General Equilibrium with Production

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Microeconomic Theory

Lecture 11

Producer Firms I

- There are N individuals; $i = 1, ..., N$
- There are M goods; $j = 1, ..., M$
- There are K firms; $k = 1, ..., K$
- Each firm has a set of production plans, i.e., production set $\mathbb{Y}^k \subset \mathbb{R}^M$
- The production set $\mathbb{Y}^k$ is the set of feasible production plans for firm k

Examples: Let $M = 3$ and $K = 2$. Suppose Firm 1 can produce six units of good 2 by using three units of good 1 and nine units of good 3, i.e.,

$$\begin{aligned} \text{Firm 1: } \mathbf{y}^1 &= (-3, 6, -9) \\ \text{Firm 2: } \mathbf{y}^2 &= (8, -3\frac{1}{2}, -14) \\ \text{Economy: } \mathbf{Y} &= (5, 2\frac{1}{2}, -23) \end{aligned}$$

Producer Firms II

A typical production plan for firm k is

$$\mathbf{y}^k = (y_1^k, ..., y_M^k),$$

where

- $y_j^k > 0$ if good j is an output produced by firm k
- $y_j^k < 0$ if good j is an input used by firm k.

Let $\mathbf{p} = (p_1, ..., p_M)$ be the given price vector. For any $\mathbf{y}^k \in \mathbb{Y}^k$, profit for firm k is

$$\pi(\mathbf{p}, \mathbf{y}^k) = p_1 y_1^k + ... + p_M y_M^k = \mathbf{p}.\mathbf{y}^k.$$

So, PMP for firm k is:

$$\max_{\mathbf{y}^k \in \mathbb{Y}^k} \{p_1 y_1^k + ... + p_M y_M^k\}, \textit{i.e.},$$

$$\max_{\mathbf{y}^k \in \mathbb{Y}^k} \{\mathbf{p}.\mathbf{y}^k\}$$

Producer Firms III

Let

$$\Pi^k(\mathbf{p}) = \max_{\mathbf{y}^k \in \mathbb{Y}^k} \pi(\mathbf{p}, \mathbf{y}^k)$$

$\Pi^k(\mathbf{p})$ homogenous function of degree 1 in $\mathbf{p}$.

Assume, for each firm k,

- $\mathbf{0} \in \mathbb{Y}^k \subset \mathbb{R}^M$, i.e., firm can always earn Zero profit
- So, profit is non-negative
- $\mathbb{Y}^k$ is closed and bounded and strictly convex
  - For any given price vector $\mathbf{p}$, the profit maximizing choice/production plan is unique
  - For different price vector $\mathbf{p}'$, profit maximizing choice/production plan will be different, in general.

Aggregate Production Plans I

Let

- $\mathbf{y}^k = (y_1^k, ..., y_M^k)$ be a feasible production plan for firm $k = 1, ..., K$.
- Corresponding to the above production plan, the aggregate production plan for the economy is given by

$$\mathbf{Y} = (\mathbf{y}^1, ..., \mathbf{y}^K), \text{ where } \mathbf{y}^k = (y_1^k, ..., y_M^k). \tag{1}$$

Corresponding to the production plan (1), the total production of good j is given by

$$\sum_{k=1}^{K} y_j^k.$$

So, corresponding to the production plan (1), the total 'output' vector is:

$$(\sum_{k=1}^{K} y_1^k, ..., \sum_{k=1}^{K} y_M^k) = \sum_{k=1}^{K} \mathbf{y}^k = \mathbf{Y}$$

Aggregate Production Plans II

- if $\sum_{k=1}^{K} y_j^k > 0$, good j is a net output for the economy
- if $\sum_{k=1}^{K} y_j^k < 0$, good j is a net input for the economy.

The aggregate production possibility set (for the entire economy) is

$$\mathbb{Y} = \left\{ \mathbf{Y} \mid \mathbf{Y} = \sum_{k=1}^{K} \mathbf{y}^k, \text{ where } \mathbf{y}^k \in \mathbb{Y}^k \right\}.$$

That is, if $\hat{\mathbf{Y}} \in \mathbb{Y}$, then there exist production plans $\hat{\mathbf{y}}^1, ..., \hat{\mathbf{y}}^k, ..., \hat{\mathbf{y}}^K$ such that

- $\hat{\mathbf{y}}^k \in \hat{\mathbb{Y}}^k$, i.e., $\hat{\mathbf{y}}^k$ is a feasible plan for firm $k = 1, ..., K$; and
- $\hat{\mathbf{Y}} = \sum_{k=1}^{K} \hat{\mathbf{y}}^k$

Aggregate Production Plans III

Proposition

Given the above assumptions on $\mathbb{Y}^k$ and PMP for firms,

- $\mathbf{0} \in \mathbb{Y} \subset \mathbb{R}^M$
- $\mathbb{Y}$ *is closed and bounded and strictly convex*

Question

Why $\mathbb{Y}$ is a bounded set?

Proposition

Given the above assumptions on $\mathbb{Y}^k$, for any given price vector $\mathbf{p} = (p_1, .., p_M) >> (0, ..., 0)$, the following PMP has a unique solution:

$$\max_{\mathbf{y}^k \in \mathbb{Y}^k} \{\mathbf{p}.\mathbf{y}^k\} \text{ for every } k=1,...,K$$

Aggregate Production Plans IV

Proposition

Given the above assumptions on $\mathbb{Y}$*, for any given price vector* $\mathbf{p} = (p_1, .., p_M) >> (0, ..., 0)$*, the following PMP has a unique solution:*

$$\max_{\mathbf{Y} \in \mathbb{Y}} \{\mathbf{p}.\mathbf{Y}\}$$

Efficient Production: Two definitions I

Definition

(Definition 1): Production Plan $\mathbf{Y} = (\mathbf{y}^1, ..., \mathbf{y}^K) \in \mathbb{Y}$ is 'efficient' if there is no other plan $\mathbf{Z} = (\mathbf{z}^1, ..., \mathbf{z}^K)$ such that: $\mathbf{Z}$ is feasible, i.e., $\mathbf{Z} \in \mathbb{Y}$, and

$$\begin{aligned} \sum_{k=1}^{K} z_j^k &\geq \sum_{k=1}^{K} y_j^k, \text{ for all goods } j \\ \sum_{k=1}^{K} z_j^k &> \sum_{k=1}^{K} y_j^k, \text{ for at least one good } j \end{aligned}$$

Remark

Suppose kth good is a net input. In that case, $\sum_{k=1}^{K} z_j^k > \sum_{k=1}^{K} y_j^k$ implies that the production plan requires smaller quantity of this input.

Efficient Production: Two definitions II

Suppose $\mathbf{Y} = (\mathbf{y}^1, \mathbf{y}^2)$:

$$\begin{aligned} \text{firm 1: } \mathbf{y}^1 &= (-2, 4, -6) \\ \text{firm 2: } \mathbf{y}^2 &= (8, -4, -14) \end{aligned}$$

So, $\mathbf{Y} = \sum_1^2 \mathbf{y}^i = (\sum_1^2 y_1, \sum_1^2 y_2, \sum_1^2 y_3) = (6, 0, -20)$

Let $\mathbf{Z} = (\mathbf{z}^1, \mathbf{z}^2)$:

$$\begin{aligned} \text{firm 1: } \mathbf{z}^1 &= (-2, 4, -6) \\ \text{firm 2: } \mathbf{z}^2 &= (8, -3\frac{1}{2}, -14) \end{aligned}$$

So, $\mathbf{Z} = \sum_1^2 \mathbf{z}^i = (\sum_1^2 z_1, \sum_1^2 z_2, \sum_1^2 z_3) = (6, \frac{1}{2}, -20)$

Efficient Production: Two definitions III

Definition

(Definition 2): For given price vector $\mathbf{p} = (p_1, ..., p_M)$, production Plan $\mathbf{Y} = (\mathbf{y}^1, ..., \mathbf{y}^K) \in \mathbb{Y}$ is efficient if it solves

$$\max_{\mathbf{Y}' \in \mathbb{Y}} \{\mathbf{p}.\mathbf{Y}'\}, \textit{i.e.},$$

$$\mathbf{p}.\mathbf{Y} \geq \mathbf{p}.\mathbf{Y}' \quad \text{for all } \mathbf{Y}' \in \mathbb{Y}$$

Question

Does (D1) imply (D2)? Does (D2) imply (D1)?

Production Equilibrium I

Total supply is given by

$$\sum_{i=1}^{N} \mathbf{e}^i + \sum_{k=1}^{K} \mathbf{y}^k$$

Let

- $\bar{\mathbf{p}} = (\bar{p}_1, ..., \bar{p}_M)$ be a price vector.
- $\bar{\mathbf{Y}} = (\bar{\mathbf{y}}^1, ..., \bar{\mathbf{y}}^K)$ be a production plan for the economy.

Definition

$(\bar{\mathbf{Y}}, \bar{\mathbf{p}})$ is a competitive 'production' equilibrium only if: For all $k = 1, ..., K$,

$$\bar{\mathbf{y}}^k \quad \text{solves} \quad \max_{\mathbf{y}^k \in \mathbb{Y}^k} \{\bar{\mathbf{p}}.\mathbf{y}^k\}.$$

Production Equilibrium II

Proposition

Take any price vector $\bar{\mathbf{p}} = (\bar{p}_1, ..., \bar{p}_M)$*. If* $\bar{\mathbf{y}}^k \in \mathbb{Y}^k$ *solves*

$$\max_{\mathbf{y}^k \in \mathbb{Y}^k} \{\bar{\mathbf{p}}.\mathbf{y}^k\} \text{ for } k = 1, ..., K.$$

Let $\bar{\mathbf{Y}}$*, where* $\bar{\mathbf{Y}} = \sum_{k=1}^{K} \bar{\mathbf{y}}^k$*. Then, there is NO other plan* $\mathbf{Z} = (\mathbf{z}^1, ..., \mathbf{z}^K)$ *such that:* $\mathbf{Z}$ *is feasible, i.e.,* $\mathbf{Z} \in \mathbb{Y}$*, and*

$$\sum_{k=1}^{K} z_j^k \geq \sum_{k=1}^{K} \bar{y}_j^k, \text{ for all goods } j$$

$$\sum_{k=1}^{K} z_j^k > \sum_{k=1}^{K} \bar{y}_j^k, \text{ for at least one good } j$$

Production Equilibrium III

Proposition

Take any price vector $\bar{\mathbf{p}} = (\bar{p}_1, ..., \bar{p}_M)$. *Suppose production plans* $\bar{\mathbf{y}}^1, ..., \bar{\mathbf{y}}^K$ *are such that* $\bar{\mathbf{y}}^k \in Y^k$ *solves*

$$\max_{\mathbf{y}^k \in \mathbb{Y}^k} \{\bar{\mathbf{p}}.\mathbf{y}^k\} \text{ for all } k = 1, ..., K$$

. *Then,* $\bar{\mathbf{Y}} = \sum_{k=1}^{K} \bar{\mathbf{y}}^k$ *solves*

$$\max_{\mathbf{Y} \in \mathbb{Y}} \{\bar{\mathbf{p}}.\mathbf{Y}\}$$

That is, individual profit maximization leads to total profit maximization. Why?

Question

What assumption is made about Externality?

Production Equilibrium IV

Proposition

Take any price vector $\bar{\mathbf{p}} = (\bar{p}_1, ..., \bar{p}_M)$. *If* $\bar{\mathbf{Y}}$ *solves*

$$\max_{\mathbf{Y} \in \mathbb{Y}} \{\bar{\mathbf{p}}.\mathbf{Y}\}$$

then there exist production plans $\bar{\mathbf{y}}^1, ..., \bar{\mathbf{y}}^K$ *such that* $\bar{\mathbf{y}}^k \in Y^k$, $\bar{\mathbf{Y}} = \sum_{k=1}^K \bar{\mathbf{y}}^k$, *and* $\bar{\mathbf{y}}^k$ *solves*

$$\max_{\mathbf{y}^k \in \mathbb{Y}^k} \{\bar{\mathbf{p}}.\mathbf{y}^k\} \text{ for all } k = 1, ..., K$$

Proof: Let $\bar{\mathbf{Y}}$ solve $\max_{\mathbf{Y} \in \mathbb{Y}} \{\bar{\mathbf{p}}.\mathbf{Y}\}$. That is,

$$(\forall \mathbf{y} \in \mathbb{Y})[\bar{\mathbf{p}}.\bar{\mathbf{Y}} \geq \bar{\mathbf{p}}.\mathbf{Y}]$$

Let $\bar{\mathbf{y}}^1, ..., \bar{\mathbf{y}}^K$ are such that $\bar{\mathbf{y}} = \sum_{k=1}^K \bar{\mathbf{y}}^k$ and $\bar{\mathbf{y}}^k \in \mathbb{Y}^k$. If possible, suppose for some k, $\bar{\mathbf{y}}^k$ does not solve

$$\max_{\mathbf{y}^k \in \mathbb{Y}^k} \{\bar{\mathbf{p}}.\mathbf{y}^k\}$$

Production Equilibrium V

So, there exists some $\hat{\mathbf{y}}^k \in \mathbb{Y}^k$ such that

$$\bar{\mathbf{p}}.\hat{\mathbf{y}}^k > \bar{\mathbf{p}}.\bar{\mathbf{y}}^k$$

Now, consider the production plan $\mathbf{Z}$ such that

$$\begin{aligned} \mathbf{Z} &= (\mathbf{z}^1, ..., \mathbf{z}^k, ..., \mathbf{z}^K) \\ &= (\bar{\mathbf{y}}^1, ..., \hat{\mathbf{y}}^k, ..., \bar{\mathbf{y}}^K) \end{aligned}$$

Clearly, $\mathbf{Z} \in \mathbb{Y}$. It is easy to show that

$$\bar{\mathbf{p}}.\mathbf{Z} > \bar{\mathbf{p}}.\bar{\mathbf{Y}},$$

a contradiction.

Privatized Economy I

Privatized Economy: $(u^i(.), e^i(.), \mathbb{Y}^k, \theta^{ik})_{i\in\{1,..,N\}, j\in\{1,...,M\}, k\in\{1,...,K\}}$

- All firms are privately owned
- θ^{ik} is the (ownership) share of kth firm owned by i the individual;
- $0 \leq \theta^{ik} \leq 1$ for all $i = 1, ..., N$ and $k = 1, ..., K$
- $\sum_{i=1}^{n} \theta^{ik} = 1$ for all $k = 1, ..., K$

Now, for any given price vector, $\mathbf{p} = (p_1, ..., p_M)$, individual i solves

$$\max_{\mathbf{x}^i \in R_+^M} \; u^i(\mathbf{x}), \text{ s.t. } \; \mathbf{p}.\mathbf{x}^i \leq I^i(\mathbf{p}),$$

where

$$I^i(\mathbf{p}) = \mathbf{p}.\mathbf{e}^i + \sum_{k=1}^{K} \theta^{ik} \pi^k(\mathbf{p}).$$

Privatized Economy II

Remark

In view of the above assumptions on Production sets,

- $\pi(\mathbf{p}) \geq 0$ is bounded above and continuous in $\mathbf{p}$.
- Moreover, $I^i(\mathbf{p})$ is a continuous function of $\mathbf{p}$
- In view of the assumption on $u^i(.)$, the solution of the above UMP is unique.

Definition

The excess demand function for good j is

$$z_j(\mathbf{p}) = \sum_{i=1}^{N} x_j^i(\mathbf{p}, I^i(\mathbf{p})) - \sum_{k=1}^{K} y_j^k(\mathbf{p}) - \sum_{i=1}^{N} e_j^i.$$

So, excess demand vector is $\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), ..., z_M(\mathbf{p}))$.

Privatized Economy III

Definition

Feasible Allocation: An allocation $(\mathbf{X}, \mathbf{Y})$, where $\mathbf{Y} = (\mathbf{y}^1, ..., \mathbf{y}^K)$ is feasible, if:

- For all k, $\mathbf{y}^k \in \mathbb{Y}^k$
- $$\sum_{i=1}^{N} \mathbf{x}^i = \sum_{i=1}^{N} \mathbf{e}^i + \sum_{k=1}^{K} \mathbf{y}^k.$$

Existence of WE I

As before, in view of the assumption on $u^i(.)$ and $\mathbb{Y}^k$,

- the excess demand function is homogeneous function of $\mathbf{p}$ of degree 0.
- For equilibrium to exist $\mathbf{Y} + \sum_{i=1}^{N} \mathbf{e}^i >> \mathbf{0}$ must hold for some $\mathbf{Y} \in \mathbb{Y}$.
- As $p_j \to 0$ for some j, the excess demand becomes unbounded for one of such commodities.

Theorem

Consider an economy $(u^i(.), \mathbf{e}^i, \theta^{ik}, Y^j)$, where $i = 1, ..., N$, $j = 1, ..., M$ and $k = 1, ..., K$. If $u^i(.)$ and Y^j satisfy above assumptions, then there exists a price vector $\mathbf{p}^* >> \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Efficiency of WE I

Definition

A feasible allocation $(\bar{\mathbf{X}}, \bar{\mathbf{Y}})$ is Pareto optimum if there is no other allocation $(\mathbf{X}, \mathbf{Y})$ such that

$$\begin{aligned}
\sum_{i=1}^{N} \mathbf{x}^i &= \sum_{i=1}^{N} \mathbf{e}^i + \sum_{k=1}^{K} \mathbf{y}^k; \\
u^i(\mathbf{x}^i) &\geq u^i(\bar{\mathbf{x}}) \text{ for all } i \in \{1, ..., N\} \text{ and} \\
u^j(\mathbf{x}^j) &> u^j(\bar{\mathbf{x}}) \text{ for some } j \in \{1, ..., N\}
\end{aligned}$$

Theorem

Consider an economy $(u^i(.), \mathbf{e}^i, \theta^{ik}, \mathbf{y}^k)$*, where* $i = 1, ..., N$ *and* $k = 1, ..., K$*. If* $u^i(.)$ *is strictly increasing, then every WE is Pareto optimum.*

Proof: Suppose $(\bar{\mathbf{X}}, \bar{\mathbf{Y}})$ along with $\mathbf{p}^*$ is WE. Therefore,

Efficiency of WE II

$$\sum_{i=1}^{N} \bar{\mathbf{x}}^i = \sum_{i=1}^{N} \mathbf{e}^i + \sum_{k=1}^{K} \bar{\mathbf{y}}^k$$

Suppose $(\bar{\mathbf{X}}, \bar{\mathbf{Y}})$ is not PO. So, there is some $(\mathbf{X}, \mathbf{Y})$, such that

$$\begin{aligned} \sum_{i=1}^{N} \mathbf{x}^i &= \sum_{i=1}^{N} \mathbf{e}^i + \sum_{k=1}^{K} \mathbf{y}^k; \\ u^i(\mathbf{x}^i) &\geq u^i(\bar{\mathbf{x}}^i) \text{ for all } i \in \{1, ..., N\} \text{ and} \\ u^j(\mathbf{x}^j) &> u^j(\bar{\mathbf{x}}^j) \text{ for some } j \in \{1, ..., N\} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{p}^*.\mathbf{x}^i &\geq \mathbf{p}^*.\bar{\mathbf{x}}^i \text{ for all } i \in \{1, ..., N\} \text{ and} \\ \mathbf{p}^*.\mathbf{x}^j &> \mathbf{p}^*.\bar{\mathbf{x}}^i \text{ for some } j \in \{1, ..., N\} \end{aligned}$$

Efficiency of WE III

$$\mathbf{p}^*.\sum_{i=1}^{N}\mathbf{x}^i > \mathbf{p}^*.\sum_{i=1}^{N}\bar{\mathbf{x}}^i$$

$$\mathbf{p}^*.\left(\sum_{j=1}^{M}\mathbf{y}^j + \sum_{i=1}^{M}\mathbf{e}^j\right) > \mathbf{p}^*.\left(\sum_{j=1}^{M}\bar{\mathbf{y}}^j + \sum_{j=1}^{M}\mathbf{e}^j\right)$$

$$\mathbf{p}^*.\sum_{j=1}^{M}\mathbf{y}^j > \mathbf{p}^*.\sum_{j=1}^{M}\bar{\mathbf{y}}^j$$

a contradiction. Why?

Efficiency of WE IV

Theorem

Consider an economy $(u^i(.), \mathbf{e}^i, \theta^{ik}, Y^j)$, where $i = 1, ..., N$ and $k = 1, ..., K$. Suppose, $u^i(.)$ and Y^j satisfy above assumptions, and $\mathbf{y} + \sum_{i=1}^N \mathbf{e}^i >> \mathbf{0}$ for some $\mathbf{y} \in \mathbb{Y}$. If $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is any feasible PO allocation, then there exist

- *'Cash' transfers T^i, $i = 1, .., N$ such that $\sum_{i=1}^N T^i = 0$*
- *A price vector $\bar{\mathbf{p}} = (\bar{p}_1, ..., \bar{p}_M)$*

such that

1. *$\bar{\mathbf{x}}^i$ maximizes $u^i(\mathbf{x}^i)$ s.t. $\bar{\mathbf{p}}.\mathbf{x}^i \leq \bar{\mathbf{p}}.\mathbf{e}^i + \sum_{k=1}^K \theta^{ik}\pi^k(\bar{\mathbf{p}}) + T^i$ for all $i = 1, ..., N$*
2. *$\bar{\mathbf{y}}^k$ maximizes $\bar{\mathbf{p}}.\mathbf{y}^k$ for all $k = 1, ..., K$*
3. *$\mathbf{z}^j(\bar{\mathbf{p}}) = 0$ for all $j = 1, ..., M$*