Number and Stability of Walrasian Equilibria

Ram Singh

Microeconomic Theory

Lecture 9-10

Excess Demand Function: Basics I

Consider a $N \times M$ economy: Let, $M = 2$ and $\mathbf{p} = (p, 1)$ be a price vector, where $p > 0$.

Let $\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}))$ be the excess demand function.

$\mathbf{p}^*$ is an equilibrium price vector if and only if

$$z_1(\mathbf{p}^*) = 0 \textit{ and } z_2(\mathbf{p}^*) = 0.$$

That is, iff

$$\begin{aligned} z_1(\mathbf{p}^*) &= 0 \\ \vdots &= \vdots \\ z_M(\mathbf{p}^*) &= 0; \end{aligned}$$

Clearly,

$$[(z_1(\mathbf{p}), z_2(\mathbf{p})) = (0, 0)] \text{ iff } z_1(\mathbf{p}) = 0$$

Excess Demand Function: Basics II

Lemma

For $M = 2$, Price vector $\mathbf{p} = (p, 1)$ is equilibrium price vector of a 2×2 economy iff $z_1(\mathbf{p}) = 0$. That is, iff $z_{M-1}(\mathbf{p}) = 0$

For any $N \times M$ economy, consider

- a price vector say $\mathbf{p} = (p_1, p_2, ..., p_M)$
- another price vector $\mathbf{p}' = \frac{1}{p_M}\mathbf{p} = (\frac{p_1}{p_M}, \frac{p_2}{p_M}, ..., 1) = (p'_1, p'_2, ..., 1)$
- Individual and aggregate demand under $\mathbf{p}'$ will be exactly the same as under $\mathbf{p}$.

So, WLOG we can consider vectors in the set

$$\mathbb{P} = \{\mathbf{p} | \mathbf{p} \in \mathbb{R}^M_{++}, \text{ and } p_M = 1\}$$

Excess Demand Function: Basics III

Let

$$\mathbf{p}_{\smallfrown M} = (p_1, ..., p_{M-1}) \text{ and}$$

$$\mathbf{z}_{\smallfrown M} = (z_1(\mathbf{p}), ..., z_{M-1}(\mathbf{p}))$$

Therefore,

$$\mathbf{p} = (\mathbf{p}_{\smallfrown M}, 1) \text{ and}$$

$$\mathbf{z} = (\mathbf{z}_{\smallfrown M}, \mathbf{z}_M)$$

Again, a price vector $\mathbf{p} = (p_1, ..., p_{M-1}, 1)$ is an equilibrium price vector if it solves the $M \times M$ system $\mathbf{z} = (\mathbf{z}_{\smallfrown M}, \mathbf{z}_M) = \mathbf{0}$, i.e., if it solves the system:

$$\begin{aligned} z_1(\mathbf{p}) &= 0 \\ \vdots &= \vdots \\ z_{M-1}(\mathbf{p}) &= 0 \\ z_M(\mathbf{p}) &= 0. \end{aligned}$$

Excess Demand Function: Basics IV

From Walras Law: $p_1 z_1(\mathbf{p}) + ... + p_{M-1} z_{M-1}(\mathbf{p}) + p_M z_M(\mathbf{p}) = 0$. If

$$\begin{aligned} z_1(\mathbf{p}) &= 0 \\ \vdots &= \vdots \\ z_{M-1}(\mathbf{p}) &= 0; \end{aligned}$$

then $z_M(\mathbf{p}) = 0$.

Proposition

A price vector $\mathbf{p} = (p_1, ..., p_{M-1}, 1)$ *is an equilibrium price vector iff it solves the following system of* $M - 1$ *equations:* $\mathbf{z}_{\backsim M}(\mathbf{p}) = \mathbf{0}$*, i.e., iff it solves the system:*

$$\begin{aligned} z_1(\mathbf{p}) &= 0 \\ \vdots &= \vdots \\ z_{M-1}(\mathbf{p}) &= 0. \end{aligned}$$

Local Uniqueness of WE: Two Goods I

For a $N \times 2$ economy:

Definition

An **equilibrium** price vector $\mathbf{p} = (p_1, 1)$ is called **regular** if $z_1'(\mathbf{p}) \neq 0$.

Definition

An $N \times 2$ economy is regular if every **equilibrium** price vector $\mathbf{p} = (p_1, 1)$ is regular.

Theorem

A regular equilibrium price vector $\mathbf{p} = (p_1, 1)$ *is locally unique. That is, there exists an* $\epsilon > 0$ *such that: for every* $\mathbf{p}' = (p_1', 1)$, $\mathbf{p}' \neq \mathbf{p}$, *and* $\|\mathbf{p}' - \mathbf{p}\| < \epsilon$, *we have*

$$\mathbf{z}(\mathbf{p}') \neq \mathbf{0}.$$

Local Uniqueness of WE: Two Goods II

Proof Suppose, $\mathbf{p} = (p_1, 1)$ is an equilibrium price vector, i.e.,

$$\mathbf{z}(\mathbf{p}) = \mathbf{0}, i.e., \; z_1(\mathbf{p}) = 0.$$

Now, consider an infinitesimal change in $\mathbf{p}$, say $\mathbf{dp} \neq \mathbf{0}$. Let $\mathbf{dp} = (dp_1, 0)$, $dp_1 < \epsilon$ and

$$\mathbf{p}' = \mathbf{p} + \mathbf{dp} = (p_1 + dp_1, 1)$$

Since $\mathbf{p} = (p_1, 1)$ is **regular**, we have $z_1'(\mathbf{p}) \neq 0$. Therefore,

$$dp_1 z_1'(\mathbf{p}) \neq 0.$$

Using Taylor series approximation, we can write

$$z_1(\mathbf{p}') \approx z_1(\mathbf{p}) + dp_1 z_1'(\mathbf{p}) \neq 0.$$

Therefore,

$$\begin{aligned} z_1(\mathbf{p}') &\neq 0, i.e., \\ \mathbf{z}(\mathbf{p}') &\neq \mathbf{0}. \end{aligned}$$

That is, $\mathbf{p}'$ is not WE.

Number of a WE: Two goods I

Let

$$\mathbb{E} = \{\mathbf{p} | \mathbf{p} \in \mathbb{P}, \text{ and } \mathbf{z}(\mathbf{p}) = \mathbf{0}\}.$$

Note: $\mathbb{E} \subset\subset \mathbb{P} \subseteq \mathbb{R}^M_{++}$.

Remark

If an economy is regular, the set $\mathbb{E}$ is discrete.

Proposition

When 'Boundary conditions' on $\mathbf{z}(\mathbf{P})$ *hold,* $\mathbb{E}$ *is bounded.*

Proof: Suppose, $\mathbf{p}^* = (p_1^*, 1) \in \mathbb{E}$ is a equilibrium price vector, i.e., $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

For a two goods Economy: Boundary conditions on $z(.)$ imply that

- $z_1(.) > 0$ for very small p_1
- $z_1(.) < 0$ for very high p_1

Number of a WE: Two goods II

- Therefore, p_1^* is finite and bounded away from 0 and ∞
- That is, $\mathbf{p}^*$ is finite and bounded away from $\mathbf{0}$

Therefore, the set $\mathbb{E}$ is bounded.

Proposition

Assuming that $\mathbf{z}$ *is continuous in* $\mathbf{p}$, $\mathbb{E}$ *is* compact *- bounded and closed.*

Hint: Consider a sequence of prices in $\mathbb{E}$.

- the sequence is bounded
- it has a convergent sub-sequence - From Bolzano-Weierstrass Theorem, Every bounded sequence in R^n has a convergent subsequence.
- Let $\bar{\mathbf{p}}$ be the limit of the subsequence
- Since $\mathbf{z}$ is continuous $\mathbf{z}(\bar{\mathbf{p}}) = \mathbf{0}$, so $\bar{\mathbf{p}} \in \mathbb{E}$
- So, $\mathbb{E}$ is closed.

Number of a WE: Two goods III

Next, we use the following result:

Theorem

If a set is compact and discrete, then it has to be finite.

Theorem

If an economy is regular, there are only finitely many equilibrium prices.

Since $\mathbb{E}$ is bounded, closed and discrete, it is a finite set.

Theorem

If an economy is regular and the 'boundary conditions' on $\mathbf{z}(\mathbf{P})$ *hold, then*

- *Either there will be a unique equilibrium*
- *The number of equilibria will be odd.*

Local Uniqueness of WE: M Goods I

Define the $M-1 \times M-1$ matrix of first order derivatives:

$$Dz_{\frown M}(\mathbf{p}) = \begin{pmatrix} \frac{\partial z_1(\mathbf{p})}{\partial p_1} & \frac{\partial z_1(\mathbf{p})}{\partial p_2} & \cdots & \frac{\partial z_1(\mathbf{p})}{\partial p_{M-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z_{M-1}(\mathbf{p})}{\partial p_1} & \frac{\partial z_{M-1}(\mathbf{p})}{\partial p_2} & \cdots & \frac{\partial z_{M-1}(\mathbf{p})}{\partial p_{M-1}} \end{pmatrix}$$

Definition

An equilibrium price vector $\mathbf{p} = (p_1, ..., p_{M-1}, 1)$ is **regular** if the $M-1 \times M-1$ matrix, $D\mathbf{z}_{\frown M}(.)$, is non-singular at $\mathbf{p} = (p_1, ..., p_{M-1}, 1)$.

Definition

An economy is regular if every **equilibrium** price vector $\mathbf{p} = (p_1, ..., p_{M-1}, 1)$ is regular.

Local Uniqueness of WE: M Goods II

Theorem

A regular equilibrium price vector $\mathbf{p} = (p_1, ..., p_{M-1}, 1)$ *is locally unique. That is, there exists an* $\epsilon > 0$ *such that: for every* $\mathbf{p}' = (p'_1, ..., p'_{M-1}, 1)$*,* $\mathbf{p}' \neq \mathbf{p}$*, and* $\|\mathbf{p}' - \mathbf{p}\| < \epsilon$*, we have*

$$\mathbf{z}(\mathbf{p}') \neq \mathbf{0}.$$

Proof Suppose, $\mathbf{p} = (p_1, ..., p_{M-1}, 1)$ is an equilibrium price vector, i.e.,

$$\mathbf{z}(\mathbf{p}) = \mathbf{0}.$$

Now, consider an infinitesimal change in $\mathbf{p}$, say $\mathbf{dp} \neq \mathbf{0}$. Let $\mathbf{dp} = (dp_1, ..., dp_{M-1}, 0)$, and

$$\mathbf{p}' = \mathbf{p} + \mathbf{dp} = (p_1 + dp_1, ..., p_{M-1} + dp_{M-1}, 1)$$

Local Uniqueness of WE: M Goods III

Since, $D\mathbf{z}_{\frown M}(\mathbf{p})$ is non-singular, we have

$$D\mathbf{z}_{\frown M}(\mathbf{p})\mathbf{dp}_{\frown M} = \begin{pmatrix} \frac{\partial z_1(\mathbf{p})}{\partial p_1} & \frac{\partial z_1(\mathbf{p})}{\partial p_2} & \cdots & \frac{\partial z_1(\mathbf{p})}{\partial p_{M-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z_{M-1}(\mathbf{p})}{\partial p_1} & \frac{\partial z_{M-1}(\mathbf{p})}{\partial p_2} & \cdots & \frac{\partial z_{M-1}(\mathbf{p})}{\partial p_{M-1}} \end{pmatrix} \begin{pmatrix} dp_1 \\ dp_2 \\ \vdots \\ dp_{M-1} \end{pmatrix} \quad (1)$$

$$D\mathbf{z}_{\frown M}(\mathbf{p})\mathbf{dp}_{\frown M} \neq \mathbf{0}. \quad \text{Why?}$$

$$\mathbf{z}_{\frown M}(\mathbf{p}') \approx \mathbf{z}_{\frown M}(\mathbf{p}) + D\mathbf{z}_{\frown M}(\mathbf{p})\mathbf{dp}_{\frown M} \neq \mathbf{0}.$$

Therefore,

$$\begin{aligned} \mathbf{z}_{\frown M}(\mathbf{p}') &\neq \mathbf{0}, \textit{i.e.}, \\ \mathbf{z}(\mathbf{p}') &\neq \mathbf{0}. \end{aligned}$$

That is, $\mathbf{p}'$ is not WE.

Number of a WE I

Let

$$\mathbb{E} = \{\mathbf{p} | \mathbf{p} \in \mathbb{P}, \text{ and } \mathbf{z}(\mathbf{p}) = \mathbf{0}\}.$$

Remark

Note: $\mathbb{E} \subset\subset \mathbb{P} \subseteq \mathbb{R}^M_{++}$. Also, if an economy is regular, the set $\mathbb{E}$ is discrete.

Proposition

When 'Boundary conditions' on $\mathbf{z}(\mathbf{P})$ *hold,* $\mathbb{E}$ *is bounded.*

Proof: Suppose, $\mathbf{p}^* = (p_1^*, ..., p_{M-1}^*, 1) \in \mathbb{E}$ is an equilibrium price vector, i.e., $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

In general, if the 'boundary conditions' hold, then there exists $r > 0$ such that: For all $j = 1, .., J$

$$\frac{1}{r} < p_j^* < r.$$

Number of a WE II

Therefore, the set $\mathbb{E}$ is bounded.
Assuming that $\mathbf{z}$ is continuous in $\mathbf{p}$, $\mathbb{E}$ is closed.
Hint: Consider a sequence of prices in $\mathbb{E}$ and use continuity of $\mathbf{z}$.
As earlier, if a set is compact and discrete, then it has to be finite.

Theorem

If an economy is regular, there are only finitely many equilibrium prices.

Since $\mathbb{E}$ is bounded, closed and discrete, it is a finite set.

Theorem

If an economy is regular and the 'boundary conditions' on $\mathbf{z}(\mathbf{P})$ hold, then

- *Either there will be a unique equilibrium*
- *The number of equilibria will be odd.*

Unique WE: Conditions? I

For an individual consumer. Let

- $u(.)$ is a continuous, strictly increasing and strictly quasi-concave utility function
- $u^* = v(\mathbf{p}, I)$, and
- $\mathbf{x} : \mathbb{R}^{M+1}_{++} \mapsto \mathbb{R}^M_+$ be the (Marshallian) demand function generated by $u(.)$.
- $\mathbf{x}^H : \mathbb{R}^{M+1}_{++} \mapsto \mathbb{R}^M_+$ be the associated Hicksian demand function.
- $\frac{\partial x_j(\mathbf{p},I)}{\partial p_k} = \frac{\partial x_j^H(\mathbf{p},u^*)}{\partial p_k} - x_k(\mathbf{p}, I)\frac{\partial x_j(\mathbf{p},I)}{\partial I}$, i.e.,
- $\frac{\partial x_j^H(\mathbf{p},u^*)}{\partial p_k} = \frac{\partial x_j(\mathbf{p},I)}{\partial p_k} + x_k(\mathbf{p}, I)\frac{\partial x_j(\mathbf{p},I)}{\partial I}$.

Unique WE: Conditions? II

We know that

$$\frac{\partial x_j^H(\mathbf{p}, u^*)}{\partial p_j} = \left(\frac{\partial x_j(\mathbf{p}, u^*)}{\partial p_j}\right)_{du=0} = \frac{\partial x_j(\mathbf{p}, I)}{\partial p_j} + x_j(\mathbf{p}, I)\frac{\partial x_j(\mathbf{p}, I)}{\partial I} < 0.$$

Let,

$$D(\mathbf{x}^H) = \begin{pmatrix} \frac{\partial x_1^H(\mathbf{p},u)}{\partial p_1} & \dots & \frac{\partial x_1^H(\mathbf{p},u)}{\partial p_M} \\ \vdots & \dots & \vdots \\ \frac{\partial x_M^H(\mathbf{p},u)}{\partial p_1} & \dots & \frac{\partial x_M^H(\mathbf{p},u)}{\partial p_M} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 E(\mathbf{p},u)}{\partial p_1^2} & \dots & \frac{\partial^2 E(\mathbf{p},u)}{\partial p_M \partial p_1} \\ \vdots & \dots & \vdots \\ \frac{\partial^2 E(\mathbf{p},u)}{\partial p_1 \partial p_M} & \dots & \frac{\partial^2 E(\mathbf{p},u)}{\partial p_M^2} \end{pmatrix}$$

We can write

$$D(\mathbf{x}^H) = \begin{pmatrix} \frac{\partial x_1(\mathbf{p},I)}{\partial p_1} + x_1(\mathbf{p}, I)\frac{\partial x_1(\mathbf{p},I)}{\partial I} & \dots & \frac{\partial x_1(\mathbf{p},I)}{\partial p_M} + x_M(\mathbf{p}, I)\frac{\partial x_1(\mathbf{p},I)}{\partial I} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_M(\mathbf{p},I)}{\partial p_1} + x_1(\mathbf{p}, I)\frac{\partial x_M(\mathbf{p},I)}{\partial I} & \dots & \frac{\partial x_M(\mathbf{p},I)}{\partial p_M} + x_M(\mathbf{p}, I)\frac{\partial x_M(\mathbf{p},I)}{\partial I} \end{pmatrix}$$

Unique WE: Conditions? III

We know that

- Matrix $D(\mathbf{x}^H)$ is negative semi-definite. Why?
- Moreover, for every pair $(\mathbf{p}, I)$, there is unique 'equilibrium' for the consumer. Why

Theorem

If $\mathbf{x} : \mathbb{R}_{++}^{M+1} \mapsto \mathbb{R}_{+}^{M}$, is a demand function generated by a continuous, strictly increasing and strictly quasi-concave utility function, **then** $\mathbf{x}$ *satisfies*

- *budget balancedness,*
- *symmetry and*
- *negative semi-definiteness.*

Unique WE: Conditions? IV

Theorem

If a function, $\mathbf{x}$, satisfies budget balancedness, symmetry and negative semi-definiteness, then it is a demand function $\mathbf{x} : \mathbb{R}^{M+1}_{++} \mapsto \mathbb{R}^{M}_{+}$

- *generated by some continuous, strictly increasing and strictly quasi-concave utility function.*

That is, if matrix

$$D\{\mathbf{x}\} = \begin{pmatrix} \frac{\partial x_1(\mathbf{p},I)}{\partial p_1} + x_1(\mathbf{p},I)\frac{\partial x_1(\mathbf{p},I)}{\partial I} & \cdots & \frac{\partial x_1(\mathbf{p},I)}{\partial p_M} + x_M(\mathbf{p},I)\frac{\partial x_1(\mathbf{p},I)}{\partial I} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_M(\mathbf{p},I)}{\partial p_1} + x_1(\mathbf{p},I)\frac{\partial x_M(\mathbf{p},I)}{\partial I} & \cdots & \frac{\partial x_M(\mathbf{p},I)}{\partial p_M} + x_M(\mathbf{p},I)\frac{\partial x_M(\mathbf{p},I)}{\partial I} \end{pmatrix}$$

is symmetric and negative semi-definite, then

Unique WE: Conditions? V

- There is a continuous, strictly increasing and strictly quasi-concave utility function $u(.)$ that induces demand function $\mathbf{x}$.
- The 'equilibrium of the consumer' is unique.
- Moreover, if good j is normal, then $\frac{\partial x_j(\mathbf{p})}{\partial p_j} < 0$.

Question

Does a similar result hold for the aggregate demand function? That is, does a similar result hold at the aggregate level?

Let

$$D\{\mathbf{z}(\mathbf{p})\} = \left\{ \frac{\partial z_j(\mathbf{p})}{\partial p_k} \right\}, \quad j, k = 1, ..., J - 1.$$

At aggregate level, we have:

Unique WE: Conditions? VI

Theorem

WE is unique, if the matrix

- $D\{\mathbf{z}(.)\}$ *is negative semi-definite at all* $\mathbf{p} \in \mathbb{E}$*, i.e.,*
- $D\{-\mathbf{z}(.)\}$ *has a positive determinant at all* $\mathbf{p} \in \mathbb{E}$*.*

Note, now Slutsky equation is:

$$\left(\frac{\partial x_j^i(\mathbf{p}^*)}{\partial p_k}\right)_{du^i=0} = \frac{\partial x_j^i(\mathbf{p}^*)}{\partial p_k} + \left(x_k^i(\mathbf{p}^*) - e_k^i\right)\left(\frac{\partial x_j^i(\mathbf{p}^*)}{\partial I}\right)_{dp=0}$$

We have seen that even for 2×2 economy,

Unique WE: Conditions? VII

Remark

- $D\{\mathbf{z}(.)\}$ is not necessarily negative semi-definite, even if $D(\mathbf{x}^H)$ is negative semi-definite.
- Moreover, even if the goods are all normal, $\frac{\partial z_j(\mathbf{p})}{\partial p_j} < 0$ may not hold.