Competitive Equilibrium and its Existence*

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This Write-up is not for circulation

1 Basics

1.1 Individual UMP

Let us start with the utility maximization problem of the individuals in the economy. We assume that individual consumers are price-takers. Let the set of price vectors be $\mathbf{p} = (p_1, ..., p_M) \in \mathbb{R}^M_{++}$. That is, $(p_1, ..., p_M) > (0, ..., 0)$. Now, the consumer i will choose the her optimum bundle by solving:

$$\max_{\mathbf{x} \in \mathbb{R}^J_+} u^i(\mathbf{x}) \quad s.t. \quad \mathbf{p}.\mathbf{x} \leq \mathbf{p}.\mathbf{e}^i$$

We impose the following standard assumptions on utility functions.

Assumption 1 For all $i \in I$, u^i is continuous, strongly increasing, and strictly quasi-concave on $\mathbb{R}^M_+$.

Consider two bundles $\mathbf{x} = (x_1, ..., x_M)$ and $\mathbf{x}' = (x'_1, ..., x'_M)$. We write $\mathbf{x}' \geq \mathbf{x}$, if $x'_j \geq x_j$ for all $j \in \{1, ..., M\}$ and $x'_j > x_j$ for some $j \in \{1, ..., M\}$. That is, $\mathbf{x}' \geq \mathbf{x}$ if quantity of each good is (weakly) higher in $\mathbf{x}'$ compared to the bundle $\mathbf{x}$; and, quantity of at least one good is strictly greater in the bundle $\mathbf{x}'$ compared to the bundle $\mathbf{x}$.

The utility function, u^i, is said to be strongly increasing if for any two bundles $\mathbf{x}$ and $\mathbf{x}'$

$$\mathbf{x}' \geq \mathbf{x} \Rightarrow u^i(\mathbf{x}') > u^i(\mathbf{x}).$$

In view of monotonicity of the preferences, for given $\mathbf{p} = (p_1, ..., p_M) >> (0, ..., 0)$, consumer i solves:

$$\max_{\mathbf{x} \in \mathbb{R}^M_+} u^i(\mathbf{x}) \quad s.t. \quad \mathbf{p}.\mathbf{x} = \mathbf{p}.\mathbf{e}^i \tag{1}$$

From the first part of the course, you know that when $u^i(.)$ satisfies assumptions listed above, the following result holds.

*References are: Arrow and Debreu (1954), and McKenzie (2008); Arrow and Hahn (1971). Jehle and Reny (2008).

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Theorem 1 *Under the above assumptions on $u^i(.)$, for every $(p_1, ..., p_M) > (0, ..., 0)$, (1) has a unique solution, say $\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i)$.*

Note: For each $i = 1, ..., N$,

$$\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) : \mathbb{R}^M_{++} \mapsto \mathbb{R}^M_{+};$$

Note that we allow consumption of non negative amount of goods, and as such we do not insist that each good be consumed in strictly positive quantity. However, $\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) = (x^i_1(\mathbf{p}, \mathbf{p}.\mathbf{e}^i), x^i_2(\mathbf{p}, \mathbf{p}.\mathbf{e}^i), ..., x^i_M(\mathbf{p}, \mathbf{p}.\mathbf{e}^i))$.

Theorem 2 *Under the above assumptions on $u^i(.)$, for every $(p_1, ..., p_M) > (0, ..., 0)$,*

- $\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i)$ *is continuous in* $\mathbf{p}$ *over* $\mathbb{R}^M_{++}$.
- *For all $i = 1, 2, ..., N$, we have: $\mathbf{x}^i(t\mathbf{p}) = \mathbf{x}^i(\mathbf{p})$, for all $t > 0$. That is, demand of each good j by individual i satisfies the following property:*

$$x^i_j(t\mathbf{p}) = x^i_j(\mathbf{p}) \text{ for all } t > 0.$$

Question 1 *Given that $u^i(.)$ is strongly increasing,*

- *is $\mathbf{x}^i(\mathbf{p})$ continuous over $\mathbb{R}^M_{+}$?*
- *is the demand function $x^i_j(\mathbf{p})$ defined at $p_j = 0$?*

1.2 Excess Demand Function

Definition 1 *The excess demand for jth good by the ith individual is give by:*

$$\mathbf{z}^i_j(\mathbf{p}) = x^i_j(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - e^i_j.$$

The aggregate excess demand for jth good is give by:

$$\mathbf{z}_j(\mathbf{p}) = \sum_{i=1}^{N} x^i_j(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - \sum_{i=1}^{N} e^i_j.$$

So, Aggregate Excess Demand Function is:

$$\mathbf{z}(\mathbf{p}) = (\mathbf{z}_1(\mathbf{p}), ..., \mathbf{z}_M(\mathbf{p})),$$

Theorem 3 *Under the above assumptions on $u^i(.)$, for any $\mathbf{p} >> \mathbf{0}$,*

- $\mathbf{z}(.)$ *is continuous in* $\mathbf{p}$
- $\mathbf{z}(t\mathbf{p}) = \mathbf{z}(\mathbf{p})$, *for all* $t > 0$
- $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$. *(*the Walras' Law*)*

For any given price vector $\mathbf{p}$, we have

$$\begin{aligned}\mathbf{p}.\mathbf{x}^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - \mathbf{p}.\mathbf{e}^i &= 0, i.e., \\ \sum_{j=1}^{M} p_j[x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - e_j^i] &= 0.\end{aligned}$$

This gives:

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{M} p_j[x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - e_j^i] &= 0, i.e., \\ \sum_{j=1}^{M}\sum_{i=1}^{N} p_j[x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - e_j^i] &= 0, i.e., \\ \sum_{j=1}^{M} p_j\left[\sum_{i=1}^{N} x_j^i(\mathbf{p}, \mathbf{p}.\mathbf{e}^i) - \sum_{i=1}^{N} e_j^i\right] &= 0\end{aligned}$$

That is, if we add the accounting worth of the excess demand/supply across all the goods, it will add up to zero. Note that this does not mean that there is no excess demand or supply. Depending on the price vector, there can be excess demand for some goods and excess supply for some other goods. However, the accounting of worth of the excess demands and excess supplies will cancel each other.

Re-writing the last equality, we get:

$$\begin{aligned}\sum_{j=1}^{M} p_j z_j(\mathbf{p}) &= 0, i.e., \\ \mathbf{p}.\mathbf{z}(\mathbf{p}) &= 0\end{aligned}$$

So,

$$p_1 z_1(\mathbf{p}) + p_2 z_2(\mathbf{p}) + ... + p_{j-1} z_{j-1}(\mathbf{p}) + p_{j+1} z_{j+1}(\mathbf{p}) + +p_M z_M(\mathbf{p}) = -p_j z_j(\mathbf{p})$$

For a price vector $\mathbf{p} >> \mathbf{0}$,

- if $z_{j'}(\mathbf{p}) = 0$ for all $j' \neq j$, then $z_j(\mathbf{p}) = 0$
- For two goods case,
$$p_1 z_1(\mathbf{p}) = -p_2 z_2(\mathbf{p}).$$

So,
$$z_1(\mathbf{p}) > 0 \Rightarrow z_2(\mathbf{p}) < 0; \text{ and } z_1(\mathbf{p}) = 0 \Rightarrow z_2(\mathbf{p}) = 0$$

1.3 Walrasian Equilibrium

Definition 2 *Walrasian Equilibrium Price: A price vector* $\mathbf{p}^*$ *is equilibrium price vector, if for all* $j = 1, ..., M$,

$$\mathbf{z}_j(\mathbf{p}^*) = \sum_{i=1}^{N} x_j^i(\mathbf{p}^*, \mathbf{p}^*.\mathbf{e}^i) - \sum_{i=1}^{N} e_j^i = 0, \quad i.e., \text{ if}$$

$$\mathbf{z}(\mathbf{p}^*) = \mathbf{0} = (0, ..., 0).$$

Two goods: food and cloth

Let (p_f, p_c) be the price vector. We can work with $\mathbf{p} = (\frac{p_f}{p_c}, 1) = (p, 1)$. Since, we know that for all $t > 0$:

$$\mathbf{z}(t\mathbf{p}) = \mathbf{z}(\mathbf{p})$$

Therefore, we have

$$pz_f(\mathbf{p}) + z_c(\mathbf{p}) = 0.$$

Assumptions:

- $z_i(\mathbf{p})$ is continuous for all $\mathbf{p} >> \mathbf{0}$, i.e., for all $p > 0$.
- there exists small $p = \epsilon > 0$ s.t. $z_f(\epsilon, 1) >> 0$ and another $p' > \frac{1}{\epsilon}$ s.t. $z_f(p', 1) << 0$.

2 Existence of Walrasian Equilibrium: General Case

As demonstrated above, the individual demand functions are homogenous functions of degree zero. That is, for all $i = 1, 2, ..., N$, $\mathbf{x}^i(t\mathbf{p}) = \mathbf{x}^i(\mathbf{p})$, for all $t > 0$. Moreover, the excess demand function is also homogenous function of degree zero. So, it has the following property: $\mathbf{z}(t\mathbf{p}) = \mathbf{z}(\mathbf{p})$, for all $t > 0$.

Without any loss of generality, we can restrict attention to the following set of prices:

$$\mathbb{P}_\epsilon = \left\{ \mathbf{p} = (p_1, ..., p_M) | \sum_{j=1}^{M} p_j = 1 \text{ and } p_j \geq \frac{\epsilon}{1 + 2M} \right\},$$

where $\epsilon > 0$.

Note that the set $\mathbb{P}_\epsilon$ contains its boundaries. So, it is closed. Moreover, it is easily seen that the $\mathbb{P}_\epsilon$ is non-empty, bounded, and convex set for all $\epsilon \in (0, 1)$.

Theorem 4 Suppose $u^i(.)$ satisfies the above assumptions, and $\mathbf{e} >> \mathbf{0}$. Let $\{\mathbf{p}^s\}$ be a sequence of price vectors in $\mathbb{R}^M_{++}$, such that

- $\{\mathbf{p}^s\}$ converges to $\bar{\mathbf{p}}$, where
- $\bar{\mathbf{p}} \in \mathbb{R}^M_+$, $\bar{\mathbf{p}} \neq \mathbf{0}$, but for some j, $\bar{p}_j = 0$.

Then, for some good k with $\bar{p}_k = 0$, the sequence of excess demand (associated with $\{\mathbf{p}^s\}$), say $\{z_k(\mathbf{p}^s)\}$, is unbounded above.

Theorem 5 *Under the above assumptions on u^i, there exists a price vector $\mathbf{p}^* >> \mathbf{0}$, such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.*

2.1 WE: Proof

We are familiar with the properties of the excess demand function $z_j(\mathbf{p})$ for every good, $j = 1, ..., M$. In the proof we will use this function to derive some other functions that will be useful in proving the result. First of all, let us define a function,

$$\bar{z}_j(\mathbf{p}) = \min\{z_j(\mathbf{p}), 1\}. \tag{2}$$

Note by its specification, $\bar{z}_j(\mathbf{p}) = \min\{z_j(\mathbf{p}), 1\} \leq 1$. Therefore, we have

$$0 \leq \max\{\bar{z}_j(\mathbf{p}), 0\} \leq 1.$$

Next, we want to define a function $f(\mathbf{p}) = (f_1(\mathbf{p}), ..., f_M(\mathbf{p})) : \mathbb{P}_\epsilon \mapsto \mathbb{P}_\epsilon$. Note that $f(\mathbf{p}) : \mathbb{P}_\epsilon \mapsto \mathbb{P}_\epsilon$ if and only if the two conditions are met. First, $f_1(\mathbf{p}) \geq\geq \frac{\epsilon}{1+2M}$ should hold for every $j = 1, .., M$. Second, $\sum_{j=1}^M f_j(\mathbf{p}) = 1$.

Suppose, we specify a function such that: For $j = 1, .., M$,

$$f_j(\mathbf{p}) = \frac{\epsilon + p_j + \max\{\bar{z}_j(\mathbf{p}), 0\}}{\epsilon M + 1 + \sum_{j=1}^M \max\{\bar{z}_j(\mathbf{p}), 0\}} = \frac{N_j(\mathbf{p})}{D(\mathbf{p})},$$

For this specification, we have $\sum_{j=1}^M f_j(\mathbf{p}) = 1$. Moreover, using the facts that $\max\{\bar{z}_j(\mathbf{p}), 0\} \leq 1$, $\epsilon < 1$ and $p_j > 0$, you can check that the following inequalities hold:

$$f_j(\mathbf{p}) \geq \frac{N_j(\mathbf{p})}{\epsilon M + 1 + M.1} \geq \frac{\epsilon}{\epsilon M + 1 + M.1} \geq \frac{\epsilon}{1 + 2M}.$$

Therefore, both of the above conditions are satisfied. So,

$$f(\mathbf{p}) = (f_1(\mathbf{p}), ..., f_M(\mathbf{p})) : \mathbb{P}_\epsilon \mapsto \mathbb{P}_\epsilon.$$

Also, since $D(\mathbf{p}) \geq 1 > 0$, the function $f(\mathbf{p})$ is a well defined and continuous function defined over a compact and convex domain. Therefore, by the Brouwer's fixed-point theorem, a 'Fixed Point' exists. That is, there exists $\mathbf{p}^\epsilon$ such that

$$f(\mathbf{p}^\epsilon) = \mathbf{p}^\epsilon, i.e.,$$

For all $j = 1, .., M$, we have: $f_j(\mathbf{p}^\epsilon) = p_j^\epsilon$. Using the full form of $f_j(.)$, this implies that for all $j = 1, .., M$,

$$\begin{aligned}
\frac{\epsilon + p_j + \max\{\bar{z}_j(\mathbf{p}), 0\}}{\epsilon M + 1 + \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}), 0\}} &= p_j^\epsilon, i.e., \\
p_j^\epsilon[M\epsilon + \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^\epsilon), 0\}] &= \epsilon + \max\{\bar{z}_j(\mathbf{p}^\epsilon), 0\}. \qquad (3)
\end{aligned}$$

Next, we let $\epsilon \to 0$. Consider the sequence of price vectors $\{\mathbf{p}^\epsilon\}$, as $\epsilon \to 0$.

- Sequence $\{\mathbf{p}^\epsilon\}$, as $\epsilon \to 0$, has a convergent subsequence, say $\{\mathbf{p}^{\epsilon'}\}$. Why?
- Let $\{\mathbf{p}^{\epsilon'}\}$ converge to $\mathbf{p}^*$, as $\epsilon \to 0$.
- Clearly, $\mathbf{p}^* \geq \mathbf{0}$. Why?

Suppose, $p_k^* = 0$. Recall, we have

$$p_k^{\epsilon'} \left[M\epsilon' + \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^{\epsilon'}), 0\} \right] = \epsilon' + \max\{\bar{z}_k(\mathbf{p}^{\epsilon'}), 0\}. \qquad (4)$$

as $\epsilon' \to 0$ while the LHS converges to 0, since $\lim_{\epsilon' \to 0} p_k^{\epsilon'} = 0$ and term $[M\epsilon' + \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^{\epsilon'}), 0\}]$ on LHS is bounded.

However, the RHS takes value 1 infinitely many times. Why? This is a contradiction, because the equality in (4) holds for all values of ϵ'. Therefore, $p_j^* > 0$ for all $j = 1, .., M$. That is,

$$\mathbf{p}^* >> \mathbf{0}, i.e.,$$

$$\lim_{\epsilon \to 0} \mathbf{p}^\epsilon = \mathbf{p}^* >> \mathbf{0}.$$

In view of continuity of $\bar{z}(\mathbf{p})$ over $\mathbb{R}_{++}^M$,
from (4) we get (by taking limit $\epsilon \to 0$):
For all $j = 1, .., M$

$$\begin{aligned}
p_j^* \sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^*), 0\} &= \max\{\bar{z}_j(\mathbf{p}^*), 0\}, i.e., \\
z_j(\mathbf{p}^*) p_j^* \left(\sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^*), 0\} \right) &= z_j(\mathbf{p}^*) \max\{\bar{z}_j(\mathbf{p}^*), 0\}, i.e., \\
\sum_{j=1}^{M} z_j(\mathbf{p}^*) p_j^* \left(\sum_{j=1}^{M} \max\{\bar{z}_j(\mathbf{p}^*), 0\} \right) &= \sum_{j=1}^{M} z_j(\mathbf{p}^*) \max\{\bar{z}_j(\mathbf{p}^*), 0\}, i.e.,
\end{aligned}$$

$$\sum_{j=1}^{M} z_j(\mathbf{p}^*) \max\{\bar{z}_j(\mathbf{p}^*), 0\} = 0. \tag{5}$$

You can verify that, given the definition of $\bar{z}_j(\mathbf{p}^*)$:

$$\begin{aligned} z_j(\mathbf{p}^*) > 0 &\Rightarrow \max\{\bar{z}_j(\mathbf{p}^*), 0\} > 0; \\ z_j(\mathbf{p}^*) \leq 0 &\Rightarrow \max\{\bar{z}_j(\mathbf{p}^*), 0\} = 0. \end{aligned}$$

Suppose, for some j, we have $z_j(\mathbf{p}^*) > 0$, then we will have

$$\sum_{j=1}^{M} z_j(\mathbf{p}^*) \max\{\bar{z}_j(\mathbf{p}^*), 0\} > 0. \tag{6}$$

But, this is a contradiction in view of (5). Therefore:

For any $j = 1, .., M$, we have

$$z_j(\mathbf{p}^*) \leq 0. \tag{7}$$

Suppose, $z_k(\mathbf{p}^*) < 0$ for some k. We know

$$p_1^* z_1(\mathbf{p}^*) + ... + p_k^* z_k(\mathbf{p}^*) + ... + p_M^* z_M(\mathbf{p}^*) = 0.$$

Since $p_j^* > 0$ for all $j = 1, .., M$.

$z_k(\mathbf{p}^*) < 0$ implies: There exists k', such that

$$z_{k'}(\mathbf{p}^*) > 0, \tag{8}$$

which is a contradiction in view of (7). Therefore,

$$\begin{aligned} \text{For all } \; j = 1, .., M, \text{ we have:} z_j(\mathbf{p}^*) &= 0, i.e., \\ \mathbf{z}(\mathbf{p}^*) &= \mathbf{0}. \end{aligned}$$