Fundamental Theorems of Welfare Economics

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Lecture 6

First Fundamental Theorem

The First Fundamental Theorem of Welfare Economics:
Consider an exchange economy $(u^i, \mathbf{e}^i)_{i \in N}$.

Theorem

If u^i is strongly increasing for all $i = 1, .., N$, then $W((u^i, \mathbf{e}^i)_{i \in N}) \subseteq C((u^i, \mathbf{e}^i)_{i \in N})$. That is,

- *Every WE/Competitive equilibrium is Pareto optimum;*
- *Every WE/Competitive equilibrium is in the Core.*

Question

- *What if the Core allocations are highly unequal*
- *Can markets lead to equitable outcomes?*

Question

Suppose:

- $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ is any feasible Pareto optimum allocation.
- $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ may or may not be equitable across individuals

Question

- *If desired, can* $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ *be achieved as a competitive equilibrium?*
- *If yes, what are the conditions for* $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ *to be achieved as a competitive equilibrium?*

An Example I

Consider a 2×2 economy:

- $u^1(.) = x_1^1 + 2x_2^1$, and $u^2(.) = x_1^2 x_2^2$
- Therefore, $MRS_1 = \frac{1}{2}$ and $MRS_2 = \frac{x_2^2}{x_1^2}$
- Let $e^1(.) = (1, \frac{1}{2})$, and $e^2(.) = (0, \frac{1}{2})$
- Assume that individuals act as price-takers

Given any price vector, in equi. person 2 will consume (x_1^2, x_2^2) such that: $MRS_2 = \frac{p_1}{p_2}$ and all income is spent.

That is, the demanded bundle (x_1^2, x_2^2) will be such that: $\frac{x_2^2}{x_1^2} = \frac{p_1}{p_2}$, i.e,

$$\begin{aligned} p_2.x_2^2 &= p_1.x_1^2 \text{ and} \\ p_1 x_1^2 + p_2 x_2^2 &= p_1.0 + \frac{p_2}{2} \end{aligned}$$

An Example II

This gives us:

$$\begin{aligned} 2p_1 x_1^2 &= \frac{p_2}{2} \; i.e., \\ x_1^2 &= \frac{p_2}{4p_1}. \text{ Moreover,} \\ x_2^2 &= \frac{1}{4} \end{aligned}$$

For the 1st person, the following holds:

$$\begin{aligned} \frac{p_1}{p_2} > \frac{1}{2} \quad &\Rightarrow \quad \text{only 2nd good is demanded} \\ \frac{p_1}{p_2} < \frac{1}{2} \quad &\Rightarrow \quad \text{only 1st good is demanded} \\ \frac{p_1}{p_2} = \frac{1}{2} \quad &\Rightarrow \quad \text{any } (x_1^1, x_2^1) \text{ on the budget line can be demanded.} \end{aligned}$$

An Example III

That is, the demanded bundle (x_1^1, x_2^1) will be such that: if $(x_1^1, x_2^1) >> (0, 0)$.

$$
\begin{aligned}
MRS_1 &= \frac{p_1}{p_2}, i.e. \frac{1}{2} = \frac{p_1}{p_2} \\
p_1 x_1^1 + p_2 x_2^1 &= p_1 + \frac{p_2}{2}.
\end{aligned}
$$

Otherwise, only one good is demanded.

So, the plausible equilibrium price vector will have: $\frac{p_1}{p_2} = \frac{1}{2}$. Why?
Let $(p_1, p_2) = (1, 2)$. At this price:

- For 2nd person, the demanded bundle $\mathbf{x}^2 = (x_1^2, x_2^2) = (1/2, 1/4)$
- For 1st person, the bundle $\mathbf{x}^1 = (x_1^1, x_2^1) = (1/2, 3/4)$ lies on the budget line.

An Example IV

Therefore,
$(\mathbf{x}^1, \mathbf{x}^2)$, where $\mathbf{x}^2 = ((1/2, 1/4)$ and $\mathbf{x}^1 = (1/2, 3/4))$, along with $(p_1, p_2) = (1, 2)$ is a competitive equilibrium.

Remark

WE exists even though preferences are not strictly quasi-concave.

Question

For the above economy, suppose we are told that a WE exists. How can we find the WE?

Note

- We know that WE is PO and is a Core allocation (Why?)
- So, we can start with the set of PO points.

An Example V

The locus of tangencies of ICs, i.e, where

$$\begin{aligned} MRS_1 &= MRS_2 \text{ , } i.e. \frac{1}{2} = \frac{x_2^2}{x_1^2} \\ x_1^2 &= 2x_2^2. \end{aligned}$$

The only consistent point is

$$\mathbf{x}^1 = (1/2, 3/4) \text{ and } \mathbf{x}^2 = (1/2, 1/4).$$

Now, question is:

- Is $\mathbf{x}^1 = (1/2, 3/4)$ and $\mathbf{x}^2 = (1/2, 1/4)$ a WE?
- For what price vector, the utility maximizers persons 1 an 2 will choose $\mathbf{x}^1 = (1/2, 3/4)$ and $\mathbf{x}^2 = (1/2, 1/4)$, respectively?

Given the nature of the preferences: Try any $\mathbf{p} = (p_1, p_2)$ such that $\frac{p_1}{p_2} = \frac{1}{2}$.

Redistribution and Policy Interventions I

- At $\mathbf{x}^1 = (1/2, 3/4)$ and $\mathbf{x}^2 = (1/2, 1/4)$,

$$u^1(.) = 2 \ \& \ u^2() = 1/8$$

- Suppose we want to achieve $u^1(.) = 3/2 \ \& \ u^2() = 9/32$.
- Allocation $(\mathbf{y}^1, \mathbf{y}^2)$ where $\mathbf{y}^1 = (1/4, 5/8)$ and $\mathbf{y}^2 = (3/4, 3/8)$ can achieve this

$$u^1(1/4, 5/8) = 3/2 \ \& \ u^2(3/4, 3/8) = 9/32$$

- $(\mathbf{y}^1, \mathbf{y}^2)$ where $\mathbf{y}^1 = (1/4, 5/8)$ and $\mathbf{y}^2 = (3/4, 3/8)$ is PO

Redistribution and Policy Interventions II

Question

Can we induce $(\mathbf{y}^1, \mathbf{y}^2)$ where $\mathbf{y}^1 = (1/4, 5/8)$ and $\mathbf{y}^2 = (3/4, 3/8)$, as WE?

Yes, try this by keeping $\mathbf{p} = (p_1, p_2)$ such that $\frac{p_1}{p_2} = \frac{1}{2}$, but by choosing

$$T_1 = -\frac{1}{2} \text{ and } \; T_2 = \frac{1}{2}$$

Now, in equi. person 2 will consume (x_1^2, x_2^2) such that: $\frac{x_2^2}{x_1^2} = \frac{p_1}{p_2}$, i.e,

$$\begin{aligned} p_2.x_2^2 &= p_1.x_1^2 \text{ and} \\ p_1 x_1^2 + p_2 x_2^2 &= p_1.0 + p_2.\frac{1}{2} + T_2 \end{aligned}$$

You can check that $T_2 = \frac{1}{2}$ induces the 2nd person to buy $3/8$.

Redistribution and Policy Interventions III

When $T_1 = -\frac{1}{2}$ and $T_2 = \frac{1}{2}$, the only solution to 2's problem is

$$(y_1^2, y_2^2) = (3/4, 3/8).$$

Also, person 1 demands $\mathbf{y}^1 = (1/4, 5/8)$.

Second Fundamental Theorem

The Second Fundamental Theorem of Welfare Economics:

Theorem

If u^i is continuous, strongly increasing, and strictly quasi-concave for all $i = 1, .., N$, then any *Pareto optimum allocation, $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$, such that $\mathbf{y}^i >> \mathbf{0}$,*

- *can be achieved as competitive equilibrium with suitable transfers.*
- *That is, $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ is a WE with suitable transfer.*
- *With suitable transfers, market can achieve any of the socially desirable allocation as competitive equilibrium.*

2nd Theorem: Transfer of Goods

Suppose, $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ is a feasible PO allocation, and we want to achieve allocation as $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ a competitive equilibrium. There are two solutions.

Choose, $\acute{\mathbf{e}} = (\acute{\mathbf{e}}^1, ..., \acute{\mathbf{e}}^N)$ such that: For all $i = 1, ..., N$

$$\mathbf{y}^i = \mathbf{e}^i + \acute{\mathbf{e}}^i.$$

It can be easily seen that there exists a price vector such that $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ a competitive equilibrium. Let $\acute{\mathbf{p}} = (p'_1, ..., p'_M)$ be such a price vector.

Remark

$\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ a WE if we choose any $\hat{\mathbf{e}} = (\hat{\mathbf{e}}^1, ..., \hat{\mathbf{e}}^N)$ such that the new endowment vectors $(\mathbf{e}^1 + \hat{\mathbf{e}}^1, ..., \mathbf{e}^N + \hat{\mathbf{e}}^N)$ lies on the budget line generated by the price vector $\mathbf{p}' = (p'_1, ..., p'_M)$.

2nd Theorem: Cash Transfer I

Consider an exchange economy $(u^i, \mathbf{e}^i)_{i \in N}$. Let, $\mathbf{x} = (\mathbf{x}^1, ..., \mathbf{x}^N)$ be an the equilibrium without transfers. Clearly, for some $\mathbf{p}$, we have: For all $i = 1, ..., N$

$$\mathbf{p}.\mathbf{x}^i = \mathbf{p}.\mathbf{e}^i$$

Now, consider 'cash' transfers; individual i gets T_i.

Let, $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^N)$ be the equilibrium after 'cash' transfers. Now for some $\mathbf{p}'$ we have: For all $i = 1, ..., N$

$$\mathbf{p}'.\mathbf{y}^i = \mathbf{p}'.\mathbf{e}^i + T^i, i.e.,$$

for all $i = 1, ..., N$

$$\sum_{j=1}^{M} p'_j y^i_j = \sum_{j=1}^{M} p'_j e^i_j + T^i \tag{1}$$

2nd Theorem: Cash Transfer II

That is,

$$\sum_{i=1}^{N}\left(\sum_{j=1}^{M}p'_j y_j^i\right) = \sum_{i=1}^{N}\left(\sum_{j=1}^{M}p'_j e_j^i\right) + \sum_{i=1}^{N} T^i, i.e,$$

$$\begin{aligned}
\sum_{i=1}^{N} T^i &= \sum_{j=1}^{M}\left(\sum_{i=1}^{N}p'_j y_j^i\right) - \sum_{j=1}^{M}\left(\sum_{i=1}^{N}p'_j e_j^i\right) \\
&= \sum_{j=1}^{M}p'_j\left(\sum_{i=1}^{N}y_j^i - \sum_{i=1}^{N}e_j^i\right) \\
&= 0.
\end{aligned}$$