = 0$$

**Observations**

1. Partial derivatives of $f$ exists at $\bar{x}$ and $\gamma = Df(\bar{x})$.

Proof: Since the limit of $\frac{[f(y) - f(\bar{x}) - \gamma \cdot (y - \bar{x})]}{||y - \bar{x}||}$ exists, all sequences have the same limit. We take the sequence $\bar{x} + te_i$ where $t \to 0$. Hence $\lim_{t \to 0} \frac{[f(\bar{x}+te_i) - f(\bar{x}) - \gamma \cdot (te_i)]}{||te_i||} = 0$
$\Rightarrow \lim_{t \to 0} \frac{f(\bar{x}+te_i) - f(\bar{x})}{t} = \gamma \cdot (e_i) \Rightarrow \frac{\partial f}{\partial x_i}(\bar{x}) = \gamma_i$

2, If $h$ is small then $f(\bar{x} + h)$ is approximated by $[f(\bar{x}) + Df(\bar{x}) \cdot h]$. Equivalently $[f(\bar{x} + h) - f(\bar{x})]$ is approximated by $Df(\bar{x}) \cdot h$, which is called **Total derivative**.

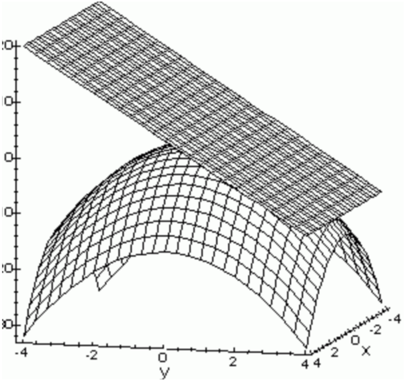
3 $f$ is continuous at $\bar{x}$.

Proof: For all $y \neq \bar{x}$, we can write
$[f(y) - f(\bar{x})] = \frac{[f(y) - f(\bar{x}) - \gamma \cdot (y - \bar{x})]}{||y - \bar{x}||} ||y - \bar{x}|| + Df(x) \cdot (y - \bar{x})$. Taking limit as $y \to \bar{x}$, we get our result.

# Derivative of function from $\mathcal{R}^l$ to $\mathcal{R}$

**Result** (without proof): Take $f : A \to \mathcal{R}$, where $A \subseteq \mathcal{R}^l$. If partial derivatives of $f$ exists and are continuous in some neighbourhood $V_\epsilon(\bar{x})$ around $\bar{x}$, then $f$ is differentiable at $\bar{x}$.

**Diagram of Tangent plane**:



# Derivative of function from $\mathcal{R}^l$ to $\mathcal{R}$

**Chain rule**:
Sometime we shall deal with situations where $x_1, x_2, \ldots, x_l$ are functions of a parameter $t \in \mathcal{R}$. Then we can write a composite function $g(t) = f(x_1(t), \ldots, x_l(t))$, where $g : \mathcal{R} \to \mathcal{R}$. We may want to know how $g$ changes with $t$. When $f$ and $x_k$ are continuously differentiable for all $k$, This is given by
$g'(t_0) = Df(x(t_0)) \cdot x'(t_0)$, where $x'(t_0) = (x_1'(t_0), x_2'(t_0), \ldots, x_l'(t_0))$
(We omit the proof, which is similar to chain rule in $\mathcal{R} \to \mathcal{R}$)

**Second order partial derivative** of $f$ with respect to $x_i$ and $x_j$ at $\bar{x}$ is defined as
$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(\bar{x}) = \left(\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)\right)(\bar{x})$ for all $i = 1, 2, \ldots l$ and $j = 1, 2, \ldots, l$

If first and second order partial derivative of $f$ exists for all for all $x \in A$ and these partial derivatives are continuous functions in $A$ then we say that $f \in \mathcal{C}^2(A)$.

If $f \in \mathcal{C}^2(A)$ then $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(\bar{x}) = \left(\frac{\partial^2 f}{\partial x_j \partial x_i}\right)(\bar{x})$ for all $i, j$.

Second order partial derivative matrix is also called **Hessian matrix**

$$D^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_l} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_l} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_l \partial x_1} & \frac{\partial^2 f}{\partial x_l \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_l \partial x_l} \end{bmatrix}$$