Solution - MID SEM 1

Question 1

(i) To show: $\forall\epsilon > 0, \exists\delta > 0$ s.t. $|x - y| < \delta$ gives $|\phi(x) - \phi(y)| < \epsilon$

Given: $|\phi(x) - \phi(y)| \leq (x - y)^2$ (1)

Choose $\delta = \sqrt{\epsilon}$.

Using (1), $|\phi(x) - \phi(y)| \leq (x - y)^2 < \delta^2 = \epsilon$

Hence proved.

(ii) $|\phi(x) - \phi(y)| \leq (x - y)^2$

$\iff -(x - y)^2 \leq \phi(x) - \phi(y) \leq (x - y)^2$

dividing both sides by $(x - y)$

$-(x - y) \leq \frac{\phi(x)-\phi(y)}{x-y} \leq (x - y)$

on taking limits

$\lim_{x\to y}[-(x - y)] \leq \lim_{x\to y} \frac{\phi(x)-\phi(y)}{x-y} \leq \lim_{x\to y}(x - y)$

Now, $\lim_{x\to y}[(x - y)] = 0$

Therefore by squeeze theorem

$\phi'(x) = 0$

(iii) Choose any $x < y$. By Intermediate Value Theorem, $\frac{\phi(x)-\phi(y)}{x-y} = \phi'(z)$ for some $z \in (x, y)$

Since $\phi'(z) = 0 \forall z$, we have $\phi(x) = \phi(y)$.

Question 2

Since the objective function is increasing in both x_1 and x_2 in the neighbourhood of 0 (check partials of the objective function with respect to x_1 and x_2), the maximum can not be obtained at the boundary points (that is either $x_1 = 0$ or $x_2 = 0$). If it exists, maximum will be an interior solution.

$\pi(p, w_1, w_2) = max_{x_1,x_2>0}[p(\sqrt{x_1} + \sqrt{x_2}) - (w_1x_1 + w_2x_2)]$.

FOC: $\frac{\partial\pi}{\partial x_1} = \frac{p}{2\sqrt{x_1}} - w_1$

$\frac{\partial\pi}{\partial x_2} = \frac{p}{2\sqrt{x_2}} - w_2$

Critical Point: $(x_1^*, x_2^*) = ((\frac{p}{2w_1})^2, (\frac{p}{2w_2})^2)$

SOC: $\frac{\partial^2 \pi}{\partial x_1^2} = -\frac{p}{4}(x_1)^{\frac{3}{2}}$.

$\frac{\partial^2 \pi}{\partial x_2^2} = -\frac{p}{4}(x_2)^{\frac{3}{2}}$

$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 0$

$$H(x_1, x_2) = \begin{bmatrix} -\frac{p}{4}(x_1)^{-\frac{3}{2}} & 0 \\ 0 & -\frac{p}{4}(x_2)^{-\frac{3}{2}} \end{bmatrix}$$

Evaluating $H(x_1, x_2)$ at the critical point:

$$H(x_1^*, x_2^*) = \begin{bmatrix} -\frac{p}{4}(\frac{2w_1}{p})^3 & 0 \\ 0 & -\frac{p}{4}(\frac{2w_2}{p})^3 \end{bmatrix}$$

$H(x_1, x_2)$ is a negative definite (Show either by the signs of principle leading minors or $z^T H z < 0$ for $z \neq 0$). Hence, it is a strict local maximum. This is also Global Maximum as the objective function decreases in x_1 and x_2 (check partials) as $x_1, x_2 \to \infty$

Question 3

(i) Counterexample - $A = (0, 1)$ and $\alpha = 1$. Here, $a < \alpha$ forall $a \in A$ but Sup $A = 1 = \alpha$.

(ii) Let $\epsilon > 0$ be arbitrary. Then since $\{x_n\} \to x$, there exist $N \in \mathcal{N}$ such that for all $n \geq N$, we have $|x_n - x| < \epsilon$. Then if $n \geq N$, the reverse triangle inequality shows:

$||x_n| - |x|| \leq |x_n - x| < \epsilon$

We have shown that for all $\epsilon > 0$, there exists $N \in \mathcal{N}$ such that if $n = N$, then $||x_n| - |x|| < \epsilon$. Therefore $\{|x_n|\} \to |x|$.

Question 4

(i) There are many example which will work. For instance,$I_n = (0, \frac{1}{n})$. Note that $I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n$. Also, $\cap_{n=1}^{\infty} I_n = \phi$. The intersection is an empty set as required.

(ii) $f : \Re \rightarrow \Re$

$$f(x_1, x_2) = \begin{cases} log(x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

This function is NOT one-one. $f(-1) = f(-2) = 0$ and $-1 \neq -2$. Also, this function is onto, take any point in the Range, we can always find its pre-image in the domain. Let $y = log(x)$, then $10^y = x$ and we can find always find x for any y in $\Re$.

Question 5

(i) $\forall \epsilon > 0,\ \exists x \in B_\epsilon(p)$ s.t. $x \in L.$ (Since p is a limit point of L)

Since $x \in L$, x is a limit point of E (Definition of L)

Take $\epsilon' = \epsilon - d(x, p)$

$\Rightarrow B_{\epsilon'}(x) \subset B_\epsilon(p)$ (1)

$\exists y \in B_{\epsilon'}(x)$ s.t. $y \in E.$ (Since x is a limit point of E)

$\Rightarrow y \in B_\epsilon(p)$ s.t. $y \in E$ (from (1))

Hence proved.

(ii) We first prove that $E \cup L$ is a closed set. $E \cup L$ is a closed set iff it contains all its limit points. Let s be a limit point of $E \cup L \Rightarrow$ Either s is a limit point of E or a limit point of L. If s is a limit point of L then by (1), s is also a limit point of E. Therefore s must be a limit point of E.

Hence $s \in L$ and $s \in E \cup L$.

Now we show that $E \cup L$ the smallest closed set containing E. Take any $L' \subset L$. L' does not have all limit points of E. Hence $E \cup L'$ is not a closed set.