C. Non-linear Difference and Differential Equations: Linearization and Phase Diagram Technique

So far we have discussed methods of solving linear difference and differential equations. Let us now discuss the case of **nonlinear** difference and differential equations.

The first point to be noted here is that it is extremely difficult to derive an exact solution to a non-linear difference or differential equation. However, two techniques are often used to draw some qualitative inference about the behaviour of the dynamic system: one of these is the linearization technique, and the other is the phase diagram technique.

C.1 Linearization of non-linear difference/differential equations and local stability analysis:

(a) Single non-linear equation:

Consider any nonlinear function of a single variable *x*: $f(x): D \rightarrow R$ where *D* and *R* are the domain and range of the function respectively. Let $\hat{x} \in D$ be some given value of the variable. Then by Taylor's Theorem, the function $f(x)$ can be expanded around $\hat{x}$ in the following way:

$$f(x) = f(\hat{x}) + \frac{f'(\hat{x})}{1!}(x-\hat{x}) + \frac{f'(\hat{x})}{2!}(x-\hat{x})^2 + \frac{f'''(\hat{x})}{3!}(x-\hat{x})^3 + \ldots\ldots\ldots$$

Now a linear approximation of the non-linear function $f(x)$ around the point $\hat{x}$ is given by:

$$f(x) \cong f(\hat{x}) + \frac{f'(\hat{x})}{1!}(x-\hat{x})$$

.

i.e., $f(x) \cong f'(\hat{x})x + [f(\hat{x}) - f'(\hat{x})\hat{x}]$ (21)

Note the linear function given above is only an approximation of the $f(x)$ function around $\hat{x}$, i.e., it resembles the $f(x)$ function only in a small neighbourhood of $\hat{x}$. In general this linear function does *not* closely approximate the $f(x)$ function for all values of *x*. Thus whatever conclusion we draw on the basis of this linear approximation will only be valid ***locally*** around $\hat{x}$.

Linearization technique is often used to convert a non-linear difference or differential equation into a linear form. Generally the non-linear equation is linearly approximated around its steady state value. This allows us to derive some conclusions about the time path of the variable in the neighbourhood of the steady state and thus its local stability property.

First let us consider a non-linear *difference* equation of the form:

$$x_t = f(x_{t-1}) \qquad (22)$$

(We are ignoring the parameter vector $\boldsymbol{\alpha}$ for the time being).

We know that the steady state of the above difference equation is defined as $x_t = x_{t-1} = \bar{x}$, i.e., $\bar{x} = f(\bar{x})$. Suppose there is indeed a $\bar{x}$ that solves this equation. In other words, suppose a steady state exists. Then linearizing the $f(x_{t-1})$ equation around $\bar{x}$, we get a linear differential equation of the form:

$$\begin{aligned} x_t = f(x_{t-1}) &\cong f(\bar{x}) + f'(\bar{x})(x_{t-1} - \bar{x}) \\ &= f'(\bar{x})x_{t-1} + [f(\bar{x}) - f'(\bar{x})\bar{x}] \\ &= ax_{t-1} + b \end{aligned} \tag{22′}$$

Solution: $x_t = C(a)^t + \bar{x}$, *C* an arbitrary constant.

Stability depends on the term *a*, i.e., on the term $f'(\bar{x})$.

If $|f'(\bar{x})| < 1$ the system is *locally* stable; if $|f'(\bar{x})| > 1$ the system is *locally* unstable.

We can proceed to analyse the local stability property of a non-linear *differential* equation in an analogous manner. Consider a non-linear differential equation of the form:

$$\frac{dx}{dt} = f(x) \tag{23}$$

The steady state of this differential equation is defined by $\frac{dx}{dt} = 0$, i.e., $f(x) = 0$. Suppose there is a $\bar{x}$ that satisfies this steady state condition. Then linearizing $f(x)$ around $\bar{x}$, we can write the differential equation (23) as:

$$\begin{aligned} \frac{dx}{dt} &\cong f(\bar{x}) + f'(\bar{x})(x - \bar{x}) \\ &= f'(\bar{x})x - f'(\bar{x})\bar{x} \quad [\text{since } f(\bar{x}) = 0\,] \\ &= ax - b \end{aligned} \tag{23′}$$

Solution: $x(t) = C\exp^{at} + \bar{x};\quad C$ an arbitrary constant.

Once again stability depend on term *a*, i.e., on $f'(\bar{x})$.

If $f'(\bar{x}) < 0$, the system is *locally* stable; if $f'(\bar{x}) > 0$, the system is *locally* unstable.

(b) 2 dimensional system of non-linear equations:

Let us first consider the following non-linear system of *differential* equations:

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}\right\} \tag{24}$$

The steady state of this system is defined as $(\bar{x}, \bar{y})$ such that $f(\bar{x}, \bar{y}) = 0$ and $g(\bar{x}, \bar{y}) = 0$. Suppose that a steady state exists. Then for the two variable functions $f(x, y)$ and $g(x, y)$, we can derive the linear approximations around $(\bar{x}, \bar{y})$ as:

$$f(x, y) \cong f(\bar{x}, \bar{y}) + f_x(\bar{x}, \bar{y})(x - \bar{x}) + f_y(\bar{x}, \bar{y})(y - \bar{y})$$

$$g(x, y) \cong g(\bar{x}, \bar{y}) + g_x(\bar{x}, \bar{y})(x - \bar{x}) + g_y(\bar{x}, \bar{y})(y - \bar{y})$$

Noting that $f(\bar{x}, \bar{y}) = 0$ and $g(\bar{x}, \bar{y}) = 0$ (from the definition of the steady state) and simplifying, we can transform the non-linear system given in (24) to a corresponding linear system of the following form:

$$\left.\begin{aligned}\frac{dx}{dt} &= f_x(\bar{x},\bar{y})x + f_y(\bar{x},\bar{y})y - [f_x(\bar{x},\bar{y})\bar{x} + f_y(\bar{x},\bar{y})\bar{y}] = a_{11}x + a_{12}y - b_1 \\ \frac{dy}{dt} &= g_x(\bar{x},\bar{y})x + g_y(\bar{x},\bar{y})y - [g_x(\bar{x},\bar{y})\bar{x} + g_y(\bar{x},\bar{y})\bar{y}] = a_{21}x + a_{22}y - b_2\end{aligned}\right\} \quad (24')$$

The coefficient matrix of this linear system is given by the following *Jacobian* matrix of partial derivatives of $f(x,y)$ and $g(x,y)$ (with the derivatives being evaluated at the steady state):

$$A = \begin{bmatrix} f_x(\bar{x},\bar{y}) & f_y(\bar{x},\bar{y}) \\ g_x(\bar{x},\bar{y}) & g_y(\bar{x},\bar{y}) \end{bmatrix} \quad (25)$$

If the explicit function forms of $f(x,y)$ and $g(x,y)$ were known to us, then we could directly calculate the steady state values of *x* and *y* and find out the exact numerical values of these partial derivatives evaluated at that steady state. Thus we could then find the exact form of the co-efficient matrix and proceed to solve the system of linear differential equations given in (24′) following the method discussed earlier.

But even when the exact functional forms of $f(x,y)$ and $g(x,y)$ are not known, we can still say something about the stability of the system without actually solving the system, provided we are given certain information about these partial derivatives. To see how, first recall that for a 2×2 system of linear differential equations, stability of the system depends crucially on the characteristics roots of the co-efficient matrix. If the co-efficient matrix has real characteristic roots (either distinct or repeated), then the dynamic system is stable if the roots are negative; is unstable if the roots are positive, and is a saddle point if one root is positive and one is negative. On the other hand, if the coefficient matrix has complex characteristic roots of the general form $u \pm iv$, then the stability depends on the real part of the complex root *u*: the system is oscillatory stable (cycles with decreasing amplitudes) if $u < 0$; the system is oscillatory unstable (cycles with increasing amplitudes) if $u > 0$; and the system will show uniform cycles with constant amplitudes (neither stable nor unstable) if $u = 0$. Thus in order to be able to say something about the (local) stability of system given in (24′), we have to examine the characteristics roots of the co-efficient matrix given in (25).

Note that the characteristic equation for the Jacobian matrix given in (25) is:

$$\begin{vmatrix} f_x(\bar{x},\bar{y}) - \lambda & f_y(\bar{x},\bar{y}) \\ g_x(\bar{x},\bar{y}) & g_y(\bar{x},\bar{y}) - \lambda \end{vmatrix} = 0$$

i.e., $[f_x(\bar{x},\bar{y}) - \lambda]\ [g_y(\bar{x},\bar{y}) - \lambda] - f_y(\bar{x},\bar{y}).g_x(\bar{x},\bar{y}) = 0$

Simplifying,

$$\lambda^2 - [f_x(\bar{x},\bar{y}) + g_y(\bar{x},\bar{y})] + [f_x(\bar{x},\bar{y}).g_y(\bar{x},\bar{y}) - f_y(\bar{x},\bar{y}).g_x(\bar{x},\bar{y})] = 0$$

i.e., $\lambda^2 - [\text{Trace}A]\lambda + \text{Det}A = 0$ (26)

Now form the theories of quadratic equations, we know that for any quadratic equation of the form $x^2 - px + q = 0$ has two roots x_1 and x_2 such that $x_1 + x_2 = p$ and $x_1.x_2 = q$. Applying this theorem to (26) (which is a quadratic equation in λ), we know that the two characteristics roots would be such that $\lambda_1 + \lambda_2 = \text{Trace}A$ and $\lambda_1.\lambda_2 = \text{Det}A$.

We can now proceed to discuss different alternative (and mutually exclusive) possibilities:

CASE I: $(\text{Trace}A)^2 - 4\text{Det}A \geq 0$ (implying that **the roots are real**)

Subcase (a): Det*A*<0; Trace*A* could be anything (i.e., greater than, equal to, or less than zero).
Note that Det*A*<0 implies $\lambda_1\lambda_2 < 0$, i.e., λ_1 and λ_2 are have opposite signs. Thus one of them would be positive and one negative. Hence in this case the equilibrium will be locally a *saddle point*.
Subcase (b): Det*A*>0; Trace*A*>0.
In this case Det*A*>0 implying $\lambda_1\lambda_2 > 0$, i.e., λ_1 and λ_2 are of same sign – either both are positive or both are negative. But it also given that Trace*A*>0 implying $\lambda_1 + \lambda_2 > 0$. Thus both λ_1 and λ_2 must be positive. Hence in this case the equilibrium is locally *unstable*.

Subcase (c): Det*A*>0; Trace*A*<0.
In this case once again $\text{Det}A = \lambda_1\lambda_2 > 0$; hence λ_1 and λ_2 have the same signs. But now Trace*A*<0 implying $\lambda_1 + \lambda_2 < 0$. Thus both λ_1 and λ_2 must be negative. Hence in this case the equilibrium is locally *stable*.

Remark. The determinant of the co-efficient matrix can never be zero for any linearly independent system. Thus Det*A* is either positive or negative.

Remark. Note that when the roots are real (i.e., $(\text{Trace}A)^2 - 4\text{Det}A \geq 0$), the possibility that Det*A*>0 and Trace*A*=0 cannot arise because when Trace*A*=0, then the roots will be real if and only if $-4\text{Det}A > 0 \Rightarrow \text{Det}A < 0$.

CASE II: $(\text{Trace}A)^2 - 4\text{Det}A < 0$ (implying that the **roots are complex**)
In this case, the roots will have the general form $a \pm ib$. Therefore,

$$\text{Trace}A = \lambda_1 + \lambda_2 = (a+ib) + (a-ib) = 2a$$

$$\text{Det}A = \lambda_1\lambda_2 = (a+ib)(a-ib) = a^2 + b^2$$

Now recall that for a two dimensional system of linear differential equations with complex roots, the stability of the system depends on the real part of the complex root a. Therefore, in this case, we can determine the stability property of the system by simply looking the Trace*A.*
$\text{Trace}A > 0 \Rightarrow a > 0$; hence the system will be unstable (unstable oscillations)
$\text{Trace}A < 0 \Rightarrow a < 0$; hence the system will be stable (stable oscillations)
$\text{Trace}A = 0 \Rightarrow a = 0$; hence the system will exhibit uniform cycles which are neither stable nor unstable.

Let us next consider a non-linear system of *difference* equations, given by

$$\left.\begin{aligned} x_t &= f(x_{t-1}, y_{t-1}) \\ y_t &= g(x_{t-1}, y_{t-1}) \end{aligned}\right\} \qquad (27)$$

The steady state of this system of difference equations is defined as $(\bar{x}, \bar{y})$ such that $\bar{x} = f(\bar{x}, \bar{y})$and $\bar{y} = g(\bar{x}, \bar{y})$. Suppose that a steady state exists. Then for the two variable functions $f(x_{t-1}, y_{t-1})$ and $g(x_{t-1}, y_{t-1})$, we can derive the linear approximations around $(\bar{x}, \bar{y})$ as:

$$f(x_{t-1}, y_{t-1}) \cong f(\bar{x}, \bar{y}) + f_x(\bar{x}, \bar{y})(x_{t-1} - \bar{x}) + f_y(\bar{x}, \bar{y})(y_{t-1} - \bar{y})$$

$$g(x_{t-1}, y_{t-1}) \cong g(\bar{x}, \bar{y}) + g_x(\bar{x}, \bar{y})(x_{t-1} - \bar{x}) + g_y(\bar{x}, \bar{y})(y_{t-1} - \bar{y})$$

Using these two linear approximations, we can transform the nonlinear system of difference equation given in (27) to the following linear system:

$$\left.\begin{aligned} x_t &= f_x(\bar{x}, \bar{y})x_{t-1} + f_y(\bar{x}, \bar{y})y_{t-1} + [f(\bar{x}, \bar{y}) - f_x(\bar{x}, \bar{y})\bar{x} + f_y(\bar{x}, \bar{y})\bar{y}] \\ y_t &= g_x(\bar{x}, \bar{y})x_{t-1} + g_y(\bar{x}, \bar{y})y_{t-1} + [g(\bar{x}, \bar{y}) - g_x(\bar{x}, \bar{y})\bar{x} + g_y(\bar{x}, \bar{y})\bar{y}] \end{aligned}\right\} \qquad (27')$$

Like the differential equation case, the coefficient matrix of this linear system is again given by the Jacobian matrix:

$$A = \begin{bmatrix} f_x(\bar{x}, \bar{y}) & f_y(\bar{x}, \bar{y}) \\ g_x(\bar{x}, \bar{y}) & g_y(\bar{x}, \bar{y}) \end{bmatrix}$$

We know that the stability of the system once again depends on the characteristic roots of this co-efficient matrix. However unlike the differential equation case, if the roots are real, stability condition now requires that the absolute values of both these roots be less than unity. If the roots are real and have absolute values greater than unity, the system in unstable. If one of the real roots have absolute value greater than unity and the other one has absolute value less than unity, then equilibrium is a saddle point. Also recall that when the roots are complex and of the general form $u \pm iv$, then stability depend on the modulus $r = \sqrt{u^2 + v^2}$. The system will show stable oscillations if $r < 1$; will show unstable oscillations if $r > 1$; and will be characterised by uniform oscillations (neither stable nor unstable) if $r = 1$.

Given the co-efficient matrix *A,* we can again derive the characteristic equation as

$$\lambda^2 - [f_x(\bar{x}, \bar{y}) + g_y(\bar{x}, \bar{y})] + [f_x(\bar{x}, \bar{y}).g_y(\bar{x}, \bar{y}) - f_y(\bar{x}, \bar{y}).g_x(\bar{x}, \bar{y})] = 0$$

i.e., $\lambda^2 - [\mathrm{Trace}A]\lambda + \mathrm{Det}A = 0$.

As before, the characteristic roots would be such that $\lambda_1 + \lambda_2 = \mathrm{Trace}A$ and $\lambda_1.\lambda_2 = \mathrm{Det}A$. Thus as we did in the differential equation case, by examining the signs of Trace*A* and Det*A*, we can derive some conclusions about the signs of the characteristics roots. However that information is now not sufficient for stability. In order to be able to say something about the stability of the system, we have to check whether the characteristic roots are greater or less than unity in absolute value. To determine whether λ_1 and λ_2 are less than unity in absolute value, we use some additional conditions that are discussed below.

CASE I: $(\mathrm{Trace}A)^2 - 4\mathrm{Det}A \geq 0$ (implying that **the roots are real**)

Consider the real line and take the two points +1 and –1 as two reference points:

$$-\infty \quad \text{———} \underset{-1}{|} \text{———} \underset{+1}{|} \text{———} \quad +\infty$$

There are six possibilities here:

(i) Both λ_1 and λ_2 lies in (-1,1), which implies that the equilibrium is stable.

(ii) One λ lies in (-1,1), the other one in (+1, ∞), which implies that the equilibrium is a saddle point.
(iii) One λ lies in (-1,1), the other one in (-∞, -1), which implies that the equilibrium is a saddle point.
(iv) One λ lies in (+1, ∞), the other one in (-∞, -1), which implies that the equilibrium is unstable.
(v) Both λ_1 and λ_2 lies in (+1, ∞), which implies that the equilibrium is unstable.
(vi) Both λ_1 and λ_2 lies in (-∞, -1), which implies that the equilibrium is unstable.

But how do we know which of these cases are true? Can we determine that from the Trace and the Determinant of the co-efficient matrix, as we did for the differential equation case? The answer is "yes", but in order to do that we need to know something more than just the signs of the Trace and the Determinant. To be more precise, we need to know whether λ_1 and λ_2 lie on the same side or the opposite sides of 1 and –1 respectively.

Note that
(i) If λ_1 and λ_2 lie on the same side of +1 (either both lie to the left of +1, or both to the right), then $(1-\lambda_1)(1-\lambda_2)>0$, i.e., $\lambda_1\lambda_2-(\lambda_{1+}\lambda_2)+1>0$.
(ii) If λ_1 and λ_2 lie on the opposite sides of +1 (one to the left and one to the right), then $(1-\lambda_1)(1-\lambda_2)<0$, i.e., $\lambda_1\lambda_2-(\lambda_{1+}\lambda_2)+1<0$.
(iii) If λ_1 and λ_2 lie on the same side of –1 (either both lie to the left of –1, or both to the right), then $(1+\lambda_1)(1+\lambda_2)>0$, i.e., $\lambda_1\lambda_2+(\lambda_{1+}\lambda_2)+1>0$.
(iv) if λ_1 and λ_2 lie on the same side of –1 (one to the left and the other to the right), then $(1+\lambda_1)(1+\lambda_2)<0$, i.e., $\lambda_1\lambda_2+(\lambda_{1+}\lambda_2)+1<0$.
Noting that $\lambda_1+\lambda_2=\mathrm{Trace}A$ and $\lambda_1.\lambda_2=\mathrm{Det}A$, we can write these conditions in terms of Trace*A* and Det*A* as:
(i) $\mathrm{Det}A-\mathrm{Trace}A+1>0$ implies λ_1 and λ_2 lie on the same side of +1;
(ii) $\mathrm{Det}A-\mathrm{Trace}A+1<0$ implies λ_1 and λ_2 lie on the opposite sides of +1;
(iii) $\mathrm{Det}A+\mathrm{Trace}A+1>0$ implies λ_1 and λ_2 lie on the same side of –1;
(iv) $\mathrm{Det}A+\mathrm{Trace}A+1<0$ implies λ_1 and λ_2 lie on the opposite sides of –1.

Now let us consider all the possible cases:

Case (I-a): $\mathrm{Det}A-\mathrm{Trace}A+1>0$; $\mathrm{Det}A+\mathrm{Trace}A+1>0$
Then λ_1 and λ_2 lie on the same side of +1 as well as –1.
There are mutually exclusive possibilities here: either $\lambda_1,\lambda_2\in(+1,+\infty)$; or $\lambda_1,\lambda_2\in(-1,-\infty)$; or $\lambda_1,\lambda_2\in(-1,+1)$. λ_1 and λ_2 *must* lie in one of these sets; no other situation is possible.
In order to know precisely in which set λ_1 and λ_2 belong to, we look at the information given about Trace*A* and Det*A*.
$|\mathrm{DetA}|<1\Leftrightarrow|\lambda_1\lambda_2|<1$, hence $\lambda_1,\lambda_2\in(-1,+1)$; so the equilibrium is stable.
$\mathrm{DetA}>1$ and $\mathrm{Trace}A>0\Leftrightarrow\lambda_1,\lambda_2\in(+1,+\infty)$; so the equilibrium is unstable.

$\text{DetA}>1$ and $\text{Trace}A < 0 \Leftrightarrow \lambda_1, \lambda_2 \in (-1, -\infty)$; so the equilibrium is unstable.

Case (I-b): $\text{Det}A - \text{Trace}A + 1 < 0\,;\text{Det}A + \text{Trace}A + 1 < 0$
Then λ_1 and λ_2 lie on the opposite sides of +1 as well as –1.
The only possibility here : one λ lies in (+1, ∞), the other one in (-∞, -1), which implies that the equilibrium is unstable.

Case (I-c): $\text{Det}A - \text{Trace}A + 1 > 0\,;\text{Det}A + \text{Trace}A + 1 < 0$
Then λ_1 and λ_2 lie on the same side of +1, but opposite sides of –1.
The only possibility here is: one λ lies in (-1,1), the other one in (-∞, -1), which implies that the equilibrium is a saddle point.

Case (I-d): $\text{Det}A - \text{Trace}A + 1 < 0\,;\text{Det}A + \text{Trace}A + 1 > 0$
Then λ_1 and λ_2 lie on the opposite sides of +1, but on the same side of –1.
The only possibility here is: one λ lies in (-1,1), the other one in (+∞, +1), which implies that the equilibrium is a saddle point.

CASE II: $(\text{Trace}A)^2 - 4\text{Det}A < 0$ (implying that the **roots are complex**)
If the roots are complex they have the general form $u \pm iv$. Therefore,

$$\text{Trace}A = \lambda_1 + \lambda_2 = (a+ib) + (a-ib) = 2a$$

$$\text{Det}A = \lambda_1 \lambda_2 = (a+ib)(a-ib) = a^2 + b^2$$

As was mentioned before, in the difference equation case with complex roots, stability depends on the term $\sqrt{a^2+b^2}$. Thus we can determine the stability property of the system by looking at the determinant alone.
If $\text{Det}A = a^2 + b^2 < 1$, then $\sqrt{a^2+b^2} < 1$, so the system will show *stable* oscillations;
If $\text{Det}A = a^2 + b^2 > 1$, then $\sqrt{a^2+b^2} > 1$, so the system will show *unstable* oscillations;
If $\text{Det}A = a^2 + b^2 = 1$, then $\sqrt{a^2+b^2} = 1$, so the system will show uniform oscillations which are neither stable nor unstable.

C.2 Phase Diagram Analysis:

Sometimes along with linearization technique, a diagrammatic method is used in order to derive some qualitative conclusions about the behaviour of the dynamic system over time. This graphical method is known as the phase diagram technique and is often used to analyse the behaviour of non-linear dynamic equations.

(a) Phase portrait for single difference or differential equation:

In the one dimensional (single equation) case, the phase diagram technique typically involves plotting the state variable in horizontal axis and the changes in the value of the state variable along the vertical axis.
First consider a difference equation of the form

$$x_{t-1} = f(x_t)$$

The change in the state variable *x* is defined as

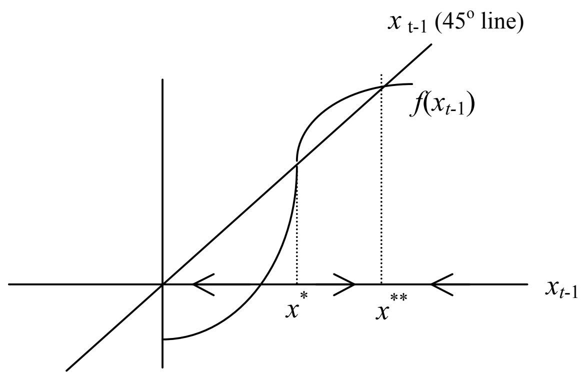
$$\Delta x = x_t - x_{t-1}$$
$$= f(x_{t-1}) - x_{t-1} \tag{28}$$

Note that $\Delta x > 0$ means $x_t > x_{t-1}$, i.e., *x* is increasing over time.

Similarly, $\Delta x < 0$ means *x* is decreasing over time.

And $\Delta x = 0$ means *x* is constant over time, which in turn implies that the system is at its steady state.

Now we want to plot the change in *x* corresponding to different values of the state variable itself. From (28), we see that change in *x* is reflected in the difference between $f(x_{t-1})$ and x_{t-1}. Thus we can plot these two functions separately as $y = f(x_{t-1})$ and $\hat{y} = x_{t-1}$ and observe their difference in order to get an idea about Δx. If we plot the function $\hat{y} = x_{t-1}$, with x_{t-1} in the horizontal axis, we will get the 45^o line. However plotting $y = f(x_{t-1})$ with x_{t-1} in the horizontal axis requires certain information about the slope the curvature of the *f* function. The diagram below depicts a hypothetical *f* function as an example.



Note that in the diagram the $f(x_{t-1})$ curve intersects the 45^o line (representing x_{t-1}) twice – at x^* and x^{**} respectively. At these two *x* values $\Delta x = f(x_{t-1}) - x_{t-1} = 0$; hence they represent the two steady state points.

Also note that in a close neighbourhood of x^*, if we take an *x* value to the left of x^*, $f(x_{t-1})$ lies below the 45^o line implying $\Delta x < 0$. Hence *x* is decreasing in that region. We draw an arrow pointing towards the left to indicate that *x* is decreasing here. On the other hand, if we take an *x* value to the right of x^*(in a small neighbourhood of x^*), $f(x_{t-1})$ lies above the 45^o line implying $\Delta x > 0$. Hence *x* is increasing in that region. Once again we draw an arrow pointing towards the right to indicate that *x* is increasing here.

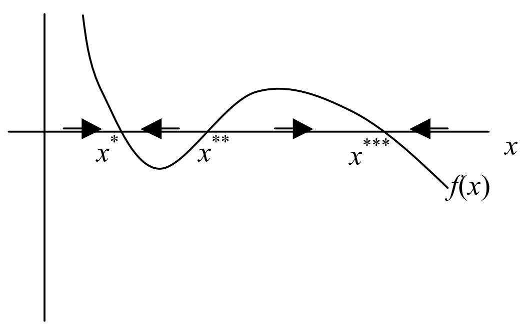
Similarly, one can see that in a small neighbourhood of the second equilibrium point x^{**}, $\Delta x > 0$ if we take an *x* value lying to its left and $\Delta x < 0$ if we take an *x* value lying to its right. Hence we can draw the arrows accordingly.

Thus the diagram above – complete with the direction of movements of *x* (the arrows) in different regions – enables us to derive graphically that x^* is a locally unstable equilibrium (because both the arrows on either side of x^* point outwards – indicating that *x* is moving away from x^*) and x^{**} is a locally stable equilibrium (because both the arrows in its either side point inwards indicating that *x* is moving towards x^{**}).

The phase diagram for a differential equation can be constructed in an analogous manner. Consider the differential equation

$$\frac{dx}{dt} = f(x).$$

The phase diagram plots the change in *x* corresponding to different values of the state variable. Here the change in *x* over time is directly measured by the $f(x)$ function. If $f(x) > 0$, $\frac{dx}{dt} > 0$ - implying *x* in increasing over time; if $f(x) < 0$, $\frac{dx}{dt} < 0$ - implying *x* in decreasing over time; and if $f(x) = 0$, $\frac{dx}{dt} = 0$ - implying *x* remains constant (the steady state). If we are given ceratin information regarding the slope and curvature of $f(x)$, we can draw the phase portrait for this differential equation. The diagram below again depicts a hypothetical *f* function as an illustration:



In the diagram the $f(x)$ curve intersects the horizontal axis thrice, denoting three steady state points: x^*, x^{**} and x^{***} respectively.
If we consider a small neighbourhood of the first equilibrium point, to its left $f(x) > 0$ and to its right $f(x) < 0$ – implying that *x* in increasing to its left and decreasing to its right. Thus again we draw the arrows to indicate the direction of movement of *x*.
Similarly, we can find out the direction of movements of *x* in close neighbourhoods of the other equilibrium points as well and draw the arrows accordingly.
The complete phase diagram now tells us that x^* and x^{***} are locally stable equilibrium points while x^{**} is locally unstable.

(b) Phase portrait of a two dimensional system :

In the two dimensional case, the phase diagram typically traces the curves along which the values of the state variables remain unchanged over time. Using these two curves as the reference points, one then identifies the regions where the state variables are changing and determines the direction of change for each of the state variable.

Let us first consider the 2×2 system of difference equations:

$$x_t = f(x_{t-1}, y_{t-1})$$
$$y_t = g(x_{t-1}, y_{t-1})$$

As before, we can denote the changes in *x* and *y* respectively as:

$$\Delta x = f(x_{t-1}, y_{t-1}) - x_{t-1} = \hat{f}(x_{t-1}, y_{t-1})$$
$$\Delta y = g(x_{t-1}, y_{t-1}) - y_{t-1} = \hat{g}(x_{t-1}, y_{t-1})$$

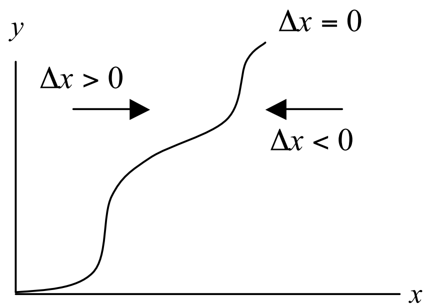
Now we trace the curves $\Delta x = 0$ and $\Delta y = 0$, measuring *x* along the horizontal axis and *y* along the vertical axis.

Note that the slopes of these two curves in the (*x*,*y*) plane is given by $\left.\frac{dy}{dx}\right|_{\Delta x=0} = \frac{1-f_x}{f_y}$ and $\left.\frac{dy}{dx}\right|_{\Delta y=0} = \frac{g_x}{1-g_y}$ respectively. We cannot draw these curves unless we are given some information about these partial derivatives.

For illustrative purposes, let us assume that $f_y > 0; f_x < 1; g_x < 0; g_y < 1$ for all $x, y > 0$.

This implies that $\hat{f}_y = f_y > 0; \hat{f}_x = f_x - 1 < 0; \hat{g}_x = g_x < 0; \hat{g}_y = g_y - 1 < 0$.

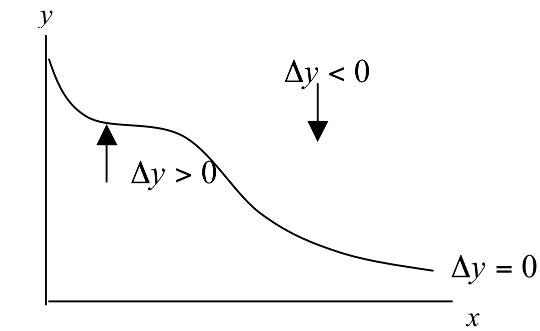
Then in the positive quadrant $\Delta x = 0$ is an upward sloping curve as shown below:



Let us now consider the region above $\Delta x = 0$. Recall that along any point on the $\Delta x = 0$ curve, $\hat{f}(x, y) = 0$. Now in the region above this curve, for each value of *x*, y is higher than the corresponding *y* value along $\Delta x = 0$. Since $\hat{f}_y > 0$, it implies that in the region above $\Delta x = 0$, $\hat{f}(x, y) > 0$. Thus $\Delta x > 0$ and *x* is increasing in this region. As before, we draw an arrow pointing to the right to indicate that *x* in increasing here.

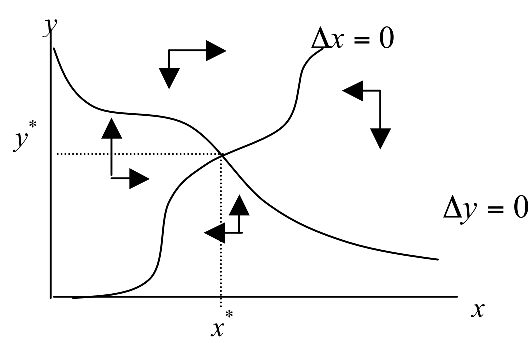
Using analogous argument, we can show that in the region below $\Delta x = 0$ curve, $\hat{f}(x, y) < 0$ – implying $\Delta x < 0$ and *x* is decreasing in this region. Thus we draw an arrow here pointing to the left.

On the other hand, given the conditions on the partial derivatives, $\Delta y = 0$ is a downward sloping curve in the (*x*,*y*) plane, as depicted below:



Now consider the region above $\Delta y = 0$ curve. For any *x*, the *y* value in this region is higher than the corresponding *y* along $\Delta y = 0$. Since $\hat{g}_y < 0$, it implies that in this region $\hat{g}(x, y) < 0$, i.e., $\Delta y < 0$. Thus *y* in decreasing here. We draw an arrow pointing down to indicate the *y* in decreasing in this region.

By analogous argument, we can show that in the region below $\Delta y = 0$ curve, $\hat{g}(x, y) > 0$; hence *y* is increasing here and we draw an arrow point up to indicate that. Combining these two diagrams we can draw the complete phase portrait for this two dimensional system of difference equations as follows:



The point of intersection between the two curves denote s the steady state such the at this point (and only at this point) both $\Delta x = 0$ and $\Delta y = 0$. If we start from any other point, the direction of arrows tells us which way *x* and *y* would move. In this specific case we find that all the arrows point towards the equilibrium; hence the equilibrium is stable here.
If the directions of the arrows are such that they point outward, away from the equilibrium, then the diagram tells us that the equilibrium is unstable.

If the arrows are such that from some regions they point towards the equilibrium, and from other regions, they point outward – away from the equilibrium, then we can conclude that the equilibrium is a saddle point.

To draw the phase portrait of a two dimensional system differential equations, we can proceed in similar way. Consider the following system:

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

We first draw the curves $f(x, y) = 0$ and $g(x, y) = 0$ in the (*x*,*y*) plane and using these two curves as reference points, determine the direction of movements of *x* and *y* for different regions lying above or below these two curves. Again the point of intersection of the two curves would denote the steady state and the direction of arrows will tell us whether the steady state is stable, unstable or a saddle point, exactly as in difference equation case.