Conditions for Unique Walrasian Equilibrium

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Lecture 8

Gross Substitutes I

Suppose,

- There are two goods
- Consider three price vectors: $\mathbf{p} = (p_1, p_2) = (2, 1)$, $\mathbf{p}' = (p'_1, p'_2) = (3, 1)$ and $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2) = (3, 2)$.

Let, $\mathbf{x}^i(\mathbf{p})$, be the demand function for individual i.

Question

Suppose, the above goods are 'gross substitutes' for individual i.

- *How will $x_2^i(\mathbf{p}')$ compare with $x_2^i(\mathbf{p})$?*
- *How will $x_2^i(\bar{\mathbf{p}})$ compare with $x_2^i(\mathbf{p})$?*

- Let $\lambda = \max_j\{\frac{\bar{p}_j}{p_j}\}$, $j = 1, 2$.
- Note here $\lambda = \frac{\bar{p}_2}{p_2} = 2$

Gross Substitutes II

- Also, $\lambda \mathbf{p} \geq \bar{\mathbf{p}}$. Since $(4, 2) \geq (3, 2)$.

Question

What can we say about the individual demand for the two goods at these two price vectors $\lambda \mathbf{p} = (4, 2)$ *and* $\bar{\mathbf{p}} = (3, 2)$*?*

Question

What can we say about the individual demand for the two goods at the price vectors $\mathbf{p} = (2, 1)$ *and* $\lambda \mathbf{p} = (4, 2)$*?*

Gross Substitutes III

Next, consider two price vectors

- $\mathbf{p} = (p_1, p_2, p_3) = (3, 2, 1)$ and $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2, \bar{p}_3) = (5, 1, 4)$

Question

What can we say about the excess demand at these two price vectors?

- Let $\lambda = \max_j \{\frac{\bar{p}_j}{p_j}\}$, $j = 1, .., 3$.
- Note here $\lambda = \max\{\frac{5}{3}, \frac{1}{2}, \frac{4}{1}\} = \frac{\bar{p}_3}{p_3} = 4$
- Also, $\lambda \mathbf{p} \geq \bar{\mathbf{p}}$. Since $(12, 8, 4) \geq (5, 1, 4)$.

Remark

$\mathbf{z}(\lambda \mathbf{p}) = \mathbf{z}(\mathbf{p})$, i.e., $\mathbf{z}(4\mathbf{p}) = \mathbf{z}(\mathbf{p})$.

Gross Substitutes IV

Consider the following price vectors

- $\mathbf{p} = (p_1, p_2, p_3) = (3, 2, 1)$, $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3) = (12, 8, 4)$ and $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2, \bar{p}_3) = (5, 1, 4)$.

Question

- *What can we say about the excess demand for 3rd good at prices* $\hat{\mathbf{p}}$ *and* $\bar{\mathbf{p}}$*? That is,*
- *How is* $z_3(\hat{\mathbf{p}})$ *expected to compare with* $z_j(\bar{\mathbf{p}})$*?*

Note:

- $\hat{\mathbf{p}} = \lambda \mathbf{p}$ and $\hat{\mathbf{p}} \geq \bar{\mathbf{p}}$
- $\hat{p}_3 = \lambda p_3 = \bar{p}_3$.

GS and No of WE I

Question

What are the assumptions needed for the aggregate demand function to exist?

- Aggregate demand function $\mathbf{x}(.)$ will exist iff if the individual demand function, i.e., $\mathbf{x}^i(.)$, exists for all $i = 1, .., N$.
- $\mathbf{x}^i(.)$ exists if the underlying utility function satisfies the assumption of continuity, strong monotonicity and strict quasi-concavity.

GS and No of WE II

Definition

Aggregate demand function, $z(.)$, satisfies condition of 'Gross Substitutes' (GS) if for all $\hat{\mathbf{p}}, \bar{\mathbf{p}} \in \mathbb{R}^M_{++}$, such that $\hat{\mathbf{p}} \geq \bar{\mathbf{p}}$ and $\hat{\mathbf{p}} \neq \bar{\mathbf{p}}$:

$$\hat{p}_j = \bar{p}_j \Rightarrow z_j(\hat{\mathbf{p}}) > z_j(\bar{\mathbf{p}}).$$

Theorem

If $Z(.)$ satisfies condition of GS, then there is unique WE.

WLOG, we can consider vectors in the set

$$\mathbb{P} = \{\mathbf{p} | \mathbf{p} \in \mathbb{R}^M_{++}, \text{ and } p_M = 1\}.$$

Proof: Suppose, WE is not unique. If possible, suppose $\mathbf{p}, \mathbf{p}' \in \mathbb{E}$. Moreover, $\mathbf{p} \neq \mathbf{p}'$.

GS and No of WE III

Let

$$\begin{aligned}\lambda &= \max_j \left\{ \frac{\acute{p}_j}{p_j} \right\} \text{ for } j = 1, .., M. \\ &= \max \left\{ \frac{\acute{p}_1}{p_1}, \frac{\acute{p}_2}{p_2}, ..., \frac{\acute{p}_M}{p_M} \right\}\end{aligned}$$

Suppose, $\frac{\acute{p}_k}{p_k} \geq \frac{\acute{p}_j}{p_j}$ for all $j = 1, .., M$. That is,

$$\lambda = \frac{\acute{p}_k}{p_k}$$

Clearly, $\lambda \mathbf{p} \geq \mathbf{p}'$, and $p_k \lambda = \acute{p}_k$. Let $\bar{\mathbf{p}} = \lambda \mathbf{p}$.

- This means $\bar{\mathbf{p}} \geq \mathbf{p}'$ and $\bar{p}_k = \acute{p}_k$.
- Hence

$$z_k(\bar{\mathbf{p}}) > z_k(\mathbf{p}').$$

GS and No of WE IV

But $z_k(\mathbf{p}') = 0$. Therefore,

$$z_k(\bar{\mathbf{p}} = \lambda \mathbf{p}) > 0,$$

which is a contradiction. **Why?**

Since $\mathbf{p} \in \mathbb{E}$, therefore

$$z_k(\mathbf{p}) = 0.$$

Since $\bar{\mathbf{p}} = \lambda \mathbf{p}$,

$$z_k(\bar{\mathbf{p}}) = z_k(\mathbf{p}) = 0.$$

WARP and no of WE I

Let,

- $\mathbf{x} = \mathbf{x}(\mathbf{p})$ denote the bundle demanded at price $\mathbf{p}$; where $\mathbf{x} = (x_1, ...x_m)$.
- $\mathbf{x}' = \mathbf{x}(\mathbf{p}')$ denote the bundle demanded at price $\mathbf{p}'$; - $\mathbf{x}' = (x_1', ...x_m')$.

Therefore,

- $\mathbf{p}.\mathbf{x} = \mathbf{p}.\mathbf{x}(\mathbf{p})$ is the expenditure incurred at price $\mathbf{p}$.
- $\mathbf{p}'.\mathbf{x}' = \mathbf{p}'.\mathbf{x}(\mathbf{p}')$ is the expenditure incurred at price $\mathbf{p}'$.

The demand satisfies Weak Axiom of Revealed Preference (WARP), if

$$\mathbf{p}.\mathbf{x}' \leq \mathbf{p}.\mathbf{x} \Rightarrow \mathbf{p}'.\mathbf{x}' < \mathbf{p}'.\mathbf{x}.$$

- $\mathbf{p}.\mathbf{x}' \leq \mathbf{p}.\mathbf{x}$ implies that the bundle $\mathbf{x}'$ was affordable at price $\mathbf{p}$.
- $\mathbf{p}'.\mathbf{x} > \mathbf{p}'.\mathbf{x}'$ implies that the bundle $\mathbf{x} = \mathbf{x}(\mathbf{p})$ is strictly more expensive (than $\mathbf{x}' = \mathbf{x}(\mathbf{p}')$) at price $\mathbf{p}'$.

WARP and no of WE II

Restating: The demand satisfies WARP, if

$$\begin{aligned} \mathbf{p}.\mathbf{x}' \leq \mathbf{p}.\mathbf{x} \quad &\Rightarrow \quad \mathbf{p}'.\mathbf{x}' < \mathbf{p}'.\mathbf{x}. \\ \mathbf{p}(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) \leq 0 \quad &\Rightarrow \quad \mathbf{p}'(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) < 0. \end{aligned}$$

Theorem

If the aggregate demand function satisfies the WARP, then the WE is unique.

WARP and no of WE III

Define (aggregate) vectors:

$$\begin{aligned}
\mathbf{x} &= \left(\sum_{i=1}^{N} x_1^i, \sum_{i=1}^{N} x_2^i, ..., \sum_{i=1}^{N} x_M^i\right) \\
\mathbf{e} &= \left(\sum_{i=1}^{N} e_1^i, \sum_{i=1}^{N} e_2^i, ..., \sum_{i=1}^{N} e_M^i\right) \\
\mathbf{z} &= \mathbf{x} - \mathbf{e} = \left(\sum_{i=1}^{N}(x_1^i - e_1^i), ..., \sum_{i=1}^{N}(x_M^i - e_M^i)\right) \\
\mathbf{z} &= \mathbf{x} - \mathbf{e} = (z_1, ..., z_M)
\end{aligned}$$

WARP and no of WE IV

Proof: Suppose, WE is not unique. If possible, suppose $\mathbf{p}, \mathbf{p}' \in \mathbb{E}$, and $\mathbf{p} \neq \mathbf{p}'$. Note for any price vectors $\mathbf{p}$ and $\mathbf{p}'$, we have:

$$\begin{aligned} \mathbf{p}.\mathbf{z}(\mathbf{p}) &= 0, \textit{i.e.}, \\ \mathbf{p}.(\mathbf{x}(\mathbf{p}) - \mathbf{e}) &= 0. \end{aligned} \tag{1}$$

Since $\mathbf{p}'$ is an equi. price vector, $\mathbf{z}(\mathbf{p}') = \mathbf{x}(\mathbf{p}') - \mathbf{e} = 0$, i.e., $\mathbf{x}(\mathbf{p}') = \mathbf{e}$. Therefore, the assumption $\mathbf{p}' \in \mathbb{E}$ gives us

$$\begin{aligned} \mathbf{p}.(\mathbf{x}(\mathbf{p}) - \mathbf{x}(\mathbf{p}')) &= 0, \textit{i.e.}, \\ \mathbf{p}.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) &= 0. \end{aligned} \tag{2}$$

From WARP, we know that

$$\mathbf{p}.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) \leq 0 \Rightarrow \mathbf{p}'.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) < 0. \tag{3}$$

(2) and (3) give us,

$$\mathbf{p}'.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) < 0. \tag{4}$$

WARP and no of WE V

Similarly, we get:

$$\begin{aligned} \mathbf{p}'.\mathbf{z}(\mathbf{p}') &= 0, \textit{i.e.}, \\ \mathbf{p}'.(\mathbf{x}(\mathbf{p}') - \mathbf{e}) &= 0 \\ \mathbf{p}'.(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) &= 0 \qquad (5) \end{aligned}$$

which is a contradiction, since in view of (4), we have

$$\mathbf{p}'(\mathbf{x}(\mathbf{p}') - \mathbf{x}(\mathbf{p})) < 0.$$

- The assumption that there are two price vectors $\mathbf{p}, \mathbf{p}' \in \mathbb{E}$ leads to a contradiction.
- There cannot be two or more equilibrium price vectors.