Introductory Mathematical Economics (002)

Part II (Dynamics)

Lecture Notes (MAUSUMI DAS**)**

DIFFERENCE AND DIFFERENTIAL EQUATIONS:

Some Definitions:

State Vector: At any given point of time *t*, a dynamic system is typically described by a dated n-vector of real numbers, **x**(*t*), which is called the *state vector* and the elements of this vector are called *state variables*.

(As we shall see later, in dynamic problems, there could be other type of variables, which are called *control variables*. The appropriate definition of a state variable for a given problem is not always unique. For the time being however we shall identify all those variables whose changes we wish to observe over time as state variables.)

State Space: The *state space X* is a subset of $\mathbf{R}^n$ that contains all feasible state vectors of the system .

A. Difference Equation

A ***difference equation of order m*** in a time dependent variable x_t is an equation of the form

$$F(t, x_t, x_{t-1,} ..., x_{t-m}; \boldsymbol{\alpha}) = 0$$

where *F* is a function that for each *t* and α, maps points in $\mathbf{R}^{(m+1)}$ to $\mathbf{R}$.

In general the above equation can be written in an explicit form as

$$x_t = f(t; x_{t-1}, x_{t-2}, ..., x_{t-m}; \boldsymbol{\alpha})$$

where for each *t* and α, the function *f* maps points in $\mathbf{R}^m$ to $\mathbf{R}$.

In other words, the *f* function relates the state variable at time *t* to its *m* number of previous values.

The ***order*** of a difference equation is the difference between the largest and the smallest time subscript appearing in the equation.

A difference equation is said to be ***linear*** if *f* is a linear function of the state variables.

A difference equation is said to be ***autonomous*** if the time variable *t* does not enter as a separate argument in the *f* function.

We can define a system of difference equations in similar fashion:

A ***system of difference equations of order m*** in an *n*-dimensional vector of time dependent variables $\mathbf{x}_t$ is defined as

$$F(t; \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathbf{x}_{t-m}; \boldsymbol{\alpha}) = \theta$$

where θ is a *n*-dimensional vector of zeros and the function *F* now maps points in $\mathbf{R}^{(m+1)n}$ to $\mathbf{R}^n$.

Remark. Any difference equation of higher order can be reduced to a system of difference equations of first order by introducing additional equations and variables. For example, consider the difference equation of order 2:

$$x_t = f(x_{t-1}, x_{t-2})$$

Let us define a new variable, $y_t = x_{t-1}$. Then the above second order difference equation can be expressed as system of first order difference equations in variables *x* and *y* in the following way:

$$x_t = f(x_{t-1}, y_{t-1})$$
$$y_t = x_{t-1} + 0.y_{t+1} = \hat{f}(x_{t-1}, y_{t-1})$$

A.1 Solving a Linear First Order Difference Equation

(a) AUTONOMOUS EQUATIONS:

In this section we shall be dealing with an *autonomous, linear*, first order difference equation of the form

$$x_t = ax_{t-1} + b \qquad (1)$$

where the initial value of the variable at time 0 (i.e., x_0) is given.

Superposition Principle: The general solution to a linear difference equation given in (1) can be written as

$$x_t^g = x_t^c + x_t^p$$

where x_t^c is the general solution to the corresponding homogeneous equation $x_t = ax_{t-1}$, and x_t^p is *any* particular solution to (1). x_t^c is called the ***complementary function*** and x_t^p is called a ***particular solution***.

First let us look at the homogeneous equation

$$x_t = ax_{t-1} \qquad (1')$$

which we solve to get the complementary function. We shall use the *method of iteration* to find a solution.

Note that $x_t = ax_{t-1}; x_{t-1} = ax_{t-2}; x_{1-2} = ax_{t-3}$ and so on. Hence substituting for each of the preceding values, we can write $x_t = ax_{t-1} = a^2 x_{t-2} \ldots$, and eventually $x_t = a^t x_0$. Since we know x_0 and *a*, we can find out the exact value of *x* at any point of time *t.* Thus $x_t = a^t x_0$ is a solution to (1′).

We have derived a solution here in terms of the initial value x_0. If instead the value of *x* at some other point of time was known to us (for example, suppose we knew the value of *x* at time *s*, x_s), then we could have derived the exact solution in terms of that given value. Thus the general solution to the homogenous equation (1′) (or the ***complementary function***) is given by ca^t, where *c* is an arbitrary constant whose value is to be determined by the given initial or any other boundary condition.

Having derived the complementary function as $x_t^c = ca^t$, let us now find a particular solution of (1). Remember that x_t^p is ***any*** particular solution to the difference equation (1). So consider the simplest possible solution where *x* is a constant. We want to know whether $x_t = k \text{ (a constant)}$ could be a solution to (1). Note that if *x* remains constant over time, then $x_{t-1} = k$ as well. Putting these values in (1), we get

$$k = ak + b \Rightarrow k = \frac{b}{1-a}.$$

Therefore $x_t = k$ would indeed be a solution to (1) *if and only if* $k = \frac{b}{1-a}$. Thus we have found a ***particular solution*** to (1), given by $x_t^p = \frac{b}{1-a}$.

Hence by superposition principle, the ***general solution*** to the autonomous first order difference equation in (1) is given by

$$x_t^g = ca^t + \frac{b}{1-a}$$

where *c* is an arbitrary constant whose value is to be determined by the given initial or any other boundary condition.

Remark. Observe that while the complementary function is defined for any value of a, the particular solution is not defined when $a = 1$. Thus if $a = 1$, the trial solution $x_t = k$ will not work; we have to proceed with some other trial solution. When $a = 1$, equation (1) would be of the form

$$x_t = x_{t-1} + b \qquad (1'')$$

To find the particular solution in this case, let us try another possible solution, $x_t = kt$. Then $x_{t-1} = k(t-1)$. If this is indeed a solution to (1″), then it should satisfy (1″). Putting these values in (1″), we get

$$kt = k(t-1) + b \Rightarrow k = b$$

Thus $x_t = bt$ would be a solution to (1″). Hence when $a = 1$, the general solution to a differential equation of the form $x_t = ax_{t-1} + b$ is given by $x_t = c + bt$.

(b) NON-AUTONOMOUS EQUATIONS:

Let us now consider a *non-autonomous*, *linear* first-order difference equation of the form

$$x_t = ax_{t-1} + b_t \qquad (2)$$

The superposition principal holds for (2) as well. Note that the homogeneous component of (2) is the same as that of (1); hence they have the same complementary function given by ca^t. Thus we just have to find a particular solution to (2).

Note that the earlier method of trying a constant value of *x* as a solution will not work here because the term *b* is changing over time; so no x_t and x_{t-1} could be the same (unless b_t is constant, which makes the difference equation autonomous).

If we know the exact time path of b_t, then we could proceed with some trial solution, depending on the functional form of b_t. For example, let $b_t = B(b)^t$, where B and b are given parameters. Then the non-autonomous first order difference equation given in (2) becomes

$$x_t = ax_{t-1} + B(b)^t \qquad (2')$$

Since the term b_t following a specific time path, the simplest possibility is that the variable x_t also follows a similar time path. So let us try a solution of the form $x_t = k(b)^t$, where *k* is an unknown constant. If this is indeed a solution to (2′), then

$$k(b)^t = ak(b)^{t-1} + B(b)^t$$

The above condition will be satisfied if and only if $k = \frac{Bb}{b-a}$. Thus we have found a particular solution to (2′) as $x_t^p = \frac{Bb}{b-a}(b)^t$. Thus the ***general solution*** to (2′) is given by

$$x_t = ca^t + \frac{Bb}{b-a}(b)^t .$$

Remark. Observe that the particular solution given above is defined if and only if $b \neq a$. When $b = a$, then the difference equation (2′) takes the following form:

$$x_t = ax_{t-1} + B(a)^t \qquad (2'')$$

Let us now try a solution of the form $x_t = kt(a)^t$. If this a solution to (2′′), then

$$kt(a)^t = ak(t-1)(a)^{t-1} + B(a)^t$$

Solving we get $k = B$. Therefore the particular solution in this case is given by $x_t^p = Bt(a)^t$, and the general solution is

$$x_t = (c + Bt)a^t .$$

If the particular form of b_t is not known then we cannot derive a solution using this kind of a trial and error method. There are two more general ways, which are often used in economic dynamics to derive a particular solution to non-autonomous equations like (2): one method involves iteration backward and use of some initial condition to arrive at the exact solution; the other method involves iteration forward, and use of some terminal condition. The particular solutions thus obtained are called the ***backward solution*** and the ***forward solution*** respectively.

(i) Backward Solution:

Iterating equation (2) backward for *n* periods (i.e., up to the (*t*–*n*)th period),

$$\begin{aligned} x_t &= ax_{t-1} + b_t \\ &= a^2 x_{t-2} + ab_{t-1} + b_t \\ &= a^3 x_{t-3} + a^2 b_{t-2} + ab_{t-1} + b_t \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= a^n x_{t-n} + (a^{n-1} b_{t-n+1} + a^{n-2} b_{t-n+2} + \ldots + ab_{t-1} + b_t) \end{aligned}$$

i.e., $x_t = a^n x_{t-n} + \sum_{i=0}^{n-1} a^i b_{t-i}$

If we iterate backward up to starting point when the system was set in motion (i.e., setting (*t*–*n*)=0), and if initial value of the variable is known to us, then we can derive the particular solution as:

$$x_t^p = a^t x_0 + \sum_{i=0}^{t} a^i b_{t-i} .$$

On the other hand the complementary function is already given by ca^t, where *c* is an arbitrary constant.

Thus the ***general backward solution*** to equation (2) would be given by

$$x_t = c_B\, a^t + \sum_{i=0}^{t} a^i b_{t-i}$$

where $c_B = c + x_0$ is an arbitrary constant.

(ii) Forward Solution:

Another particular solution results if we iterate equation (2) forward rather than backward. Note that we can also write (2) as

$$x_t = \frac{1}{a} x_{t+1} - \frac{1}{a} b_{t+1}.$$

Iterating the above equation forward for *n* periods (that is, up to the (*t*+*n*)th period), we get

$$\begin{aligned} x_t &= \frac{1}{a} x_{t+1} - \frac{1}{a} b_{t+1} \\ &= \left(\frac{1}{a}\right)^2 x_{t+2} - \left(\frac{1}{a^2} b_{t+2} + \frac{1}{a} b_{t+1}\right) \\ &= \left(\frac{1}{a}\right)^3 x_{t+3} - \left[\left(\frac{1}{a}\right)^3 b_{t+3} + \left(\frac{1}{a}\right)^2 b_{t+2} + \frac{1}{a} b_{t+1}\right] \\ &\quad \ldots\ldots\ldots\ldots \\ &= \left(\frac{1}{a}\right)^n x_{t+n} - \sum_{i=1}^{n}\left[\left(\frac{1}{a}\right)^i b_{t+i}\right] \end{aligned}$$

If we know where the system is headed *n* periods from now, i.e., if we are given the terminal value of the variable , x_{t+n}, then we can derive a particular solution of (2) directly from the above equation. For example if we reach the terminal time T after $(t+n)$ periods and we know the terminal value of x, given by x_T, then we can write the particular solution as:

$$x_t^p = a^t \frac{x_T}{a^T} - \sum_{i=1}^{T-t}\left[\left(\frac{1}{a}\right)^i b_{t+i}\right]$$

In many cases however there is no well-defined finite terminal point; hence the terminal value is often not defined. Letting $n \to \infty$, we get

$$x_t = \lim_{n\to\infty}\left[\left(\frac{1}{a}\right)^n x_{t+n}\right] - \left(\frac{1}{a}\right)\sum_{i=0}^{\infty}\left[\left(\frac{1}{a}\right)^i b_{t+1+i}\right]$$

The above equation will give us a forward-looking particular solution to (2), provided (a) the limit exists, and (b) the infinite sum converges.

For finite horizon cases (with given terminal points and terminal values), we can write the ***general forward solution*** to (2) as

$$x_t = c_F a^t - \sum_{i=1}^{T-t}\left[\left(\frac{1}{a}\right)^i b_{t+i}\right]$$

where $c_F = (c + \frac{x_T}{a^T})$ is an arbitrary constant.

A.2 Autonomous First Order Difference Equation: Steady States and Stability

Consider an autonomous first order difference equation of the form

$$x_t = f(x_{t-1}, \boldsymbol{\alpha}) \qquad (3)$$

Steady state: A point $\bar{x} \in X$ is a steady state of the difference equation given in (3) if it is a *fixed point* of the map *f*, that is, if $\bar{x} = f(\bar{x}, \boldsymbol{\alpha})$.

Stationary or steady state values of autonomous dynamical systems are those values, which will be preserved in perpetuity if they are attained once. These are also called rest points or long run equilibrium points.

Comment: Note that we are defining the steady state only in the context of autonomous equations. **(Why?)**

Stability of a dynamical system: A dynamical system with a steady state $\bar{x} \in X$ is said to be *asymptotically stable* if $\lim_{t\to\infty} x_t = \bar{x}$. In other words, a dynamical system is asymptotically stable if the all its trajectories approach the steady state value over time, irrespective of the initial position.

Example: Let us consider the first order linear autonomous difference equation of the form

$$x_t = ax_{t-1} + b,\ a \neq 1 \qquad (3')$$

We had derived the general solution to this equation as $x_t = ca^t + \frac{b}{1-a}$.

We want to derive its steady state value and also want to examine its stability property.

Note that here $f(x_{t-1}) = ax_{t-1} + b$. By definition, the steady state solves the equation: $\bar{x} = f(\bar{x})$ i.e., $\bar{x} = a\bar{x} + b$. Solving, we get the ***steady state*** value as $\bar{x} = \frac{b}{1-a}$, which is nothing but the ***particular solution*** that we had derived earlier.

For stability we require x_t to tend to $\bar{x}$ as *t* approaches infinity. Writing the general solution as $x_t = ca^t + \bar{x}$, we see that $\lim_{t\to\infty} x_t = \bar{x}$ if and only if $|a| < 1$. In this case the term ca^t vanishes as *t* gets larger and the system converges asymptotically to the steady state $\bar{x}$ for any value of *c* (i.e., for any initial condition).

If $a > 0$, then x_t approaches $\bar{x}$ monotonically. If $a < 0$, then the term a^t becomes positive and negative for alternate (even and odd) values of *t*, and x_t approaches $\bar{x}$ in an oscillating manner.

If $|a| > 1$, then the term ca^t explodes as *t* goes to infinity, unless *c*=0. In other words, the system will be unstable (unless we start with an initial value such that *c*=0).

Thus the ***stability condition*** for (3′) is given by $|a| < 1$.

A.3 Solving a System of First Order Difference Equations (Linear and Autonomous)

(a) *n* dimensional system:

So far we have looked at methods for solving a single linear difference equation. Now consider a system of linear and autonomous equations of the form

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + \mathbf{b} \qquad (4)$$

where ***A*** is an $n \times n$ matrix of constant coefficients and **x** is an $n \times 1$ vector (*n* dimensional column vector) of dated state variables and **b** is a $n \times 1$ vector of constant terms.

The superposition principle holds for this higher dimensional case as well. Thus we can write the general solution to the system as

$$\mathbf{x}_t^g = \mathbf{x}_t^c + \mathbf{x}_t^p$$

where $\mathbf{x}_t^c$ is a solution to the relevant homogeneous system $\mathbf{x}_t = A\mathbf{x}_{t-1}$, and $\mathbf{x}_t^p$ is a particular solution to the full system (often its steady state solution).

First let us find out a particular solution $\mathbf{x}_t^p$. As before, we try with constant values of all *x*, which would be the ***steady state values*** of these variables. Hence using the steady state condition $\overline{\mathbf{x}} = A\overline{\mathbf{x}} + \mathbf{b}$, we can directly derive the particular solution as $\mathbf{x}_t^p = (I - A)^{-1}\mathbf{b}$, where *I* is an $n \times n$ identity matrix.

Remark. Note that for such a particular solution to exist (and therefore for the steady state to exist), the (*I* – *A*) matrix must be invertible.

Remark. The steady state, if exists, will be unique. **(Why?)**

To derive the complementary function $\mathbf{x}_t^c$, consider the homogenous system

$$\mathbf{x}_t = A\mathbf{x}_{t-1} \qquad (4')$$

Observe that if we can diagonalize *A* by some similarity transformation, then we can define a similar (but much simpler) system in terms of a set of new variables, which are suitable transformation of the original variables.

Let us assume that *A* has all distinct eigenvalues. So it is diagonizable. Let these eigenvalues be denoted by $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$. Let M be the corresponding matrix of eigenvectors that diagonalizes *A*. Then $M^{-1}AM = \Lambda$, where Λ is a diagonal matrix with all the eigenvalues of *A* as its diagonal elements.

Now consider the homogeneous system of difference equations given in (4′):

$$\mathbf{x}_t = A\mathbf{x}_{t-1}$$

Let us use the following similarity transformation to define a new vector of state variables:

$$\mathbf{y} = M^{-1}\mathbf{x}.$$

Then $\mathbf{x}_t = M\mathbf{y}_t$ and $\mathbf{x}_{t-1} = M\mathbf{y}_{t-1}$.

Substituting these values to (4′):

$$M\mathbf{y}_t = AM\mathbf{y}_{t-1}$$

$$\Rightarrow M^{-1}M\mathbf{y}_t = M^{-1}AM\mathbf{y}_{t-1} = \Lambda\mathbf{y}_{t-1}$$
$$\Rightarrow \mathbf{y}_t = \Lambda\mathbf{y}_{t-1} \qquad (4'')$$

Equation (4″) represents a system of *n* independent difference equations of the form:

$$y_t^1 = \lambda_1 y_{t-1}^1$$
$$y_t^2 = \lambda_2 y_{t-1}^2$$
$$\ldots\ldots\ldots\ldots$$
$$y_t^n = \lambda_n y_{t-1}^n$$

We know that the solution to this simplified system is given by the $n \times 1$ column vector $\mathbf{y}_t$ with elements: $y_t^1 = c_1(\lambda_1)^t,\ y_t^2 = c_2(\lambda_2)^t,\ \ldots, y_t^n = c_n(\lambda_n)^t$, where $c_1,\ c_2,\ \ldots\ldots,\ c_n$ are arbitrary constants.

Thus we can now write the general solution to the system of difference equation given in (4) as

$$\mathbf{x}_t = M\mathbf{y}_t + (I - A)^{-1}\mathbf{b}$$

i.e., $\mathbf{x}_t = M\mathbf{y}_t + \bar{\mathbf{x}}$

If we expand the terms, we get solutions of the following kind:

$$x_t^1 = e_{11}c_1(\lambda_1)^t + e_{12}c_2(\lambda_2)^t + \ldots.. + e_{1n}c_n(\lambda_n)^t + \bar{x}^1$$
$$x_t^2 = e_{21}c_1(\lambda_1)^t + e_{22}c_2(\lambda_2)^t + \ldots.. + e_{2n}c_n(\lambda_n)^t + \bar{x}^2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_t^n = e_{n1}c_1(\lambda_1)^t + e_{n2}c_2(\lambda_2)^t + \ldots.. + e_{nn}c_n(\lambda_n)^t + \bar{x}^n$$

(e_{ij} is the element on the *i*-th row and *j*-th column of matrix M)

Clearly, all these variables will approach their steady state values if $|\lambda_i| < 1$ for all *i*. In this case the steady state is said to be ***stable.***

The variables will explode away from the steady state values if $|\lambda_i| > 1$ for all *i*. In this case the steady state is said to be ***unstable.***

If some of the $|\lambda_i|$ are < 1 and some of them are > 1, then the stability of the system depends crucially on the boundary conditions. The system will approach the steady state for some initial values, and will move away from the steady state for all other initial values. In this case the steady state is said to be a ***saddle point*** and the system is ***saddle point stable.***

Recall that we had assumed that all the characteristic roots of *A* matrix are distinct. If some of the roots are repeated then *A* may not be diagonizable; hence deriving the solution set would not be so easy. However, by a similarity transformation, any square matrix *A* can be converted into a matrix with the following properties:

(a) All elements below the main diagonal vanish.

(b) All elements on the main diagonal are the eigenvalues of *A* – with repeated eigen values appearing in consecutive diagonal positions. .

(c) The only elements above the main diagonal that may not vanish are those, whose column index are *i*+1 where *i* is the corresponding row index (i.e., only those elements which are adjacent to the main diagonal elements). These elements will not vanish if and only if the diagonal elements in the *i*-th and *i*+1-th positions are equal. Any such non-vanishing element has a value 1.

This latter matrix which can be derived from the similarity transformation of *A* is called the **'Jordan canonical form'** of *A*.

Note that if we could transform *A* to its canonical form, then once again solving the system would become relatively easy, because now a equation in the system will involve at the most two variables. But the procedure to identify the similarity transformation that converts a square matrix to its canonical form could be rather complex and we shall not attempt to do so except for the 2×2 case.

(b) 2 dimensional system:

Consider a two dimensional system of the form

$$\left.\begin{aligned} x_t^1 &= a_{11}x_{t-1}^1 + a_{12}x_{t-1}^2 + b^1 \\ x_t^2 &= a_{21}x_{t-1}^1 + a_{22}x_{t-1}^2 + b^2 \end{aligned}\right\} \qquad (5)$$

The co-efficient matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The first step is to find the characteristic roots and the corresponding characteristic vectors of *A*.

Case (i): the roots are real and distinct

Let the two roots be λ_1 and λ_2 with the associated characteristic vectors $\mathbf{e}_1 = \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix}$ respectively. Then the transformation matrix that diagonalizes *A* is given by $M = (\mathbf{e}_1, \mathbf{e}_2)$, which has the two eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as its columns.
We have seen that the solution to this system are given by:

$$\begin{aligned} x_t^1 &= e_{11}c_1(\lambda_1)^t + e_{12}c_2(\lambda_2)^t + \bar{x}^1 \\ x_t^2 &= e_{21}c_1(\lambda_1)^t + e_{22}c_2(\lambda_2)^t + \bar{x}^2 \end{aligned} \qquad (6)$$

where $\bar{x}_1$ and $\bar{x}_2$ are the two steady state values which are obtained by solving the two linear equations: $\bar{x}_1 = a_{11}\bar{x}_1 + a_{12}\bar{x}_2 + b^1$ and $\bar{x}_2 = a_{21}\bar{x}_1 + a_{22}\bar{x}_2 + b^2$.
The arbitrary constraints will be determined by the given boundary conditions. If we are given two initial value of the variables for time $t = 0$ as x_0^1 and x_0^2, then we can put these values in the above solutions to derive the exact values of c_1 and c_2.

As we have seen before, the system will be ***stable*** if both λ_1 and λ_2 have absolute values less than unity; will be ***unstable*** if λ_1 and λ_2 have absolute values greater than unity; and will be ***saddle point stable*** if one of the eigenvalues is greater than unity in absolute value and the other is not.

If the system is saddle point stable, then there will be a unique path such that the system will approach the steady state if and only if it follows that unique path. This path is called the ***saddle path***. Let $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Note that if we can choose the initial values in such a way that c_1 (the coefficient associated with the unstable root λ_1) vanishes, then the time paths of x_1 and x_2 (i.e., the solutions) would be given by:

$$x_t^1 = e_{12}c_2(\lambda_2)^t + \bar{x}^1$$
$$x_t^2 = e_{22}c_2(\lambda_2)^t + \bar{x}^2$$

In this case the system will tend towards the steady state as *t* goes to infinity. Thus the above equations identify the saddle path. Eliminating $c_2(\lambda_2)^t$ from both the equations, we can write the ***equation of the saddle path*** as:

$$x_2^t = \frac{e_{22}}{e_{12}}(x_1^t - \bar{x}_1) + \bar{x}_2$$

Case (ii): the roots are real and repeated

Let the repeated root be denoted by λ. In this case the matrix *A* is not diagonaizable. However, as we have mentioned before, by a similarity transformation, we can still convert *A* to its canonical form $C = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

Let the eigenvector associated with λ be $\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$. Then the invertible transformation matrix that converts *A* to *C* is given by $M = \begin{bmatrix} e_1 & v_1 \\ e_2 & v_2 \end{bmatrix}$, where the vector $v \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is obtained by solving $(A - \lambda I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$.

Now let us define a new set of variables $\mathbf{y} = M^{-1}\mathbf{x}$. Therefore, $\mathbf{x}_{t-1} = M\mathbf{y}_{t-1}$ and $\mathbf{x}_t = M\mathbf{y}_t$. Now consider the homogeneous system:

$$\mathbf{x}_t = A\mathbf{x}_{t-1}$$

By the similarity transformation,

$$M\mathbf{y}_t = AM\mathbf{y}_{t-1} \Rightarrow M^{-1}M\mathbf{y}_t = M^{-1}AM\mathbf{y}_{t-1} = C\mathbf{y}_{t-1}$$

Thus we get a new system of homogeneous differential equations given by:

$$\mathbf{y}_t = C\mathbf{y}_{t-1}$$

i.e., $$\begin{bmatrix} y_t^1 \\ y_t^2 \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} y_{t-1}^1 \\ y_{t-2}^2 \end{bmatrix} \tag{7}$$

It is easy to see that solution for the second equation is given by $y_t^2 = c_2(\lambda)^t$. Putting this solution back in the first equation, we get

$$y_t^1 = \lambda y_{t-1}^1 + c_2(\lambda)^t$$

This gives us a non-autonomous difference equation, which is similar to equation (2′) that we had discussed earlier. The homogeneous part has the solution $y_t^1 = c_1(\lambda)^t$. On the other hand, proceeding with a trial solution of the form $y_t^1 = kt(\lambda)^t$, we can derive a particular solution as $y_t^1 = c_2 t(\lambda)^t$. Hence the general solution of y_t^1 is given by

$$y_t^1 = c_1(\lambda)^t + c_2 t(\lambda)^t = (c_1 + c_2 t)(\lambda)^t$$

Once we find the solutions for y_t^1 and y_2^t, the general solutions for x_t^1 and x_t^2 can be derived as

$$\mathbf{x}_t = M\mathbf{y}_t + \bar{\mathbf{x}}.$$

Stability of the system depends on the value of λ. The system is ***stable*** if $|\lambda| < 1$ and is ***unstable*** if $|\lambda| > 1$.

Case (iii): the roots are complex conjugate

When the eigenvalues of *A* are complex, even then equation (6) would be a solution to the system. (In fact (6) is a solution to (5) whenever the roots are distinct – no matter whether they are real or complex). However, now λ_1 and λ_2 are imaginary numbers; therefore the time paths of the variables are difficult to visualize from (6). The stability property of the system also remains obscure.

However we can write the solution in a more meaningful way if we convert the matrix A to a specific form as described below.

Suppose the Matrix A has a specific form such that $a_{22} = a_{11} = a$(say) and $a_{21} = -a_{12} = b$(say). Thus the matrix A looks like this: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. One can easily verify that this specific form of A has complex conjugate eigen values given by: $a \pm ib$. Moreover it will **always** have a set of eigen vectors given by $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ i \end{pmatrix}$ respectively.

Then directly using (6), we can obtain the general solution for this special case as:

$$\begin{aligned} x_t^1 &= c_1(a+ib)^t + c_2(a-ib)^t + \bar{x}^1 \\ x_t^2 &= -ic_1(a+ib)^t + ic_2(a-ib)^t + \bar{x}^2 \end{aligned} \qquad (8)$$

Now apply a theorem of complex numbers called **De Moivre's Theorem**, which states that **for any complex conjugate numbers** $a \pm ib$**,**

$$(a \pm ib)^t = (r)^t(\cos\theta\, t \pm i \sin\theta\, t) = (r)^t \exp^{\pm i\theta\, t}$$

where $r = \sqrt{a^2 + b^2}$ **is the modulus of the complex conjugate, and** $\theta = \tan^{-1}(a/b)$.

Using the first part of this theorem, we can write the general solution given in (8) as:

$$\begin{aligned} x_t^1 &= r^t(c_3 \cos\theta\, t + c_4 \sin\theta\, t) + \bar{x}^1 \\ x_t^2 &= r^t(c_3 \sin\theta\, t - c_4 \cos\theta\, t) + \bar{x}^2 \end{aligned} \qquad (9)$$

where $c_3 = (c_1 + c_2)$ and $c_4 = -(c_1 + c_2)i$ are two arbitrary constants (not necessarily real).

It is now easy to derive the ***stability property*** of the system. Note that as *t* increases, the two terms $\cos\theta\, t$ and $\sin\theta\, t$ move in a periodic manner taking values between –1 and +1, returning to the same value after every $\frac{2\pi}{\theta}$ period. Hence x_1 and x_2 will also move in cyclically fashion. However, whether they approach the steady state over time or not depends crucially on the value of $r = \sqrt{a^2 + b^2}$. If $r < 1$, the system will approach the steady state over time, albeit cyclically (the cycles are converging in the sense the amplitude of the cycles decreases over time). If $r > 1$, the system will move away from the steady state in the form of exploding cycles of higher and higher amplitude. If $r = 1$, the system will exhibit limit cycles, moving in the same orbit period after period, neither approaching the steady state, nor moving away from it.

So far we assumed that A has a special form given by $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. What if the A matrix does not have this specific form? This is not really a problem since we know that for any Matrix A with complex conjugate roots $\alpha \pm i\beta$ and associated complex conjugate eigen vectors $\begin{pmatrix} m \pm in \\ p \pm iq \end{pmatrix}$ respectively, we can find a matrix M, given by $M = \begin{bmatrix} n & m \\ q & p \end{bmatrix}$ such that $M^{-1}AM = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$.

Then as before we can define a new set of variables $\mathbf{y} = M^{-1}\mathbf{x}$ and convert the given system into a system in terms of y as:

$$\mathbf{x}_t = A\mathbf{x}_{t-1} \Rightarrow M^{-1}\mathbf{x}_t = M^{-1}A\mathbf{x}_{t-1} \Rightarrow M^{-1}\mathbf{x}_t = M^{-1}AM \;\; M^{-1}\mathbf{x}_{t-1}$$

i.e., $\mathbf{y}_t = M^{-1}AM\mathbf{y}_{t-1}$

or, $\begin{bmatrix} y_t^1 \\ y_t^2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} y_{t-1}^1 \\ y_{t-2}^2 \end{bmatrix}$

Note now that the coefficient matrix of y system has the specific form that we just discussed and therefore it's general solution would be

$$\begin{aligned} y_t^1 &= r^t(c_3 \cos\theta\, t + c_4 \sin\theta\, t) \\ y_t^2 &= r^t(c_3 \sin\theta\, t - c_4 \cos\theta\, t) \end{aligned} \qquad (9')$$

From this, the general solution for the x system can be easily obtained as:

$$\mathbf{x}_t = M\mathbf{y}_t + \begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \end{pmatrix}$$

The stability property of the x-system will be the same as the stability property of the y system.