Problem Set 5

1. Prove that, regardless of the value of b, there is at most one point in the interval $-1 \le x \le 1$ for which $x^3 - 3x + b = 0$.

2. $f(x) = 1 - x^{\frac{2}{3}}$. Show that $f(1) = f(-1)$ but there is no $x \in [-1, 1]$ such that $f'(x) = 0$. Does this violate Rolle's theorem?

3. This is the 'general version' of 'intermediate value theorem for continuous function'.
Assume that $f : \mathcal{R} \to \mathcal{R}$ is a continuous function everywhere on a closed interval $[a, b]$. Then f takes on every value between $f(a)$ and $f(b)$ in the interval (a, b). That is for every m such that

$$\min\{f(a), f(b)\} \le m \le \max\{f(a), f(b)\}$$

there exists $c \in (a, b)$ such that $f(c) = m$. Prove this statement.

4. This is the 'general version' of 'intermediate value theorem for derivative'. Assume that $f : \mathcal{R} \to \mathcal{R}$ has derivative everywhere on an open interval (a, b). Then f' takes on every value between $f'(a)$ and $f'(b)$ in the interval (a, b). That is for every m such that

$$\min\{f'(a), f'(b)\} \le m \le \max\{f'(a), f'(b)\}$$

there exists $c \in (a, b)$ such that $f'(c) = m$. This exercise outlines a proof.
(a) Define $g(x) = \frac{f(x)-f(a)}{x-a}$ if $x \neq a$ and $g(a) = f'(a)$.
Using 'general version' of 'intermediate value theorem for continuous function' show that g takes on every value between $f'(a)$ and $g(b)$. Now use the 'mean value theorem for derivative' show that f' takes on every value between $f'(a)$ and $g(b)$.
(b) Define $h(x) = \frac{f(x)-f(b)}{x-b}$ if $x \neq a$ and $h(b) = f'(b)$.
Using 'general version' of 'intermediate value theorem for continuous function' show that h takes on every value between $f'(b)$ and $h(a)$. Now use the 'mean value theorem for derivative' show that f' takes on every value between $f'(b)$ and $h(a)$.
(c) Show that $h(a) = g(b)$ and complete the proof.

5. Sundaram:
Chapter 4: 1,2,3,4(a,b,d,e)
Chapter 5: 1,2,3(a,b,c),7,11
Chapter 6: 1,2,3,6,8,9