Social Choice Rules: Possibility and Impossibility

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Microeconomic Theory

Lecture 18

Arrow's Impossibility Theorem I

Question

Can $\sharp V = 1$*?*

Theorem

There is no SWF that satisfies conditions U, P, I and ND simultaneously.

Proof: Take any $x, y \in \mathbb{X}$. We know that if a SWF satisfies conditions U, P and I, then $\exists V \subseteq \mathbb{N}$ such that:

V is $D(x, y)$. Why?

Can we use Weak Pareto Principle here?

Arrow's Impossibility Theorem II

Let

$$\mathbb{V} = \{V | V \text{ is } D(u,v) \ \text{ for some } \ u, v \in \mathbb{X}\}.$$

Let $\bar{V} \in \mathbb{V}$ be the smallest size set. Suppose

$$\bar{V} \text{ is } D(x,y).$$

- Case 1: $\sharp\bar{V} = 1$. No need to proceed further. So, consider
- Case 2: $\sharp\bar{V} > 1$. In that case, let
- $V_1, V_2 \subset \bar{V}$ be such that: $\sharp V_1 = 1$; $V_2 = \bar{V} - V_1$. So,
- V_1 and V_2 form a partitioning of V, i.e., $V_1 \cup V_2 = \bar{V}$, and $V_1 \cap V_2 = \emptyset$
- Let $V_3 = \mathbb{N} - \bar{V}$

Arrow's Impossibility Theorem III

Consider the following:

$$(\forall i \in V_1)[xP_iy \quad \& \quad yP_iz].$$
$$(\forall j \in V_2)[yP_jz \quad \& \quad zP_jx].$$
$$(\forall k \in V_3)[xP_ky \quad \& \quad zP_kx].$$

This gives us $y\mathcal{P}z$.

Also, $y\mathcal{P}x$ or $x\mathcal{R}y$. Why?

But, $y\mathcal{P}x$ would mean

V_2 is $D(y,x)$, which means V_2 is decisive - a contradiction.

On the other hand, $x\mathcal{R}y$ means

$$x\mathcal{R}y \;\&\; y\mathcal{P}z \Rightarrow x\mathcal{P}z, i.e.,$$

Arrow's Impossibility Theorem IV

$$V_1 \text{ is } \mathrm{D(x,z)},$$

again a contradiction.

Theorem

There is no SWF that can simultaneously satisfy conditions U, P I, and ND.

Theorem

If a SWF satisfies conditions U, P and I, then $\exists i \in \mathbb{N}$ such that

$$(\forall x, y \in \mathbb{X})(\forall (R_1, ..., R_n) \in \mathbb{O}^n)[xP_iy \Rightarrow x\mathcal{P}y].$$

SWF: Examples I

Proposition

There exists a SCR that satisfies conditions U, P, I, and ND.

Let

- $N(xPy)$ number of individuals who strictly prefer x over y
- $N(xRy)$ number of individuals who weakly prefer x over y

Definition

A Method of Majority Rule is a SCR such that:

$$(\forall x, y \in \mathbb{X})[x\mathcal{R}y \Leftrightarrow [N(xPy) \geq N(yPx)], \text{ or}$$

$$(\forall x, y \in \mathbb{X})[x\mathcal{R}y \Leftrightarrow [N(xRy) \geq N(yRx)].$$

MMR satisfies all conditions but $\mathfrak{R} \notin \mathbb{O}$.

SWF: Examples II

Proposition

There exists a SWF $f : \mathbb{D} \mapsto \mathbb{O}$ that satisfies conditions U, P, and ND, but does not satisfy condition I.

Example: 'Borda count' method.

If

$Score$	$R1$	$R2$	$R3$
3	x	y	z
2	y	z	x
1	z	x	y

, the usual rank-score of each alternative is 6.

However, consider the following profile:

$Score$	$R'1$	$R'2$	$R'3$
3	x	y	z
2	y	x	x
1	z	z	y

, now the rank-score of x is 7 is maximum.

SWF: Examples III

Proposition

There exists a SWF $f : \mathbb{D} \mapsto \mathbb{O}$ *that satisfies conditions P, I, and ND, but* $\mathbb{D} \subset\subset \mathbb{O}^n$

Definition

Single Peakedness. R is single peaked if there exists a re-arrangement of alternatives in $\mathbb{X}$, say $\{y_1, y_2, ..., y_m\}$, and some y^*, say $y^* = y_k$, such that

$$\begin{aligned} j' < j \leq k \quad &\Rightarrow \quad y_j P y_{j'} \\ l' > l \geq k \quad &\Rightarrow \quad x_l P x_{l'} \end{aligned}$$

Remark: In general, y^* will differ across Preference relations.

SWF: Examples IV

Proposition

If preferences are single-peaked and number of individuals is odd, there exists a SWF $f : \mathbb{D} \mapsto \mathbb{O}$ that satisfies conditions P, I, and ND.

Answer is : MMR

Liberal Paradox I

Definition

Liberalism L: For every $i \in \mathbb{N}$, there is a pair of distinct alternatives $(x, y) \in \mathbb{X} \times \mathbb{X}$ such that

$$xP_iy \Rightarrow x\mathcal{P}y \text{ and} yP_ix \Rightarrow y\mathcal{P}x$$

Definition

Minimal Liberalism L*: For at least two individuals **Liberalism** holds.

Proposition

*No SWF can satisfy conditions U, P and L**

Suppose conditions U, P and L* hold. Let

Liberal Paradox II

- j be decisive for (x, y)
- k be decisive for (z, w)
- $xP_j y$, $zP_k w$ and $(\forall i)[wP_i x \ \& \ yP_i z]$

This gives us,

$$x\mathcal{P}y, \ z\mathcal{P}w, \ w\mathcal{P}x \text{ and } y\mathcal{P}z, i.e.,$$

$$x\mathcal{P}z, \ z\mathcal{P}w, \text{ and } w\mathcal{P}x,$$

a contradiction.

The preferences are as follows

$$\begin{array}{ccc} i & j & k \\ . & w & y \\ . & x & z \\ . & y & w \\ . & z & x \end{array},$$

Summing Up

- Implications of relaxing condition $\mathfrak{R} \in \mathbb{O}$
- Implications of relaxing/changing condition I
- Implications of relaxing/changing condition P
- Implications of relaxing/changing condition ND
- Implications of relaxing/changing condition U

There are trade-offs among

- Rationality of society
- Individual liberty
- Democracy