## Math preliminaries: Logic

<u>Examples: Proposition</u>
($P_1$) Today is hotter than yesterday.
($P_2$) Tobacco is good for health.
($P_3$) Inflation causes unemployment.

<u>Examples: These are not Proposition</u>
($i$) What is today's temperature?
($ii$) In my opinion tobacco is good for health.
($iii$) Inflation is defined as the rate at which prices for goods and services is rising.

<u>Negation of a proposition</u>
($\sim P_1$) Today is cooler than yesterday.
($\sim P_2$) Tobacco is bad for health.
($\sim P_3$) Inflation does not cause unemployment.

<u>Combining proposition</u>
Proposition $P$: Today is hotter than yesterday.
Proposition $Q$: Today is more humid than yesterday.
Proposition ($P$ & $Q$):Today is hotter and more humid than yesterday.
Proposition ($P$ or $Q$): Today is either hotter or more humid (or both) than yesterday.

<u>Proposition with quantifiers</u>
($P_4$) For all: Inflation always causes unemployment. Here $x$ is an economy and $A$ is the set of all economies.
($P_5$) There exists: There exists at least one economy where inflation causes unemployment.
($P_6$) For all: Tobacco is always bad for health. Here $x$ is an person and $A$ is the entire population.
($P_7$) There exists: There exists at least one person for whom tobacco is bad.

<u>Negation of proposition with quantifiers</u>
($\sim P_4$) There exists at least one economy where inflation does not cause unemployment.
($\sim P_5$) Inflation never causes unemployment.
($\sim P_6$) There exists at least one person for whom tobacco is good.
($\sim P_7$) Tobacco is always good for health.

<u>Logical implication</u>
1. Example of $P \Rightarrow Q$ is True:

Malaria implies fever. Here $A$ is entire population. $P_8(x)$: $x$ is suffering from Malaria; $Q_8(x)$: $x$ has fever. This Proposition is True because if one is suffering from Malaria, she must have fever.

2. Example of $P \Rightarrow Q$ is False:
GDP growth implies poverty reduction. Here $A$ is the set of all economies. $P_9(x)$: GDP has grown for an economy $x$; $Q_9(x)$: Poverty has decreased in $x$. This Proposition is false because it is possible that in an economy sum of income (GDP) of all agents have increased but income of those below poverty level has decreased.

3. Example of $P \Rightarrow Q$ is 'vacuously' True:
$S$ is my only wife $\Rightarrow$ she is my favourite wife. Here $A$ is the set of all women except $S$. $P_{10}(x)$: $x$ is my wife; $Q_{10}(x)$: I prefer $x$ to $S$. Our proposition is True because $P_{10}(x)$ is never True (as $S$ is my only wife).

4. Example of $P \Rightarrow Q$ is True but $Q \Rightarrow P$ is False:
$P_8 \Rightarrow Q_8$ is True but $Q_8 \Rightarrow P_8$ is False.

5. Example of $P \Rightarrow Q$ is True $Q \Rightarrow P$ is True. That is $P \Leftrightarrow Q$:
Suppose supply of oil is fixed. Demand of oil increases $\Leftrightarrow$ equilibrium price of oil increases

6. Example of $P \Rightarrow Q$ is False and $Q \Rightarrow P$ is False:
Both $P_9 \Rightarrow Q_9$ and $Q_9 \Rightarrow P_9$ are False.

## Math preliminaries: Function

<u>Examples of function</u>

1. $A = \{\text{Bread, Butter, Milk, Meat, Chapati}\}$, $B = \{\text{Lunch, Breakfast}\}$

$$f_1(x) = \begin{cases} \text{Breakfast} & \text{if } x = \text{Bread, Butter, Milk} \\ \text{Lunch} & \text{if } x = \text{Meat, Chapati} \end{cases}$$

2. $A = \{n \mid n \text{ is a positive and odd integer}\}$, $B = \mathcal{R}$, $f_2(n) = n + 1$.

3. $A = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$, $B = \{(u, v) \mid |u| \leq 1, |v| \leq 2\}$,
$f_3(x, y) = (x, 2y)$

<u>These are not functions from A to B</u>

$1'$. $A = \{\text{Bread, Butter, Milk, Meat, Chapati, Wine}\}$, $B = \{\text{Lunch, Breakfast}\}$

$$f_{1'}(x) = \begin{cases} \text{Breakfast} & \text{if } x = \text{Bread, Butter, Milk} \\ \text{Lunch} & \text{if } x = \text{Meat, Chapati} \end{cases}$$

$2'$. $A = \{n \mid n \text{ is a positive and odd integer}\}$, $B = \mathcal{R}$, $f_{2'}(n) = \pm\sqrt{n}$.

$3'$. $A = \{(x, y) \mid |x| \le 1, |y| \le 1\}$, $B = \{(u, v) \mid |u| \le 1, |v| \le 2\}$,
$f_{3'}(x, y) = (2x, 2y)$

<u>Examples of inverse</u>

1. $B = \{\text{Lunch, Breakfast}\}$, $A = \{\text{Bread, Butter, Milk, Meat, Chapati}\}$

$$f_1^{-1}(y) = \begin{cases} \{\text{Bread, Butter, Milk}\} & \text{if } y = \text{Breakfast} \\ \{\text{Meat, Chapati}\} & \text{if } y = \text{Lunch} \end{cases}$$

2. $f(A) = \{n \mid n \text{ is a positive and even integer}\}$, $A = \{n \mid n \text{ is a positive and odd integer}\}$,
$f_2^{-1}(n) = n - 1$.

4. $A = \{(x, y) \mid |x| \le 1, |y| \le 1\}$, $B = \{(u, v) \mid |u| \le 1, |v| \le 2\}$,
$f_3^{-1}(u, v) = (u, \frac{v}{2})$.

5. $A = \{x \in \mathcal{R} \mid x \ge 2\}$, $B = \mathcal{R}_+$, $h(x) = x^2 - 2x$.
$h^{-1}(y) = \sqrt{y + 1} + 1$.

<u>Examples of composition</u>

$A = \mathcal{R}_+$, $f(x) = x^2, g(x) = \ln(x + 1)$. $f$ and $g$ are functions from $A$ to $A$.
$g \circ f$ and $f \circ g$ are also functions from $A$ to $A$. $g(f(x)) = \ln(x^2 + 1)$ and $f(g(x)) = (\ln(x + 1))^2$

<u>Examples of different types of functions</u>

4. $f_1$ is onto but not one-to-one
5. $f_2$ above is one-to-one but not onto
6. The following function is neither onto nor one-to-one.
$A = B = \{n \mid n \text{ is a nonnegative integer}\}$,

$$f_6(x) = \begin{cases} 1 & \text{if } x = \text{is even} \\ 0 & \text{if } x = \text{is odd} \end{cases}$$

7. $f_3$ is a bijection.

8. Example showing $f(A_1) \cap f(A_2) \supset f(A_1 \cap A_2)$
$A = B = \mathcal{R}$, $A_1 = \{x \mid x \geq -1\}$, $A_2 = \{x \mid x \leq 1\}$

$$f_8(x) = \begin{cases} x+1 & \text{if } x \geq -2 \\ -x & \text{if } x < 2 \end{cases}$$

$f_8(A_1) \cap f_8(A_2) = \mathcal{R}_+$ but $f_8(A_1 \cap A_2) = [0, 2]$. Note that $f_8$ is not one-to-one.

9. More examples of inverse
Sometime it may not be possible to write the inverse with familiar operations but still the inverse can be a function. Take $A = B = \mathcal{R}_+$ and $f(x) = \sqrt{x}+x$. Check that it is a strictly increasing function and hence it must be a one-to-one function. Moreover one can show that $f$ is a bijection (try). Hence $f^{-1}$ is a function but we do not get a 'closed form'. We shall learn later (implicit function theorem) that we can still say something about the shape of $f^{-1}$.