Optimization

Example 1: Unconstrained optimization

We shall find maxima and minima of $f(x, y) = xy(1 - x - y)$ **over** $\mathcal{R}^2$.

$\frac{\partial f}{\partial x} = y(1 - 2x - y)$ and $\frac{\partial f}{\partial y} = x(1 - x - 2y)$.

$$D^2 f(x, y) = \begin{bmatrix} -2y & (1 - 2x - 2y) \\ (1 - 2x - 2y) & -2x \end{bmatrix}$$

F.O.C implies

$$\begin{aligned} y(1 - 2x - y) = &\ 0 \qquad (1) \\ x(1 - x - 2y) = &\ 0 \qquad (2) \end{aligned}$$

Solving Equation (1) and (2), we obtain four solutions:
(i) $x_1 = 0, y_1 = 0,\ f(x_1, y_1) = 0$
(ii) $x_2 = 0, y_2 = 1,\ f(x_2, y_2) = 0$
(iii) $x_3 = 1, y_3 = 0,\ f(x_3, y_3) = 0$
(iv) $x_4 = \frac{1}{3}, y_4 = \frac{1}{3},\ f(x_4, y_4) = \frac{1}{27}$

We now check S.O.C for each critical point listed above

$$(i)\ D^2 f(x_1, y_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (ii)\ D^2 f(x_2, y_2) = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$$

$$(iii)\ D^2 f(x_3, y_3) = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} \qquad (iv)\ D^2 f(x_4, y_4) = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

(i) $D^2 f(x_1, y_1)$ is an indefinite matrix: When $z = (1, 1)$, $z\big[D^2 f(x_1, y_1)\big]z^T = 2 > 0$ but for $z = (1, -1)$, $z\big[D^2 f(x_1, y_1)\big]z^T = -2 < 0$
(ii) $D^2 f(x_2, y_2)$ is an indefinite matrix: When $z = (1, -2)$, $z\big[D^2 f(x_2, y_2)\big]z^T = 2 > 0$ but for $z = (1, 1)$, $z\big[D^2 f(x_2, y_2)\big]z^T = -4 < 0$
(iii) $D^2 f(x_3, y_3)$ is an indefinite matrix: When $z = (-2, 1)$, $z\big[D^2 f(x_3, y_3)\big]z^T = 2 > 0$ but for $z = (1, 1)$, $z\big[D^2 f(x_3, y_3)\big]z^T = -4 < 0$
(iv) $D^2 f(x_4, y_4)$ is negative definite: Take any $z = (z_1, z_2)$ such that $z \neq \mathbf{0}$. Then $z\big[D^2 f(x_4, y_4)\big]z^T = -\frac{2}{3}(z_1^2 + z_2^2 + z_1 z_2) < 0$

Classification of optima

- A strict local maximum is obtained at $x_4 = \frac{1}{3}, y_4 = \frac{1}{3}$. The rest are neither local maximum nor local minimum.

- $x_4 = \frac{1}{3}, y_4 = \frac{1}{3}$ is not a global maximum. In fact global maximum does not exist. To see this, for example, take $y = 2 - x$ (one may choose other value of y). Then $f(x, y) = -x(2 - x)$, which goes to ∞ as x goes to ∞.
- Similarly global minimum does not exist. For instance, choose $y = x$. Then $f(x, y) = x^2(1 - 2x)$, which goes to -∞ as x goes to ∞.

Example 2: Optimization with Equality constraint

We shall find maxima and minima of $f(x, y) = xy(1 - x - y)$ **subject to** $2x + y = 2$.

We rewrite the constraint as $g(x, y) = 2 - 2x - y$.
$\frac{\partial g}{\partial x} = -2$ and $\frac{\partial g}{\partial y} = -1. Dg = (-2, -1)$. $D^2 g(x, y) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
λ is Lagrange multiplier. F.O.C and constraints are listed below:

$$y(1 - 2x - y) - 2\lambda = 0 \quad (3)$$
$$x(1 - x - 2y) - \lambda = 0 \quad (4)$$
$$2 - 2x - y = 0 \quad (5)$$

Replacing $y = 2 - 2x$ (from Equation (5)) in Equation (3) and (4) we obtain $\lambda = (x - 1)$ and $\lambda = 3x(x - 1)$ respectively. These give us the following critical points:
(i) $x_1 = 1, y_1 = 0, \lambda_1 = 0,$ $f(x_1, y_1) = 0$ (same as critical point (iii) in Example 1)
(ii) $x_2 = \frac{1}{3}, y_2 = \frac{4}{3}, \lambda_2 = -\frac{2}{3},$ $f(x_2, y_2) = -\frac{8}{27}$

We now check S.O.C for each critical point listed above
(i) $D^2\mathcal{L}(x_1, y_1) = D^2 f(x_1, y_1) + \lambda_1 D^2 g(x_1, y_1) = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$
$(ii) D^2\mathcal{L}(x_2, y_2) = D^2 f(x_2, y_2) + \lambda_2 D^2 g(x_2, y_2) = \begin{bmatrix} -8/3 & -7/3 \\ -7/3 & -2/3 \end{bmatrix} - 2/3 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -8/3 & -7/3 \\ -7/3 & -2/3 \end{bmatrix}$

Now, let us identify $z = (z_1, z_2)$ such that $Dg \cdot z = 0$. This implies $2z_1 + z_2 = 0$ or $z_2 = -2z_1$. Hence we have $z = (z_1, -2z_1)$ where $z_1 \neq 0$

(i) $(z_1, -2z_1)\left[D^2\mathcal{L}(x_1, y_1)\right](z_1, -2z_1)^T = -4z_1^2 < 0$
(ii) $(z_1, -2z_1)\left[D^2\mathcal{L}(x_2, y_2)\right](z_1, -2z_1)^T = 4z_1^2 > 0$

Classification of optima

- A strict local maximum is obtained at $x_1 = 1, y_1 = 0$. Note that $x_1 = 1, y_1 = 0$ was not a local maximum in Example 1 but it is a local maximum here because of the constraint. A strict local minimum is obtained at $x_2 = \frac{1}{3}, y_2 = \frac{4}{3}$.
- Global maximum and minimum do not exist even with the constraint. Replacing $y = 2 - 2x$ (from Equation (5)) in the objective function, we get, $f(x, y) = -2x(1 - x)^2$, which goes to ∞ (resp. $-\infty$) as x goes to $-\infty$ (resp. ∞).

Note that the strict local maximum in Example 1 does not appear in Example 2, because $x_4 = \frac{1}{3}, y_4 = \frac{1}{3}$ fails to satisfy the constraint.

Example 3: Optimization with Inequality constraint

We shall find maxima and minima of $f(x, y) = xy(1 - x - y)$ subject to $2x + y \leq 2$ and $x \geq 0, y \geq 0$.

We rewrite the constraint as $g^1(x, y) = (2 - 2x - y) \geq 0$, $g^2(x, y) = x \geq 0$ and $g^3(x, y) = y \geq 0$

$\lambda^1 \geq 0$, $\lambda^2 \geq 0$ and $\lambda^3 \geq 0$ are Lagrange multipliers. F.O.C and constraints are listed below:

$$y(1 - 2x - y) - 2\lambda^1 + \lambda^2 = 0 \quad (6)$$
$$x(1 - x - 2y) - \lambda^1 + \lambda^3 = 0 \quad (7)$$
$$\lambda^1(2 - 2x - y) = 0 \quad (8)$$
$$\lambda^2 x = 0 \quad (9)$$
$$\lambda^3 y = 0 \quad (10)$$

Each λ^k can be either positive or zero. Thus we consider following eight possible combinations of values of Lagrange multipliers.
$\lambda^1 > 0, \lambda^2 > 0, \lambda^3 > 0$: Then $x = 0$ (by Equation 9), $y = 0$ (by Equation 10) and $2x + y = 2$ (by Equation 8). These are not compatible. Hence we don't have a solution of this type.
$\lambda^1 = 0, \lambda^2 > 0, \lambda^3 > 0$: Once again $x = 0$ and $y = 0$. Replacing x, y and $\lambda^1 = 0$ in Equation 6, we get $\lambda^2 = 0$. A contradiction. Hence no solution of this type.

$\lambda^1 > 0, \lambda^2 = 0, \lambda^3 > 0$: Here $y = 0$. Replacing y and $\lambda^2 = 0$ in Equation 6, we get $\lambda^1 = 0$, which is a contradiction. Similarly $\lambda^1 > 0, \lambda^2 > 0, \lambda^3 = 0$ is also ruled out.
$\lambda_1 = 0, \lambda^2 = 0, \lambda^3 > 0$: Here $y = 0$. Replacing y and $\lambda^1 = 0$ in Equation 7, we get $\lambda^3 = -x(1-x)$. However g^1 and $y = 0$ gives us $x \leq 1$ while g^2 gives us $x \geq 0$. Hence $\lambda^3 = -x(1-x) \leq 0$, which is a contradiction. we can similarly rule out $\lambda^1 = 0, \lambda^2 > 0, \lambda^3 = 0$.
Thus we are left with just two more cases

$\lambda_1 > 0, \lambda^2 = 0, \lambda^3 = 0$: Since $\lambda^1 > 0$, From Equation 8, $g^1(x, y) = 0$. There are no other constraints. This is same as Example 2. Equation 6,7 and 8 reduce to Equation 3,4 and 5 of Example 2. We know there are two critical points
(i) $x_1 = 1, y_1 = 0,\ f(x_1, y_1) = 0$
(ii) $x_2 = \frac{1}{3}, y_2 = \frac{4}{3},\ f(x_2, y_2) = -\frac{8}{27}$
$\lambda_1 = 0, \lambda^2 = 0, \lambda^3 = 0$: There are no constraints, which is same as Example 1. Equation 6 and 7 reduce to Equation 1 and 2. We know there are four critical points in this case, one of which already appears above. So we note the remaining three
(iii) $x_1 = 0, y_1 = 0,\ f(x_1, y_1) = 0$
(iv) $x_2 = 0, y_2 = 1,\ f(x_2, y_2) = 0$
(v) $x_4 = \frac{1}{3}, y_4 = \frac{1}{3},\ f(x_4, y_4) = \frac{1}{27}$

Classification of optima

- Note that the restricted domain is a compact set (closed and bounded). Hence Global maximum and minimum exists. Let us compare the functional values at critical points.
- A global minimum is obtained at $x_2 = \frac{1}{3}, y_2 = \frac{4}{3}$. A global maximum is obtained at $x_4 = \frac{1}{3}, y_4 = \frac{1}{3}$.
- These results are different from Example 1 and 2, thanks to additional constraints.
- Classification of remaining three critical points can be done by comparing functional values in their respective neighbourhood. For example, take a neighbourhood of $(0, 0)$ intersected with our domain. In a small neighbourhood, $f(x, y) > 0$ for all (x, y) in the neighbourhood. Hence $(0, 0)$ is a local minimum. Check that the other two critical points are saddle points.

- We shall skip the issue of classifying critical points based on Second order condition. However one may use second order conditions of unconstrained optimization and Lagrange method (for equality constraints) as follows:
  (i) If a critical point is an interior point, that is, none of the constraint is binding at this point, then we can use second order condition for unconstrained optimization.
  (ii) For a critical point which is not an interior point, first identify all constraints which are binding. Then use second order condition of Lagrange method as if those are the only constraints.

Solution of Exercise 6.5 (Sundaram)
(a) Check that the feasible set is closed but not bounded above.

(b) Cost minimization problem: Minimize $f(x_1, x_2) = w_1x_1 + w_2x_2$ over the feasible set $S = \{(x_1, x_2) \mid \bar{y} = \sqrt{x_1x_2}, x_1 \geq 1, x_2 \geq 0\}$. Note that $\bar{y}, w_1, w_2$ are exogenous parameter. They are fixed for our minimzation problem.

Let us rewrite the constraints as (i) $h_1(x_1, x_2) = \bar{y} - \sqrt{x_1x_2} = 0$, (ii) $h_2(x_1, x_2) = x_1 - 1 \geq 0$ and (iii) $h_3(x_1, x_2) = x_2 \geq 0$. Note that there is one equality constraint and two inequality constraints. Corresponding multipliers are $\lambda_1, \lambda_2, \lambda_3$. There is no restriction on λ_1, which is multiplier of an equality constraint, while $\lambda_2 \geq 0$ and $\lambda_3 \geq 0$.

First order conditions:

$$w_1 + \lambda_1 \frac{\sqrt{x_2}}{2\sqrt{x_1}} - \lambda_2 = 0 \quad \text{(partial w.r.t } x_1) \tag{11}$$

$$w_2 + \lambda_1 \frac{\sqrt{x_1}}{2\sqrt{x_2}} - \lambda_3 = 0 \quad \text{(partial w.r.t } x_2 \tag{12}$$

$$\bar{y} - \sqrt{x_1x_2} = 0 \tag{13}$$

$$\lambda_2(x_2 - 1) = 0 \tag{14}$$

$$\lambda_3 x_2 = 0 \tag{15}$$

Now we shall find critical points:
$\lambda_3 > 0$: $x_2 = 0$ (from Equation (15)) contradicts with Equation (13).

$\lambda_2 > 0, \lambda_3 = 0$: $\lambda_2 > 0$ implies $x_1 = 1$ (from Equation (14)). From Equation (13), $x_2 = \bar{y}^2$. From Equation (12) we get $\lambda_1 = -2w_2\bar{y}$. Using this in Equation (11), we have $\lambda_2 = w_1 - w_2\bar{y}^2$. Since $\lambda_2 > 0$, this is a critical point only when $\frac{w_1}{w_2} > \bar{y}^2$.

$\lambda_2 = 0, \lambda_3 = 0$: Eliminating λ_1 from Equation (11) and (12), we get $\frac{x_2}{x_1} = \frac{w_1}{w_2}$. Replacing this in Equation (13), we get $x_1 = \bar{y}\frac{w_2}{w_1}$ and $x_2 = \bar{y}\frac{w_1}{w_2}$. Since $x_1 \geq 1$, this is a critical point only when $\frac{w_1}{w_2} \leq \bar{y}^2$.

This divide the parameter space into two parts:
If $\frac{w_1}{w_2} > \bar{y}^2$ then the only critical point is $x_1^\star = 1$, $x_2^\star = \bar{y}^2$. $f(x_1^\star, x_2^\star) = w_1 + w_2\bar{y}^2$.
If $\frac{w_1}{w_2} \leq \bar{y}^2$ then the only critical point is $\hat{x}_1 = \bar{y}\frac{w_2}{w_1}$, $\hat{x}_2 = \bar{y}\frac{w_1}{w_2}$. $f(\hat{x}_1, \hat{x}_2) = 2\bar{y}\sqrt{w_1 w_2}$.

Let us now show that our minimization problem has global minimum. Recall that the feasible set S is not compact because it is not bounded above. However since the objective function is increasing in x_1 and x_2, we can put an upper bound to S without changing the global minimum. We can formalize this argument as follows. Consider the following subset of the original feasible set.

$$S' = \left\{ (x_1, x_2) \in S \mid x_1 \leq \frac{1}{w_1} \max\{f(x_1^\star, x_2^\star), f(\hat{x}_1, \hat{x}_2)\} \text{ and } x_2 \leq \frac{1}{w_2} \max\{f(x_1^\star, x_2^\star), f(\hat{x}_1, \hat{x}_2)\} \right\}$$

Such restriction of feasible set does not change the original global minimum (if it exists) because we have only removed input bundles which are costlier than our critical points. We want to show that global minimum exists for the restricted problem, which proves that the global minimum exists for the original problem.

S' is clearly bounded above. S' is bounded below because S is bounded below. It is also easy to check that S' is closed.