Set and basic set operations

Set

A **set** is a collection of elements. Let X be a set. If x is an element in X then $x \in X$, otherwise $x \notin X$. A set with no element is called an **empty set** and denoted by $\emptyset$.

$A \subseteq B$ (A is a **subset** of B): All elements of A are member of B. Note that $\emptyset \subseteq A$ for any set A.
$A \subset B$ (A is **strict subset** of B): $A \subseteq B$ and there is at least one x such that $x \in B$ but $x \notin A$.
$A = B$ (A is **equal** to B): If $A \subseteq B$ and $B \subseteq A$ then $A = B$. That is, all elements in A and B are the same.
A^c (**Complement** of A): Suppose all sets under consideration are subset of a fixed set U (which can be context specific). $A^c = \{x \in U \mid x \notin A\}$.

Basic set operations
Union of A and B: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
Intersection of A and B: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
A **set minus** B: $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

Useful results
1. Distributive laws: (i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, ($ii$) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2. De Morgan's laws: (i) $(A \cup B)^c = A^c \cap B^c$, ($ii$) $(A \cap B)^c = A^c \cup B^c$

Set and basic set operations

Proof of Distributive law(1-i)

We want to show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. We prove this by showing $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

First, lets show that $LHS \subseteq RHS$. Take any $x \in A \cap (B \cup C)$. This means $x \in A$ and $x \in B \cup C$. Moreover $x \in B \cup C$ means either (i) $x \in B$, or (ii) $x \in C$.
If (i) is the case, then $x \in B$ and $x \in A$ together means that $x \in A \cap B$. Hence $x \in (A \cap B) \cup (A \cap C)$ because $(A \cap B) \cup (A \cap C)$ is a superset of $(A \cup B)$.
On the other hand, if (ii) is the case, then $x \in C$ and $x \in A$ means that $x \in A \cap C$. Hence, once again, $x \in (A \cap B) \cup (A \cap C)$ because $(A \cap B) \cup (A \cap C)$ is a superset of $(A \cap C)$.
Thus no matter which of case (i) or (ii) is true, we get $x \in (A \cap B) \cup (A \cap C)$. Since we started from an arbitrary member x of $A \cap (B \cup C)$, we can claim that each member of $A \cap (B \cup C)$ is also member of $(A \cap B) \cup (A \cap C)$. Hence $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Next, lets show that $RHS \subseteq LHS$. Take any $y \in (A \cap B) \cup (A \cap C)$. This means either, (i) $y \in (A \cap B)$ or (ii) $y \in (A \cap C)$. We shall show that in both case $y \in A \cap (B \cup C)$.
If (i) is true then $y \in (A \cap B)$. It means $y \in A$ and $y \in B$. But $y \in B$ makes $y \in B \cup C$, because $B \cup C$ is a superset of B. Thus we have, $y \in A$ and $y \in B \cup C$, together which means $y \in A \cap (B \cup C)$.
We can follow the same steps for case (ii). Here $y \in (A \cap C)$, means $y \in A$ and $y \in C$. But $y \in C$ makes $y \in B \cup C$, because $B \cup C$ is a superset of C. Once again $y \in A$ and $y \in B \cup C$ together means $y \in A \cap (B \cup C)$.
Irrespective of case (i) or (ii), we obtain that y is a member of $A \cap (B \cup C)$. Thus $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Logic

Proposition

A **proposition** is a statement which is either true or false. Let P be a proposition. It can have two 'truth values': True, False.
$\sim P$ is **negation** of P. $\sim P$ is a statement that is 'opposite' to P. $\sim P$ takes truth value True when P is False and $\sim P$ takes truth value False when P is True.

For any 2 propositions P and Q, there are 4 possible assignments of truth values.
(1) P is True and Q is True (2) P is True and Q is False.
(3) P is False and Q is True (4) P is False and Q is False.

We can combine 2 propositions.
$(P \mathbin{\&} Q)$ is a new proposition. This combined proposition means both P and Q. It is true in case 1 and false in cases 2, 3 and 4.
(P or Q) is another proposition. This combined proposition means either P or Q or both. So it is true in cases 1, 2, 3 and false in case of 4.

Proposition with quantifiers

Most of the time we would deal with Proposition given an underlying 'state'. Let A be the set of all relevant states. Instead of checking just P, we shall check $P(x)$ that is validity of P at x.
For all: Suppose a proposition P is true at all states $x \in A$. Our new proposition with quantifier is, $P(x)$ is True $\forall x$.
There exists: If there is at least one $x \in A$ such that P is True at state x. Our new proposition with quantifier is, $\exists x$ such that $P(x)$ is True (there exists).

Logic

Negation of proposition with quantifiers

Negation of [$P(x)$ is True $\forall x$]: $\exists x$ such that $P(x)$ is False

Negation of [$\exists x$ such that $P(x)$ is True]: $P(x)$ is False $\forall x$

Logical implication

P **implies** Q (denoted by $P \Rightarrow Q$) : This itself is a Proposition. We check validity of propositions $P(x)$ and $Q(x)$ for all $x \in A$. The Proposition $P \Rightarrow Q$ is True if for all $x \in A$ either of the following three hold:

(1) $P(x)$ is True and $Q(x)$ is True, (2) $P(x)$ is False and $Q(x)$ is True (4) $P(x)$ is False and $Q(x)$ is False.

The Proposition $P \Rightarrow Q$ is False if there exists at least one $x \in A$ such that $P(x)$ is True and $Q(x)$ is False.

Note that the set $\{x \mid P(x) \text{ is True}\}$ is a subset of $\{x \mid Q(x) \text{ is True}\}$. We also say that Q is necessary for P. Or P is sufficient for Q.

$Q \Rightarrow P$ is called the converse of the statement $P \Rightarrow Q$. There is no logical connection between these two statements. However if both are True, then we denote that by $P \Leftrightarrow Q$.

Useful observation

[$P \Rightarrow Q$] and [$\sim Q \Rightarrow \sim P$] are the same statement

Reading: SB, Appendix A1

Function, Inverse function and Composition of function

Definitions of function and related concepts

Take two sets A and B. A **function** f is a rule that maps each element of A to exactly one element in B.
That is $f(a) \in B \, \forall a \in A$. We denote this by $f : A \longrightarrow B$.
A is called the **domain** of f.

Note 1: More than one elements of A can be mapped to the same element in B.
Note 2: Some members of B may not be reached by f.

Take any $A' \subseteq A$. If A' is the restricted domain, then **image/Range** A' under f is the set $f(A') = \{b \in B \mid f(a) = b, a \in A'\}$.
Example: Indifference curve

Definitions of inverse of a function

Take a function $f : A \longrightarrow B$. Inverse of f, denoted by f^{-1}, is the route back from B to A.
For each $b \in f(A)$, $f^{-1}(b) = \{a \in A \mid f(a) = b\}$.
Note that f^{-1} is not necessarily a function. (QN: Why?)
Take any $B' \subseteq B$. **Preimage** of B' under f is $f^{-1}(B') = \{a \in A \mid f(a) = b, b \in B'\}$.

Composition of functions

We can **compose** two functions to get a new function in the following manner.
Suppose $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Then $g \circ f$ is the composite function which maps each element of A to exactly one element in C, through a temporary stop at B.
$g \circ f(a) = g(f(a)) \; \forall a \in A$.

Set operations on image and preimage

Results: Set operations on preimage

1. $B_1 \subseteq B_2 \subseteq B \Longrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$
2. $B_1, B_2 \subseteq B \Longrightarrow$

(i) $f^{-1}(B_1) \cup f^{-1}(B_2) = f^{-1}(B_1 \cup B_2)$
(ii) $f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2)$
(iii) $f^{-1}(B_1) - f^{-1}(B_2) = f^{-1}(B_1 - B_2)$

Are these results the same for image?

3. $A_1 \subseteq A_2 \subseteq A \Longrightarrow f(A_1) \subseteq f(A_2)$
4. $A_1, A_2 \subseteq A \Longrightarrow$

(i) $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$
(ii) $f(A_1) \cap f(A_2) \supseteq f(A_1 \cap A_2)$
(iii) $f(A_1) - f(A_2) \subseteq f(A_1 - A_2)$

Type of function

Definitions: Types of function

$f : A \longrightarrow B$ is **injective (one-to-one)** if $f(a) = f(a') \Longrightarrow a = a'$.

$f : A \longrightarrow B$ is **surjective (onto)** if for every $b \in B$ there exists $a \in A$ s.t. $b = f(a)$.

A function that is both one-to-one and onto is called **bijective**.

Useful results

5. If f is bijective then

(i) f^{-1} is a function from B to A

(ii) f^{-1} is bijective.

(iii) $f^{-1}(f(a)) = a \, \forall a \in A$ and $f(f^{-1}(b)) = b \forall b \in B$.

6. If f is one-to-one then $4-(ii)$ and $4-(iii)$ hold with equality.

7. If f is one-to-one then f is a bijective function from A to $f(A)$.

Definition: Cardinality of set

Cardinality is total number of elements in a set. Two sets A and B have the same cardinality, denoted by $|A| = |B|$, if there exists a bijective function from A to B.

Reading: SB 13.5

Vector

<u>Definitions of Cartesian product</u>

Take two sets A and B. The Cartesian product of A and B is,
$A \times B = \{(x, y) \mid x \in A, y \in B\}$.
A typical element of $A \times B$ is an ordered pair (x, y) where x is from A and y from B.
Similarly we can define Cartesian product of n sets, $A_1 \times A_2 \times \ldots \times A_n$.

A special case: $A_1 = A_2 \ldots = A_n$, is often denoted by A^n.
We shall focus on $\mathcal{R}^n$, where $\mathcal{R}$ is the set of all real numbers.

A vector in $\mathcal{R}^n$: $x = (x_1, x_2, \ldots, x_n)$ where $x_k \in \mathcal{R}\ \forall n$.
Some special vectors: $\mathbf{0} = (0, \ldots, 0)$, $e_j = (0, 0, \ldots, \underbrace{1}_{\text{j th position}}, \ldots, 0)$

<u>Basic operations on vector</u>

Sum: $x + y = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
$-x = (-x_1, -x_2, \ldots, -x_n)$. Hence,
Difference: $x - y = (x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$
Scalar multiplication: Let $\alpha \in \mathcal{R}$, $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$

<u>Useful observations</u>

$\alpha, \alpha_1, \alpha_2 \in \mathcal{R}$ and $x, y \in \mathcal{R}^n$
1. $x + y = y + x$
2. $(\alpha_1 + \alpha_2)x = \alpha_1 x + \alpha_2 x$
3. $\alpha(x + y) = \alpha x + \alpha y$

Inner product and Norm

Definitions of Inner product of two vectors

This operation assigns a number to each pair of vectors. This is also connected to angle between two vectors (we will skip this geometric interpretation for now).

$x \cdot y = x_1 y_1 + x_2 y_2 \ldots x_n y_n = \sum_{i=1}^{n} x_i y_i$

Useful observations: $\alpha \in \mathcal{R}$ and $x, y, z \in \mathcal{R}^n$

1. $x \cdot y = y \cdot x$
2. $x \cdot (y + z) = x \cdot y + x \cdot z$
3. $x \cdot (\alpha y) = \alpha (x \cdot y)$

Definitions of Norm

Norm is the length of a vector. It is the distance between $\mathbf{0}$ and x.

$||x|| = \sqrt{(x \cdot x)} = \sqrt{\sum_{i=1}^{n} x_i^2}$

Useful observations: $\alpha \in \mathcal{R}$ and $x \in \mathcal{R}^n$

1. $||x|| \geq 0$ and $||x|| = 0 \Leftrightarrow x = 0$
2. $||\alpha x|| = |\alpha| \, ||x||$

Two important inequalities: $x, y \in \mathcal{R}^n$

1. Cauchy-Scwarz inequality: (i) $|x \cdot y| \leq ||x|| \, ||y||$, (ii) equality hold iff $x = \alpha y$ for some $\alpha \in \mathcal{R}$
2. Triangle inequality: (i) $||x + y|| \leq ||x|| + ||y||$, (ii) equality hold iff $x = \alpha y$ for some $\alpha > 0$

Proof of Cauchy-Scwarz and Triangle inequalities

Proof of Cauchy-Scwarz:

(i) Define $z = x + ty$. We know that $z \cdot z \geq 0$ and equality hold if $z = 0$
$z \cdot z = (x + ty) \cdot (x + ty) = x \cdot x + t^2(y \cdot y) + 2t(x \cdot y) = ||x||^2 + t^2||y||^2 + 2t(x \cdot y) \geq 0$
Choose $t = -\frac{x \cdot y}{||y||^2}$ and rearrange the above inequality to obtain Cauchy-Scwarz inequality.
Qn: Why do use this particular t?

(ii) Equality holds $\Rightarrow x + ty = 0, \Rightarrow x = -ty$.
If $x = \alpha y$, check that Cauchy-Scwarz holds with equality.

Proof of Triangle inequality:

(i) $||x + y||^2 = (x + y) \cdot (x + y) = ||x||^2 + ||y||^2 + 2(x \cdot y) \leq ||x||^2 + ||y||^2 + 2(||x|| \, ||y||)$
The last inequality follows from Cauchy-Scwarz
Taking square-root, we obtain Triangle inequality.

(ii) Equality holds $\Rightarrow$ Cauchy-Scwarz hold with equality $\Rightarrow x = \alpha y$.
If $x = \alpha y$, check that Triangle inequality holds with equality.

Reading: SB, Chapter 10