**Problem Set 3**

1. Check whether following are metric space
$(i)$ $X = \mathcal{R}_+$ (set of strictly positive real numbers), $d(x,y) = |\ln\left(\frac{x}{y}\right)|$
$(ii)$ $X = \mathcal{R}^2$, $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$
$(iii)$ $X = \mathcal{R}^2$, $d(x,y) = \sqrt{\sum (x_1 - y_1)^2 + (x_2 - y_2)^2}$
$(iv)$ $S = \{f : [0,1] \to \mathcal{R} | f \text{ is continuous}\}$, $d(f,g) = \int_0^1 |f(t) - g(t)| dt$

2. Check whether the following are vector space
$(i)$ $V = \{x \in \mathcal{R}^2 | x_1 = x_2\}$ with usual binary operations. That is $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $c(x_1, x_2) = (cx_1, cx_2)$
$(ii)$ Unit circle in $\mathcal{R}^2$ with above binary operations.
$(iii)$ $V$=Set of all $n \times n$ symmetric matrices with usual matrix operations.

3. Show that union and intersection of two compact sets are compact.

4. $X$ is a metric space and $p \in E$. Show that $\epsilon$ neighbourhood of $p$ is an open set.

5. Construct a bounded set of real numbers which has exactly three limit points.

6. $X = \mathcal{R}^2$, $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ is a metric space. Draw $B_1(\mathbf{0})$, that is a neighbourhood of length 1 around $\mathbf{0} = (0,0)$.

7. Prove or provide counterexample
$(i)$ Subset of a metric space is also a metric space.
$(ii)$ Subset of a vector space is also a vector space.
$(iii)$ All open sets in $\mathcal{R}$ with Eucledian distance are convex sets.
$(iv)$ Additive identity of a vector space is unique.

8. $X = \mathcal{R}$. $d(p,q) = 1$ if $p \neq q$; $d(p,q) = 0$ otherwise.
$(a.)$ Show that this is a metric space.
$(b.)$ Identify all open sets, closed sets and compact sets in this metric space.
$(c.)$ Does Heine-Borel Theorem (a set is compact iff it is closed and bounded) hold for the above norm?

9. $V$ is a vector space with binary operations $\bigoplus$ and $\bigodot$.
$(a)$ Show that for all $x \in V$, $0 \bigodot x = \mathbf{0}$ (where $\mathbf{0}$ is additive identity of $V$).
$(b)$ Suppose $S$, a non-empty subset of $V$ satisfies the following properties: $(i)$ $a, b \in S$ implies $a \bigoplus b \in S$, $(ii)$ $a \in S$, $c \in \mathcal{R}$ implies $c \bigodot \in S$. Show that $S$ with binary operations $\bigoplus$ and $\bigodot$ is also a vector space.

10. Show that two definitions of limit point are equivalent.