Macroeconomics: A Dynamic General Equilibrium Approach

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Modern Macroeconomics: the Dynamic General Equilibrium (DGE) Approach

- Modern macroeconomics is based on a dynamic general equilibrium approach which postulates that
    - Economic agents are continuously optimizing/re-optimizing subject to their constraints and subject to their information set. They optimize not only over their current choice variables but also the choices that would be realized in future.
    - All agents have rational expectations: thus their ex ante optimal future choices would ex post turn out to be less than optimal if and only if their information set is incomplete and/or there are some random elements in the economy which cannot be anticipated perfectly.
    - The agents are atomistic is the sense that they treat the market factors as exogenous in their optimization exercise. The optimal choices of all agents are then mediated through the markets to produce an equilibrium outcome for the macroeconomy (which, by construction, is also consistent with the optimal choice of each agent).

Modern Macroeconomics: DGE Approach (Contd.)

- This approach is '**dynamic**' because agents are making choices over variables that relate to both present and future.
- This approach is '**equilibrium**' because the outcome for the macro-economy is the aggregation of individuals' equilibrium (optimal) behaviour.
- This approach is '**general equilibrium**' because it simultaneously takes into account the optimal behaviour of diiferent types of agents in different markets and ensures that all markets clear.

Modern Macroeconomics: DGE Approach (Contd.)

- The Lucas critique and the consequent logical need to develop a unified micro-founded macroeconomic framework which would allow us to accurately predict the macroeconomic outcomes in response to any external shock (policy-driven or otherwise) led to emergence of the modern dynamic general equilibrium approach.
- As before, there are two variants of modern DGE-based approach:
  - One is based on the assumption of perfect markets (the Neoclassical/RBC school). As is expected, this school is critical of any policy intervention, in particular, monetary policy interventions.
  - The other one allows for some market imperfections (the New-Keynesian school). Again, true to their ideological underpinning, this school argues for active policy intervention.

Modern Macroeconomics: DGE Approach (Contd.)

- However, both frameworks are similar in two fundamental aspects:
  - **Agents optimize over infinte horizon;** and
  - **Agents are forward looking,** i.e., when they optimize over future variable they base their expectations on all available information - including information about (future) government policies. In other words, **agents have rational expectations**.
- We now develop the choice-theoretic frameworks for households and firms under the DGE approach.
- As before, we shall assume that the economy is populated by H households with identical preferences.

Household's Choice Problem under Perfect Markets: Infinite Horizon

- Let us examine the consumption-savings choices of a household over infinite horizon when markets are perfect.
- To simplify the analysis, we shall only focus on the consumption choice of the household and ignore the labour-leisure choice (for the time being).
- At any point of time the household is endowed with one unit of labour - which it supplies inelastically to the market.
- We shall also ignore prices and the concomitant role of money and focus only on the 'real' variables.
- Let a_t^h denote the asset stock of the household at the beginning of period t.
- We shall assume that positive savings by a household in any period are invested in various forms of assets (all assets have the same return), which augments the household's asset stock in the next period.

Household's Choice Problem: Infinite Horizon (Contd.)

- As we did earlier the two period problem, we now have to similarly formulate the budget constraints of the household for every time period $t = 0, 1, 2, ...\infty$.
- Recall that at the beginning of any time period t, a household starts with its given labour endowment (1 unit) and a certain amount of asset stock a_t^h (carried forward from the past).
- The household then offers his labour in the production process to earn some wage income towards the end of the period.
- His stock of assets held at the beginning of the period (a_t^h) also generates certain interest returns during the period, denoted by $r_t a_t^h$.
- Thus the flow income of the household at time t is given by $y_t^h = w_t + r_t a_t^h$.
- Moreover, assuming that asset stocks (in particular, physical assets) get depreciated at a constant rate δ during the producation process, after the production has taken place, the household would still be in possesion of the depreciated value of its asset stock $(1 - \delta)a_t^h$.

Household's Choice Problem: Infinite Horizon (Contd.)

- The household now has to decide how much it wants to consume and how much to save.
- If the household is not allowed to borrow, then the consumption of the household would to be limited by its flow income y_t^h.
- But in this one good world, the household also has the option of eating up its existing asset stocks (which constitutes negative savings).
- Thus the maximum consumption possible in time period t is: $y_t^h + (1-\delta)a_t^h$.
- This defines the *feasible consumption set* available to the household at every point of time t as follows:

$$c_t^h \leqq y_t^h + (1-\delta)a_t^h \text{ for all } t = 0, 1, ...\infty$$

Household's Choice Problem: Infinite Horizon (Contd.)

- The flow income of the household is distributed between consumption and savings. Thus, by definition:

$$s_t^h \equiv y_t^h - c_t^h$$

- All savings are invested is buying various *new assets*, which means the asset stock of the household at the beginning of next period (period $t+1$) will be

$$a_{t+1}^h = s_t^h + (1-\delta)a_t^h$$

- Note that if the household decides to eat up its existing asset stocks (over and above its flow income) then that will constitute negative savings and would *lower* the asset base of the household over time.
- Putting all these information together, we write the *period by period budget constraint of the household* as

$$a_{t+1}^h = y_t^h - c_t^h + (1-\delta)a_t^h \text{ for all } t = 0, 1, ...\infty$$

Household's Choice Problem: Infinite Horizon (Contd.)

- In the absence of intra-household borrowing, then the representative household h's problem would given by:

$$\underset{\left\{c_t^h\right\}_{t=0}^{\infty},\left\{a_{t+1}^h\right\}_{t=0}^{\infty}}{Max.} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t^h\right); u' > 0; u'' < 0; \; 0 < \beta < 1$$

subject to

$$\begin{aligned} \text{(i) } c_t^h &\leqq w_t + r_t a_t^h + (1-\delta) a_t^h \text{ for all } t \geqq 0; \\ \text{(ii) } a_{t+1}^h &= w_t + r_t a_t^h - c_t^h + (1-\delta) a_t^h; \; a_t^h \geqq 0 \text{ for all } t \geqq 0; \end{aligned}$$

where a_0^h is historically given.

- The **atomistic** household also treats w_t and r_t as exogenous.

Household's Choice Problem: Infinite Horizon (Contd.)

- Households' utility function which is defined over *infinite horizon* requires some explanation.
- There are three alternative explanation, each associated with a different interpretaion of the intertemporal discount factor β:
  - Agents live forever: Then the utility function defines the discounted values of his 'life-time' utlity. By this definition, β is to be interpreted as the agents' innate preference for present vis-a-vis future (or their rate of time preference);
  - Each agent lives only for a single period, but in the next period an exact replica is born who inherits the parent's tastes and preferences: Then the utility function defines the weighted sum of the dynastic utility. By this definition, β is to be interpreted as a measure of the degree of intergenerational altruism of an agent;
  - An agent can *potentially* live for ever but in each period he faces an exogenous mortality shock which is i.i.d. across time: Then the utility function defines the *expected* 'life-time' utility of an agent. By this definition, β is the constant probablity of survival from each period to the next.

- In our analysis, we shall use either the first or the second definition and ignore the 'expected' utility interpretation (since we do want want any 'uncertainty' to affect the households' decision making process at this stage).
- Note that that the household is solving this problem at time 0. Therefore, in order to solve this problem the households would have to have some expectation about the entire time paths of w_t and r_t from $t = 0$ to $t \rightarrow \infty$.
- We shall however assume that households' have rational expectations. In this model with complete information and no uncertainty, rational expectation is equivalent to perfect foresight. We shall use these two terms here interchangeably.
- By virtue of the assumption of rational expectations/perfect foresight, the agents can **correctly guess** all the future values of the market wage rate and rental rate, but they still treat them as exogenous.
- As **atomistic** agents, they believe that their action cannot influence the values of these 'market' variables.

Household's Choice Problem: Infinite Horizon (Contd.)

- Notice that once we choose our consumption time path $\left\{c_t^h\right\}_{t=0}^{\infty}$, the corresponding time path of the asset level $\left\{a_{t+1}^h\right\}_{t=0}^{\infty}$ would automatically get determined from the constraint functions (and vice versa).
- So in effect in this constrained optimization problem, we only have to choose one set of variables directly. We call them the control variables. Let our control variable for this problem be $\left\{c_t^h\right\}_{t=0}^{\infty}$.
- We can always treat $c_0, c_1, c_2, \ldots\ldots$ as independent variables and solve the problem using the standard Lagrangean method.
- The only problem is that there are now infinite number of such choice variables $(c_0, c_1, c_2, \ldots\ldots, c_\infty)$ as well as infinite number of constraints (one for each time period from $t = 0, 1, 2\ldots\ldots, \infty$) and things can get quite intractable.
- Instead, we shall employ a different method - called Dynamic Programming - which simplifies the solution process and reduces it to a univariate problem.

Dynamic Optimization in Discrete Time: Dynamic Programming

- Consider the following canonical discrete time dynamic optimization problem:

$$\underset{\{x_{t+1}\}_{t=0}^{\infty},\{y_t\}_{t=0}^{\infty}}{Max.} \sum_{t=0}^{\infty} \beta^t \tilde{U}(t, x_t, y_t)$$

subject to

$$\begin{aligned} \text{(i) } y_t &\in \tilde{G}(t, x_t) \text{ for all } t \geqq 0; \\ \text{(ii) } x_{t+1} &= \tilde{f}(t, x_t, y_t);\ x_t \in X \text{ for all } t \geqq 0;\ x_0 \text{ given.} \end{aligned}$$

- Here y_t is the control variable; x_t is the state variable; $\tilde{U}$ represents the instantaneous payoff function.
- (i) specifies what values the control variable y_t is allowed to take (the feasible set), given the value of x_t at time t;
- (ii) specifies evolution of the state variable as a function of previous period's state and control variables (state transition equation).

Dynamic Programming (Contd.)

- It is often convenient to use the state transition equation given by (ii) to eliminate the control variable and write the dynamic programming problem in terms of the state variable alone:

$$\underset{\{x_{t+1}\}_{t=0}^{\infty}}{Max.} \sum_{t=0}^{\infty} \beta^t U(t, x_t, x_{t+1})$$

  subject to

$$\text{(i) } x_{t+1} \in G(t, x_t) \text{ for all } t \geqq 0; \;\; x_0 \text{ given.}$$

- We are going to focus on **stationary** dynamic programming problems, where time (t) does not appear as an independent argument in the objective or constraint function (other than in the discounting term):

$$\underset{\{x_{t+1}\}_{t=0}^{\infty}}{Max.} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1})$$

  subject to

$$\text{(i) } x_{t+1} \in G(x_t) \text{ for all } t \geqq 0; \;\; x_0 \text{ given.}$$

Stationary Dynamic Programming: Value Function

- Ideally we should be able to solve the above stationary dyanamic programming problem by employing the Lagrange method. Let us assume that such solution exists and is unique. Let $\left\{x^*_{t+1}\right\}_{t=0}^{\infty}$ denote the corresponding solution.
- Then we should be able to write the maximised value of the objective function as a function of the parameters alone, in particular as a function of x_0 :

$$\begin{aligned} V(x_0) &\equiv \underset{\{x_{t+1}\}_{t=0}^{\infty}}{Max.} \sum_{t=0}^{\infty} \beta^t U\left(x_t, x_{t+1}\right); \quad x_{t+1} \in G(x_t) \text{ for all } t \geqq 0; \\ &= \sum_{t=0}^{\infty} \beta^t U\left(x^*_t, x^*_{t+1}\right). \end{aligned}$$

- The maximized value of the objective function is called the **value function**.
- The function $V(x_0)$ represents the value function of the dynamic programming problem at time 0.

Value Function: (Contd.)

- Suppose we were to repeat this exercise again the next period i.,e. at $t = 1$.
- Now of course the time period $t = 1$ will be counted as the initial point and the corresponding initial value of the state variable will be x_1^*.
- Let τ denote the new time subscript which counts time from $t = 1$ to ∞. By construction then, $\tau \equiv t - 1$.
- When we set the new optimization exercise (relevant for $t = 1, 2...., \infty$) in terms of τ it looks exactly *similar* to the one deined in terms of t (except that $x_1^* \neq x_0$). In particular, the new value function will be given by:

$$
\begin{aligned}
V(x_1^*) &\equiv \underset{\{x_{\tau+1}\}_{\tau=0}^{\infty}}{Max.} \sum_{\tau=0}^{\infty} \beta^{\tau} U(x_\tau, x_{\tau+1}); \quad x_{\tau+1} \in G(x_\tau) \text{ for all } \tau \geqq 0; \\
&= \sum_{\tau=0}^{\infty} \beta^{\tau} U(x_\tau^*, x_{\tau+1}^*).
\end{aligned}
$$

Value Function & Principle of Optimality:

- It is important to note here that by the **Principle of Optimality** if $\left\{x^*_{t+1}\right\}_{t=0}^{\infty}$ was an optimal soution to the problem that maximises

$$\underset{\{x_{t+1}\}_{t=0}^{\infty}}{Max.} \sum_{t=0}^{\infty} \beta^t U\left(x_t, x_{t+1}\right) \quad (x_0 \text{ given}), \tag{A}$$

then $\left\{x^*_{\tau+1}\right\}_{\tau=0}^{\infty}$ such that $x^*_{\tau=1} = x^*_{t=2}$; $x^*_{\tau=2} = x^*_{t=3}, \ldots$. **must be** a solution to the problem that maximises

$$\underset{\{x_{\tau+1}\}_{\tau=0}^{\infty}}{Max.} \sum_{\tau=0}^{\infty} \beta^\tau U\left(x_\tau, x_{\tau+1}\right) \quad (x_{\tau=0} = x^*_{t=1} \text{ given}). \tag{B}$$

- Otherwise $\left\{x^*_{t+1}\right\}_{t=0}^{\infty}$ could not have been an optimal solution to problem (A) to begin with!

Value Function & Bellman Equation:

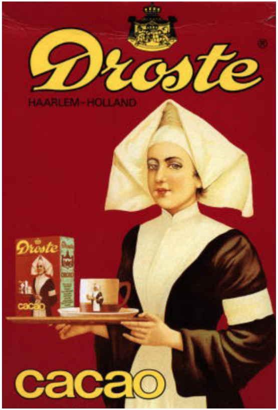
- Noting the relationship between t and τ, and noting the principle of optimality, we can immediately see that the two value functions are related in the following way:

$$
\begin{aligned}
V(x_0) &= \sum_{t=0}^{\infty} \beta^t U\left(x_t^*, x_{t+1}^*\right) \\
&= U\left(x_0, x_1^*\right) + \beta \sum_{t=1}^{\infty} \beta^{t-1} U\left(x_t^*, x_{t+1}^*\right) \\
&= U\left(x_0, x_1^*\right) + \beta \sum_{\tau=0}^{\infty} \beta^{\tau} U\left(x_\tau^*, x_{\tau+1}^*\right) \text{ (by Principle of Optimalit} \\
&= U\left(x_0, x_1^*\right) + \beta V\left(x_1^*\right).
\end{aligned}
$$

- The above relationship is the basic functional equation in dynamic programming which relates two successive value functions *recursively*. It is called the **Bellman Equation**.

Value Function & Bellman Equation: (Contd.)

- Recursive Property:



- The Bellman Equation is a recursive equation because it expresses the value function as a function of itself:

$$V(x_0) = U(x_0, x_1^*) + \beta V(x_1^*).$$

Value Function & Bellman Equation: (Contd.)

- It is important to mention here that this recursive representation of the value function may work even when
  (a) the problem is defined over finite horizon;
  (b) the discount factor itself changes over time (β_t) (as happens when, for example, when you have hyperbolic discounting);
  (c) the problem is non-stationary.
- In all these cases however one has to re-define the problem by introducing new variables (state or control) which represent these other time-dependent factors AND **re-define the value function to take into account this extra variables**.
- Having said that, for the rest of the lectures, we are going to restrict ourselves to the *special case* of a **stationary infinite horizon problem with a constant (exponential) discount factor**.

Value Function & Bellman Equation: (Contd.)

- The Bellman equation plays a crucial role in the dynammic programming technique.
- Since x_1^* is an optimal value itself, we can write the Bellman equation as:

$$V(x_0) = \underset{x_1 \in G(x_0)}{Max} \left[U(x_0, x_1) + \beta V(x_1) \right]; \; x_0 \text{ given.}$$

- Notice that it breaks down the inifinite horizon dynamic optimization problem into a two-stage problem:
  - Given x_0, what is the optimal value of x_1;
  - what is the optimal continuation path $(V(x_1))$.
- Thus it reduces the initial optimization problem with infinite number of variables and infinte number of constraints to a simple optimization exercise entailing only one variable (x_1).

Value Function & Policy Function (Contd.)

- Since the above functional relationship holds for any two successive values of the state variable,we can write the Bellman Equation more generally as:

$$V(x_t) = \underset{x_{t+1} \in G(x_t)}{Max} \left[U(x_t, x_{t+1}) + \beta V(x_{t+1}) \right] \text{ for all } x \in X.$$

- Or equivalently:

$$V(x) = \underset{\tilde{x} \in G(x)}{Max} \left[U(x, \tilde{x}) + \beta V(\tilde{x}) \right] \text{ for all } x \in X. \tag{1}$$

- The maximizer of the right hand side of equation (2) is called a **policy function**:

$$\tilde{x} = \pi(x),$$

which solves the RHS of the Bellman Equation above.

Value Function & Policy Function (Contd.)

- If we knew the exact form of the value function $V(.)$ and were it differentiable, we could have easily found the policy function by solving the following FONC (called the **Euler Equation**):

$$\tilde{x} : \frac{\partial U(x, \tilde{x})}{\partial \tilde{x}} + \beta V'(\tilde{x}) = 0. \tag{2}$$

Value Function & Policy Function (Contd.)

- **Unfortunately, the value function is not known!**
  (Recall that we introduced it as the maximized value of the objective function for a '*hypothetical*' solution to the problem. But we had not really solved the problem. Hence characteristics of this solution and therefore the characteristics of this value function are not really known!)
- In fact we do not even know whether it exists; if yes then whether it is unique, whether it is continuous, whether it is differentiable etc.
- A lot of theorems in Dynamic Programming go into establishing conditions under which a value function exists, is unique and has all the nice properties (continuity, differentibility and others).
- For now, without going into futher details, we shall simply assume that all these conditions are satisfied for our problem.
- In other words, we shall assume that **for our problem the value function exists and is well-behaved** (even though we do not know its precise form).

Value Function & Policy Function (Contd.)

- Once the existence of the value function is established (here, by assumption), we can then solve the FONC (2) (the **Euler Equation)** to get the policy function.
- But there is still one hurdle: what is the value $V'(\tilde{x})$?
- Here the Envelope Theorem comes to our rescue.

Value Function & Policy Function (Contd.)

- Recall that $V(\tilde{x})$ is nothing but the value function for the next period where $\tilde{x}$ is next period's initial value of the state variable (which is given - from next period's perspective).
- Since the Bellman equation is defined for all $x \in X$, we therefore get a similar relationship between $\tilde{x}$ and its subsequent state value $(\hat{x})$:

$$V(\tilde{x}) = \underset{\hat{x} \in G(\tilde{x})}{Max} \left[U(\tilde{x}, \hat{x}) + \beta V(\hat{x}) \right].$$

- Then applying Envelope Theorem:

$$V'(\tilde{x}) = \frac{\partial U(\tilde{x}, \hat{x})}{\partial \tilde{x}}. \tag{3}$$

- Combining the Euler Equation (2) and the Envelope Condition (3), we get the following equation:

$$\frac{\partial U(x, \tilde{x})}{\partial \tilde{x}} + \beta \frac{\partial U(\tilde{x}, \hat{x})}{\partial \tilde{x}} = 0 \text{ for all } x \in X.$$

Value Function & Policy Function (Contd.)

- Replacing $x, \tilde{x}, \hat{x}$ by their suitable time subscripts:

$$\frac{\partial U(x_t, x_{t+1})}{\partial x_{t+1}} + \beta \frac{\partial U(x_{t+1}, x_{t+2})}{\partial x_{t+1}} = 0; \;\; x_0 \text{ given.} \tag{4}$$

- Equation (4) is a difference equation which we should be able to solve to derive the time path of the state variable x_t.
- Notice that (4) is a difference equation of order 2. However we can always reduce it to a 2×2 system of first order difference equations in the following way:
  - Define a new variable: $z_{t+1} \equiv x_{t+2}$ for all t. Using this definition, we can now write the above difference equation as:

$$\text{(i)} \;\; \frac{\partial U(x_t, z_t)}{\partial z_t} + \beta \frac{\partial U(z_t, z_{t+1})}{\partial z_t} = 0;$$

  - At the same time the definition itself tells us:

$$\text{(ii)} \;\; x_{t+1} = z_t$$

  - Equations (i) and (ii) represent a 2×2 system of first order difference equations in x_t and z_t.

Value Function & Policy Function (Contd.)

- To solve this 2×2 system, we need two boundary conditions.
- One boundary condition is specified by the **given** initial value x_0 .
- Notice however that we do not have any such pre-determined initial value of the other variable z_t. So z_t constitutes a 'jump' variable.
- Jump variables are not tethered to any initial condition; often they are pinned down by a terminal condition.
- Typically in a Dynamic Programming problem such a boundary condition is provided by the following **Transversality condition (TVC)**:

$$\lim_{t \to \infty} \beta^t \frac{\partial U(x_t, x_{t+1})}{\partial x_t} x_t = 0. \tag{5}$$

Transversality Condition and its Interpretation:

- The TVC is to be read as a complementary slackness condition in the following way:
  - as $t \to \infty$, if $\beta^t \frac{\partial U(x_t, x_{t+1})}{\partial x_t} > 0$, then $x_t = 0$;
  - on the other hand, as $t \to \infty$, if $x_t > 0$, then $\beta^t \frac{\partial U(x_t, x_{t+1})}{\partial x_t} = 0$
- In interpreting the TVC, notice that $\frac{\partial U(x_t, x_{t+1})}{\partial x_t}$ captures the marginal increment in the pay-off function associated with an increase in the current stock, or its shadow price.
- The TVC states that if (present discounted value of) the shadow price is positive then at the terminal date, agents will not leave any stock unused (i.e., would not leave any postive stock at the end of the period); on the other hand, if any stock indeed remains unused at the terminal date, then it must be the case that its shadow valuation is zero.

Different Methods of Solving a DPP:

- There are other ways to solve the Bellman equation and/or the associated Euler equation:

$$V(x) = \underset{\tilde{x} \in G(x)}{Max} \left[U(x, \tilde{x}) + \beta V(\tilde{x}) \right] \text{ for all } x \in X. \qquad \text{(Bellman)}$$

$$\tilde{x} : \frac{\partial U(x, \tilde{x})}{\partial \tilde{x}} + \beta \frac{\partial U(\tilde{x}, \hat{x})}{\partial \tilde{x}} = 0. \qquad \text{(Euler)}$$

- One method entails using a 'guess & verify' approach for the policy function:
  - We start with an arbitrary guess about the policy function: $\tilde{x} = \pi(x)$. It this is indeed the policy function, then a silimilar relationship must hold between $\hat{x}$ and $\tilde{x}$ too: $\hat{x} = \pi(\tilde{x})$. Moreover, it this is indeed the policy function then it must satify the corresponding Euler euqation:

$$\frac{\partial U(x, \pi(x))}{\partial \pi(x)} + \beta \frac{\partial U(\pi(x), \pi(\pi(x)))}{\partial \pi(x)} = 0.$$

  - It our 'trial' function π satisfies the above equation, we are done. Otherwise, we make another guess.

Different Methods of Solving a DPP:

- Another methos entails using a 'guess & verify' approach for the value function itself:
  - We start with an arbitrary guess about the value function $V(x)$. If it is indeed the value function, then it must satisfy both the Bellman equation and the associated Euler equation.
  - It our 'trial' function V satisfies these two equations, we are done. Otherwise, we make another guess.
- The 'guess and verify' method works well as long as you happen to start with the right 'guess'. But there is no clear cut, well-specified way to arrive at the right 'guess'.
- So for most part of our analysis, we shall stick to the difference equation method.

Stationary Dynamic Programming: Existence & Uniqueness of Value Function

- We now provide some sufficient conditions for the Value function of the above stationary dynamic programming problem to exist, to be twice continuously differentiable, to be concave etc.
- We just state the theorems here without proof. Proofs can be found in Acemoglu (2009).

1. Let $G(x)$ be non-empty-valued, compact and continuous in all $x \in X$ where X is a compact subset of $\Re$. Also let $U : X_G \longrightarrow \Re$ is continuous, where $X_G = \left\{(x_t, x_{t+1}) \in X \times X : x_{t+1} \in G(x_t)\right\}$. Then there exits a unique and continuous function $V : X \longrightarrow \Re$ that solves the stationary dynamic programming problem specified earlier.
2. Let us further assume that $U : X_G \longrightarrow \Re$ is concave and is continuously differentiable on the interior of its domain X_G. Then the unique value function defined above is strictly concave and is differentiable.

Existence & Uniqueness of Value Function: (Contd.)

- Having specified these sufficiency condition (more for the sake of completeness than delving deeper in terms of actually verifying them), we are just going to assume that they are satisfied for our problem at hand.
- Indeed, all the economic problems that we would be looking at in this course will satisfy these sufficiency properties (although we won't prove it here).
- So we shall stop bothering about this sufficeny condition from now on and focus on applying the dynamic programming technique to the economic problems at hand.

Dynamic Programming: Reference

- When the optimization problem is non-stationary, the solution technique is similar. Only the value function will now be time-dependent.
- Corresponding Bellman equation will now be written as:

$$V(x_t, t) = \underset{x_{t+1} \in G(x_t, t)}{Max} \left[U(x_t, x_{t+1}, t) + \beta V(x_{t+1}, t+1) \right]; x_0 \text{ given}.$$

- There exist analogous theorems which ensure existence, uniqueness and differentiability fo the value function for the non-stationary dynamic programming problem.
- Interested students can look up D. Acemoglu (2009): Introduction to Modern Economic Growth, Chapter 6, for the dynamic programming technique, associated theorems and proofs.

Back to Household's Choice Problem: Infinite Horizon

- Let us now go back to the representative household's optimization problem under infinite horizon as:

$$\underset{\left\{c_t^h\right\}_{t=0}^{\infty},\left\{a_{t+1}^h\right\}_{t=0}^{\infty}}{Max.} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t^h\right); u' > 0; u'' < 0$$

subject to

$$\begin{aligned} \text{(i) } c_t^h &\leqq w_t + r_t a_t^h + (1-\delta) a_t^h \text{ for all } t \geqq 0; \\ \text{(ii) } a_{t+1}^h &= w_t + (1 + r_t - \delta) a_t^h - c_t^h; \;\; a_t^h \geqq 0 \text{ for all } t \geqq 0; \;\; a_0^h \text{ given.} \end{aligned}$$

- However in specifying the problem earlier, we assumed that there is no intra-household borrowing.
- This assumption of no borrowing is too strong, and we do not really need it for the results that follow.
- So let us relax that assumption to allow households to borrow from one another if they so wish.

Household's Choice Problem: Infinite Horizon (Contd.)

- Allowing for intra-household borrowings means that constraint (i) would no longer hold. A household can now consume beyond its current income at any point of time - by borrowing from others.
- Allowing for intra-household borrowings also means that a household now has at least two forms of assets that it can invest its savings into:
    1. physical capital (k_t^h);
    2. financial capital, i.e., lending to other households $(l_t^h \equiv -b_t^h)$.
- Let the gross interest rate on financial assets be denoted by $(1 + \hat{r}_t)$.
- Let physical capital depreciate at a constant rate δ. Then the gross interest rate on investment in physical capital is given by $(r_t + 1 - \delta)$.

Household's Choice Problem: Infinite Horizon (Contd.)

- Arbitrage in the asset market ensures that in equlibrium two interest rates are the same :

$$1 + \hat{r}_t = 1 + r_t - \delta \Rightarrow \hat{r}_t = r_t - \delta.$$

- Thus we can define the total asset stock held by the household in period t as $a_t^h \equiv k_t^h + l_t^h$.
- Notice that $l_t^h < 0$ would imply that the household is a net borrower.
- Hence the aggregate budget constraint of the household is now given by:

$$c_t^h + s_t^h = w_t + \hat{r}_t a_t^h, \text{ where } s_t^h \equiv a_{t+1}^h - a_t^h.$$

- Re-writing to eliminate s_t^h:

$$a_{t+1}^h = w_t + (1 + \hat{r}_t) a_t^h - c_t^h.$$

Household's Choice Problem: Ponzi Game

- But allowing for intra-household borrowing brings in the possibility of households' playing a **Ponzi game,** as explained below.
- Consider the following plan by a household:
  - Suppose in period 0, the household borrows a huge amount $\bar{b}$ - which would allow him to maintain a very high level of consumption at all subsequent points of time. Thus

$$b_0 = \bar{b}.$$

  - In the next period (period 1) he pays back his period 0 debt with interest by borrowing again (presumably from a different lender). Thus his period 1 borrowing would be:

$$b_1 = (1 + \hat{r}_0)b_0.$$

  - In period 2 he again pays back his period 1 debt with interest by borrowing afresh:

$$b_2 = (1 + \hat{r}_1)b_1 = (1 + \hat{r}_1)(1 + \hat{r}_0)b_0.$$

and so on.

- Notice that proceeding this way, the household effectively never pays back its initial loan $\bar{b}$; he is simply rolling it over period after period.
- In the process he is able to perpetually maintain an arbitrarily high level of consumption (over and above his current income).
- His debt however grows at the rate $\hat{r}_t$:

$$b_{t+1} = (1 + \hat{r}_t) b_t$$

  which implies that $\lim_{t \to \infty} a_t^h \simeq - \lim_{t \to \infty} b_t^h \to -\infty$.
- This kind scheme is called a **Ponzi finance scheme**.
- If a household is allowed to play such a Ponzi game, then the household's budget constraint becomes meaningless. There is effectively no budget constraint for the household any more; it can maintain any arbitrarily high consumption path by playing a Ponzi game.
- To rule this out, we impose an additional constraint on the household's optimization problem - called the **No-Ponzi Game Condition**.

- One Version of **No-Ponzi Game (NPG) Condition**:

$$\lim_{t\to\infty} \frac{a_t^h}{(1+\hat{r}_0)(1+\hat{r}_1)......(1+\hat{r}_{t-1})} \geqq 0.$$

- This No-Ponzi Game condition states that as $t \to \infty$, the present discounted value of an household's asset must be non-negative.
- Notice that the above condition rules out Ponzi finance scheme for sure.
  - If you play Ponzi game then $\lim_{t\to\infty} a_t^h \simeq - \lim_{t\to\infty} b_t^h$, when the latter term is growing at the rate $(1+\hat{r}_t)$.
  - For simplicity, let us assume interest rate is constant at some $\bar{r}$. Then $b_t^h = (1+\bar{r})^t \bar{b}$.
  - Plugging this in the LHS of the NPG condition above:

$$\lim_{t\to\infty} \frac{a_t^h}{(1+\bar{r})^t} \simeq \lim_{t\to\infty} \frac{(-b_t^h)}{(1+\bar{r})^t} = \lim_{t\to\infty} \frac{-(1+\bar{r})^t \bar{b}}{(1+\bar{r})^t} = -\bar{b} < 0.$$

  - This surely violates the NPG condition specified above.

Household's Choice Problem: No-Ponzi Game Condition (Contd.)

- At the same time the NPG condition specified above is lenient enough to allow for *some* amount of perpetual borrowing throughout one's life time as long as borrowing grows at a rate less than the corresponding interest rate.
- To see this, suppose the household's borrowing is growing at some rate $\bar{g} < \bar{r}$ such that
$$b_t^h = (1 + \bar{g})^t \bar{b}.$$
- Plugging this in the LHS of the NPG condition above:

$$\lim_{t \to \infty} \frac{a_t^h}{(1+\bar{r})^t} \simeq \lim_{t \to \infty} \frac{(-b_t^h)}{(1+\bar{r})^t} = \lim_{t \to \infty} \frac{-(1+\bar{g})^t \bar{b}}{(1+\bar{r})^t} = -\bar{b} \lim_{t \to \infty} \left(\frac{1+\bar{g}}{1+\bar{r}} \right)^t .$$

- Notice that $\bar{g} < \bar{r}$ implies that the term $\left(\dfrac{1+\bar{g}}{1+\bar{r}} \right)$ is a positive fraction and as $t \longrightarrow \infty$, $\left(\frac{1+\bar{g}}{1+\bar{r}} \right)^t \longrightarrow 0$.

Household's Choice Problem: No-Ponzi Game Condition (Contd.)

- Since $\bar{b}$ is finite, this implies that in this case

$$\lim_{t\to\infty} \frac{a_t^h}{(1+\bar{r})^t} \to 0.$$

- In other words, the NPG condition is now satisfied at the margin!
- In terms of Economics, this kind of borrowing behaviour implies that the agent in not *completely* recycling his entire accumulated debt (principle + interest): at some point the agent must have started repaying at least some part of it (though not all) from his own pocket!

Household's Choice Problem - Revisited:

- After imposing the No-Ponzi Game condition, the household' optimization problem now becomes:

$$\underset{\{c_t^h\}_{t=0}^{\infty},\{a_{t+1}^h\}_{t=0}^{\infty}}{Max.} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t^h\right)$$

subject to

$$\text{(i) } a_{t+1}^h \;=\; w_t + (1+\hat{r}_t)a_t^h - c_t^h;\; a_t^h \in \Re \text{ for all } t \geqq 0;\; a_0^h \text{ given.}$$

(ii) The NPG condition.

- Here c_t^h is the control variable and a_t^h is the state variable.

Household's Choice Problem - Revisited: (Contd.)

- We can now apply the dynamic programming technique to solve the household's choice problem.
- First let us use constraint (i) to eliminate the control variable and write the above dynamic programming problem in terms of the state variable alone:

$$\underset{\left\{a_{t+1}^h\right\}_{t=0}^{\infty}}{Max.} \sum_{t=0}^{\infty} \beta^t u\left(\left\{w_t + (1+\hat{r}_t)a_t^h - a_{t+1}^h\right\}\right)$$

- Corresponding Bellman equation relating $V(a_0^h)$ and $V(a_1^h)$ is given by:

$$V(a_0^h) = \underset{\left\{a_1^h\right\}}{Max}\left[u\left(\left\{w_0 + (1+\hat{r}_0)a_0^h - a_1^h\right\}\right) + \beta V(a_1^h)\right].$$

Household's Problem: Bellman Equation

- More generally, we can write the Bellman equation for any two time periods t and $t+1$ as:

$$V(a_t^h) = \underset{\{a_{t+1}^h\}}{Max} \left[u\left(\left\{ w_t + (1+\hat{r}_t)a_t^h - a_{t+1}^h \right\} \right) + \beta V(a_{t+1}^h) \right].$$

- Maximising the RHS above with respect to a_{t+1}^h, from the FONC:

$$u'\left(\left\{ w_t + (1+\hat{r}_t)a_t^h - a_{t+1}^h \right\} \right) = \beta V'(a_{t+1}^h) \tag{6}$$

- Notice that $V(a_{t+1}^h)$ and $V(a_{t+2}^h)$ would be related through a similar Bellman equation:

$$V(a_{t+1}^h) = \underset{\{a_{t+2}^h\}}{Max} \left[u\left(\left\{ w_{t+1} + (1+\hat{r}_{t+1})a_{t+1}^h - a_{t+2}^h \right\} \right) + \beta V(a_{t+2}^h) \right].$$

- Applying Envelope Theorem on the latter:

$$V'(a_{t+1}^h) = u'\left(\left\{ w_{t+1} + (1+\hat{r}_{t+1})a_{t+1}^h - a_{t+2}^h \right\} \right).(1+\hat{r}_{t+1}). \tag{7}$$

Household's Problem: Optimal Solutions

- Combining (5) and (6):

$$\begin{aligned} & u'\left(\left\{w_t + (1+\hat{r}_t)a_t^h - a_{t+1}^h\right\}\right) \\ = \; & \beta u'\left(\left\{w_{t+1} + (1+\hat{r}_{t+1})a_{t+1}^h - a_{t+2}^h\right\}\right)(1+\hat{r}_{t+1}). \end{aligned}$$

- The above equation implicitely defines a 2nd order difference equation is a_t^h.
- However we can easily convert it into a 2×2 system of first order difference equations in the following way.

Household's Problem: Optimal Solutions (Contd.)

- Noting that the terms inside the $u'(.)$ functions are nothing but c_t^h and c_{t+1}^h respectively, we can write the above equation as:

$$u'\left(c_t^h\right) = \beta u'\left(c_{t+1}^h\right)(1+\hat{r}_{t+1}). \tag{8}$$

- We also have the constraint function:

$$a_{t+1}^h = w_t + (1+\hat{r}_t)a_t^h - c_t^h;\ a_0^h \text{ given.} \tag{9}$$

- Equations (8) and (9) represents a 2×2 system of difference equations which **implicitly** defines the 'optimal' trajectories $\left\{c_t^h\right\}_{t=0}^{\infty}$ and $\left\{a_{t+1}^h\right\}_{t=0}^{\infty}$.
- The two boundary conditons are given by the initial condition a_0^h, and the NPG condition.

Household's Problem: Optimal Solutions (Contd.)

- If you knew how to solve a 2×2 system of difference equation, you would have been able to characterise the solution paths for $\left\{c_t^h\right\}_{t=0}^{\infty}$ and $\left\{a_{t+1}^h\right\}_{t=0}^{\infty}$ from the above two dynamic equations and associated boundary conditions.
- The precise mathematical techniques for solving difference/differential equations is taught in parallel extra lectures (tutorials) for the course.
- At this point, let us try to characterise the dynamic paths for a simple example.
- We take use this example to highlight various interesting features of the optimal solution, which will later be substantiated for the more general case using rigorous mathematical techniques.

Optimal Solution Path to Household's Problem: An Example

- Let us look at the explicit characterization of the household's optimal paths for the following specific example.
- Suppose
$$u(c) = \log c$$
- To further simplify things, let us also assume that $w_t = \bar{w}$ and $\hat{r}_t = \bar{r}$ for all t .
- Then we can immediately get two difference equations characterizing the optimal trajectories for the household as:
$$c_{t+1}^h = \beta(1+\bar{r})c_t^h \quad (10)$$
and
$$a_{t+1}^h = \bar{w} + (1+\bar{r})a_t^h - c_t^h; \; a_0^h \text{ given.} \quad (11)$$
- The two equations along with the two boundary conditons can be solved explicitly to derive the time paths of c_t^h and a_t^h.

Household's Problem: An Example - Optimal Solutions (Contd.)

- Equation (9) is a linear autonomous difference equation, which can be directly solved (by iterating backwards) to get the optimal consumption path as:

$$
\begin{aligned}
c_t^h &= \beta(1+\bar{r})c_{t-1}^h \\
&= \beta(1+\bar{r})\left[\beta(1+\bar{r})c_{t-2}^h\right] = \beta^2(1+\bar{r})^2 c_{t-2}^h \\
&= \beta^2(1+\bar{r})^2\left[\beta(1+\bar{r})c_{t-3}^h\right] = \beta^3(1+\bar{r})^3 c_{t-3}^h \\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \beta^t(1+\bar{r})^t c_0^h. \qquad (12)
\end{aligned}
$$

- **However,**we still cannot completely characterise the optimal path because we still do not know the optimal value of c_0^h. (Recall that c_0^h is **not given**; it is to be chosen optimally).
- Here the NPG condition comes in handy in identifying the optimal c_0^h.

Household's Problem: An Example - Role of the NPG Condition

- Note that the NPG condition in this case is given by:

$$\lim_{t\to\infty} \frac{a_t^h}{(1+\bar{r})^t} \geqq 0.$$

- Now let us look at asset accumulation equation at some future date $T > 0$:

$$a_{T+1}^h = \bar{w} + (1+\bar{r})a_T^h - c_T^h.$$

- Iterating backwards,

$$\begin{aligned} a_{T+1}^h &= \bar{w} + (1+\bar{r})a_T^h - c_T^h \\ &= \bar{w} + (1+\bar{r})\left[\bar{w} + (1+\bar{r})a_{T-1}^h - c_{T-1}^h\right] - c_T^h \\ &= \;.... \\ &= \sum_{t=0}^{T}\left(\bar{w}(1+\bar{r})^{T-t}\right) - \sum_{t=0}^{T}\left(c_t^h(1+\bar{r})^{T-t}\right) + (1+\bar{r})^{T+1}a_0^h. \end{aligned}$$

Household's Problem: An Example - Role of the NPG Condition (Contd.)

- Rearranging terms:

$$\frac{a^h_{T+1}}{(1+r)^T} = \sum_{t=0}^{T}\left(\frac{\bar{w}}{(1+\bar{r})^t}\right) + (1+\bar{r})a^h_0 - \sum_{t=0}^{T}\left(\frac{c^h_t}{(1+\bar{r})^t}\right)$$

- Now let $T \rightarrow \infty$. Then applying the NPG condition to the LHS, we get:

$$\sum_{t=0}^{\infty}\left(\frac{\bar{w}}{(1+\bar{r})^t}\right) + (1+\bar{r})a^h_0 - \sum_{t=0}^{\infty}\left(\frac{c^h_t}{(1+\bar{r})^t}\right) \geqq 0$$

$$\text{i.e., } \sum_{t=0}^{\infty}\left(\frac{c^h_t}{(1+\bar{r})^t}\right) \leqq \sum_{t=0}^{\infty}\left(\frac{\bar{w}}{(1+\bar{r})^t}\right) + (1+\bar{r})a^h_0. \qquad (13)$$

- Equation (13) represents the lifetime budget constraint of the household. It states that when the NPG condition is satisfied, then the discounted life-time consumption stream of the household cannot exceed the sum-total of its discounted life-time wage earnings and the returns on its initial wealth holding.

Household's Problem: An Example - Role of the NPG Condition (Contd.)

- From the above equation, it is easy to see that even though we have specified the NPG condition in the form of an inequality, the households would always satisfy it at the margin such that it holds with strict equality.
  - Since the marginal utility of consumption is positive, facing the lifetime budget constraint (as represented by equation (13)), they would optimally choose a life-time consumption stream that completely exhausts the RHS of the above equation.

Household's Problem: An Example - Role of the NPG Condition (Contd.)

- Given that equation (13) holds with strict equality, we can now identify the optimal value of c_0^h.
- We had already derived the optimal time path of c_t^h as:

$$c_t^h = \beta^t (1+\bar{r})^t c_0^h.$$

- Using this in equation (13) above, we get:

$$\sum_{t=0}^{\infty} \left(\frac{\beta^t (1+\bar{r})^t c_0^h}{(1+\bar{r})^t} \right) = \sum_{t=0}^{\infty} \left(\frac{\bar{w}}{(1+\bar{r})^t} \right) + (1+\bar{r}) a_0^h$$

$$\Rightarrow \quad \sum_{t=0}^{\infty} \left(\beta^t\right) c_0^h = \left[\sum_{t=0}^{\infty} \left(\frac{\bar{w}}{(1+\bar{r})^t} \right) + (1+\bar{r}) a_0^h \right]$$

$$\Rightarrow \quad c_0^h = (1-\beta) \left[\sum_{t=0}^{\infty} \left(\frac{\bar{w}}{(1+\bar{r})^t} \right) + (1+\bar{r}) a_0^h \right].$$

Household's Problem: An Example - Role of the NPG Condition (Contd.)

- Thus using the NPG condition we have now been able to completely characterize the optimal consumption path of the household (including the optimal value of the initial consumption level c_0^h) for this example.

Household's Problem: An Example - NPG vis-a-vis TVC

- So for this particular example, we have been able to explicitly solve for the optimal consumption path of the households.
- But there is a problem that we still need to sort out.
- Recall that while discussing the dynamic programming problem we had specified a transversality condition (TVC) as one of our boundary condition (Refer to equation (5) specified earlier).
- Then in defining the household's problem with intra-household borrowing, we have introduced the NPG condition as another boundary condition.
- So we now have a problem of plenty: for a 2×2 dynamic system, it seems that we have three boundary conditions!!!
- Between the TVC and the NPG condition, which one should we use to characterise the solution?
- As it turns out, along the optimal path the NPG condition and the TVC become equivalent.

Household's Problem: NPG vis-a-vis TVC (Contd.)

- To see this, let us take a closer look at the TVC as had been specified earlier in equation (5):

$$\lim_{t\to\infty} \beta^t \frac{\partial U(x_t, x_{t+1})}{\partial x_t} x_t = 0.$$

- In the context of the current problem of households' utility maximization exercise, this transversality condition would look as follows (**verify this**):

$$\lim_{t\to\infty} \beta^t u'(c_t^h)(1+\hat{r}_t)a_t^h = 0$$

- For our specific example with log utility and constant factor prices, this condition reduces to

$$\lim_{t\to\infty} \beta^t \frac{1}{c_t^h}(1+\bar{r})a_t^h = 0$$

Household's Problem: An Example - NPG vis-a-vis TVC (Contd.)

- Given the solution path of $c_t^h : c_t^h = \beta^t (1+\bar{r})^t c_0^h$, we can further simplify the above condition to:

$$\lim_{t \to \infty} \beta^t \frac{1}{\beta^t (1+\bar{r})^t c_0^h} (1+\bar{r}) a_t^h = 0$$

$$\Rightarrow \lim_{t \to \infty} \frac{a_t^h}{(1+\bar{r})^t} = 0 \quad \text{(since } c_0^h \text{ is finite)}$$

- But this is nothing but our earlier NPG condition - holding with strict equality!
- Thus when the household is on its optimal path, the NPG condition and the Transversality condition become equivalent - except that the NPG condition must hold with equality.
- So in identifying the optimal trajectories, we could use either of them as the relevant boundary condition.

Household's Problem with Heterogenous Agents: Does Heterogeneity Matter?

- Recall that we have assumed that all households are identical in terms of their preferences, but not necessarily in terms of their initial asset holding.
- In fact if all households were indeed identical in every respect, then allowing for intra-household borrowing and the consequent NPG condition would not have made sense: one side of the borrowing/lending market would always be missing and hence no borrowing or lending would ever take place.
- All the above conditions make sense only if households are heterogenous.
- So if households are preference-wise identical but differ in terms of their intial wealth, where does this heterogeneity show up?

Household's Problem: Does Heterogeneity Matter? (Contd.)

- To see how heterogeneity matters, let us now go back to our earlier example of log utility and constant factor returns.
- We have already seen that the dynamic equation for a household's optimal consumption path is given by:

$$c^h_{t+1} = \beta(1+\bar{r})c^h_t$$

- Thus any household with an initial wealth level of a^h_0 will have the following optimal consumption path:

$$c^h_t = \beta^t(1+\bar{r})^t c^h_0.$$

where

$$c^h_0 = (1-\beta)\left[\sum_{t=0}^{\infty}\left(\frac{\bar{w}}{(1+\bar{r})^t}\right) + (1+\bar{r})a^h_0\right].$$

- Notice that the rate of growth of consumption along the optimal path is given by $\beta(1+\bar{r})-1$, which is independent of the initial wealth!

Household's Problem: Does Heterogeneity Matter? (Contd.)

- Thus along the optimal path, consumption of all households grow at the same rate - irrespective of their initial wealth.
- The initial wealth only determines the **level** of optimal consumption: higher initial wealth means higher level of consumption.
- This is a striking result because it tells us that t*he initial wealth has no* **growth** *effect, only* **level** *effect*.
- It also tells us that when all households are following their respective optimal trajectories, the initial (relative) inequality in consumption will be maintained perpetually.

Household's Problem: Does Heterogeneity Matter? (Contd.)

- What about the growth rate of asset stocks of various households?
- Note that we should be able to solve for the time path of a_t^h (given a_0^h) by solving the following dynamic equation:

$$a_{t+1}^h = \bar{w} + (1 + \bar{r})a_t^h - [\beta(1 + \bar{r})]^t \, c_0^h$$

- This is a difference equation which linear but non-autonomous; solving this would require more elaborate technique than mere backward induction.
- We shall come back to this equation, or a more general form of this equation, later.

Household's Problem: Does Heterogeneity Matter? (Contd.)

- All these results are of course derived under the assumption of constant factor returns. When factor returns (w_t and $\hat{r}_t$) are changing over time, the consumption growth rate itself will change.
- Nonetheless, we can easily generalize the results to such non-autonomous cases. In fact all the results will go through. It is only that the dynamic equations and the associated boundary conditions will now be given by:

$$c^h_{t+1} = \beta(1 + \hat{r}_{t+1})c^h_t;$$

$$a^h_{t+1} = w_t + (1 + \hat{r}_t - \delta)a^h_t - c^h_t$$

$$a^h_0 \text{ given; } \lim_{t\to\infty} \frac{a^h_t}{(1+\hat{r}_0)(1+\hat{r}_1)(1+\hat{r}_2)...(1+\hat{r}_t)} = 0$$

- To precisely characterise the dynamic paths fo this general cases, we need more information about the precise time paths of w_t and $\hat{r}_t$, which means we shall have to discuss the production side story.

Household's Problem: Does Heterogeneity Matter? (Contd.)

- But before we move on to the production side story, we should mention something about the chracteristics of the utility function.
- Notice that in deriving all these above results, we have also made use of the log utility, which we know is special.
- Can we generalize these results to other utility functions as well?
- It turns out, all the results will go through for a broad class of utility functions called the **CRRA** variety:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma};\ \sigma \neq 1.$$

Household's Problem: Does Heterogeneity Matter? (Contd.)

- This utility function has several interesting characteristics:

1. It is associated with constant elasticity of marginal utility: $\frac{-cu''(c)}{u'(c)} = \sigma$
2. It is associated with constant relative risk aversion (as defined by the Arrow-Pratt measure of relative risk aversion): $\frac{-cu''(c)}{u'(c)} = \sigma$
3. It is associated with constant elasticity of substitution between current and future consumption: $\frac{-d\left(\frac{c_{t+1}}{c_t}\right) / \left(\frac{c_{t+1}}{c_t}\right)}{d\left(\frac{u'(c_{t+1})}{u'(c_t)}\right) / \left(\frac{u'(c_{t+1})}{u'(c_t)}\right)} = \frac{1}{\sigma}$

- In fact $u(c) = \log c$ is a special case of this particular class of CRRA utility functions. It can be shown that as $\sigma \to 1$, $\frac{c^{1-\sigma}}{1-\sigma} \to \log c$.

Household's Problem: Does Heterogeneity Matter? (Contd.)

- **An Exercise:** Assume that wages and interest rates are constant and use the dynamic programming technique to derive the dynamic equation for the optimal consumption path of an agent with an initial asset stock of a_0^h, when his utility function is of the CRRA variety, as defined above.

Production Side Story: Optimal Decisions of Firms

- Typically we do not think of the firm's choice problem as a dynamic one.
- In a perfectly competitive set up, the firm is a blackbox: it does not own any factors of production and merely decides how much labour to employ and how much capital to hire in every period so as to maximise its current profit (taking all prices as given).
- Since firms under perfect competition do not earn any positive profit, the question of investing in activities that may augment future profit does not arise.
- Thus the choice problem of a firm is essentially static - same as what we discussed earlier (in topic 2 on Microfoundations), the only difference being now that firm optimally chooses both how much labour to employ as well as how much capital to rent in.

Production Side Story: (Contd.)

- Consider M number of firms, each having access to an identical production technology.
- Each firm produces a final commodity in every period using a production function:
$$Y_t^i = F(N_t^i, K_t^i).$$
- All the Neoclassical properties of the production function specified previously are assumed to hold - including diminishing marginal products, Inada Conditions and CRS.
- The perfectly comptetitive firm takes all the prices as given, and its optimal choice of labour and capital comes out of the following static optimization exercise:

$$\underset{\{N_t^i, K_t^i\}}{Max.} \ \pi_t^i = Y_t^i - w_t N_t^i - r_t K_t^i$$

Production Side Story: (Contd.)

- Corresponding FONCs:

$$
\begin{aligned}
F_N(N_t^i, K_t^i) &= w_t; \\
F_K(N_t^i, K_t^i) &= r_t.
\end{aligned}
$$

- If the firm is asked to repeat this exercise in the next period (i.e., in period $t+1$), it will choose its optimal employment of N_{t+1}^i and K_{t+1}^i in the same fashion such that

$$
\begin{aligned}
F_N(N_{t+1}^i, K_{t+1}^i) &= w_{t+1}; \\
F_K(N_{t+1}^i, K_{t+1}^i) &= r_{t+1}.
\end{aligned}
$$

- Thus in every period, the firm will employ capital and labour so as to equate the respective marginal products with the corresponding factor prices (in real terms) at that time period.

Firms' Optimal Choices: Static vis-a-vis Dynamic

- In the above analysis, we have set up the problem of the firm as a static problem.
- But the analysis would not change even if we think of this as a dynamic problem - carried over infinite horizon.
- As long as such hiring decisions do not affect future profits, setting the optimization problem in a dynamic framework (i.e., optimizing over multiple time periods) does not bring in any extra insight over the static optimization problem.
- The firms will have meaningful dynamic choices if and only if we allow a firm to **own** the capital stock that it employ; then it will be interested in investing part of its current income/pay off/profit in augmenting its capital stock which will affect its future profitability.
- But in the present set up where the entire capital stock is owned by the households, this issue does not arise.

Bringing Households and Firms Together: The General Equilibrium Set Up

- So far we have looked at the households' problem and the firms' problem in isolation.
- Both sets of agents were assumed to be 'atomistic'; they take all the market variables as exogenously given.
- But in the aggregate economy, the market variables are not exogenous; they are determined precisely by the aggregate actions of the households and the firms.
- So we now consider the general equilibrium set up where the households' and the firms' actions are mediated through the market to generate some aggregative behaviour for the entire macroeconomy.
- The corresponding solution for the aggregate economy will be called the 'decentralized' or 'market' equilibrium solution (as opposed to an alternative scenario where production is centralized under a social planner).

General Equilibrium in the Decentralized Market Economy:

- Let us quickly revisit the household and firm specifications for this de-centralized market economy:
  - We have H single-membered households which are identical in terms of preferences **but differ in terms of their initial asset holdings**;
  - Each household is endowed with one unit of labour - which it supplies inelastically to the market in every period (there is no population growth);
  - Households are atomistic and take the market wage rate (w_t) and market the interest rate (r_t) (and the corresponding net interest rate, $\hat{r}_t = r_t - \delta$) as given. But they are endowed with perfect foresight - so they can correctly guess the entire stream of current & future wage rates $\{w_t\}_{t=0}^{t=\infty}$, as well as the current & future interest rates $\{r_t\}_{t=0}^{t=\infty}$.
  - The households own the entire labour and the capital stock in the economy. In addition, they also hold loans against one another.
  - Each household maximises its lifetime utility subject to its period by period budget constraint.

General Equilibrium: Decentralized Market Economy (Contd.)

- On the production side:
  - There are M **identical** firms endowed with a technology to produce the final commodity.
  - The technology uses capital and labour as inputs; it exhibits diminishing returns with respect to each of the inputs; it is also CRS in both the inputs.
  - The firms do not own any capital or labour; they hire labour and capital from the market to carry out production in each period.
  - The firms operate under perfect competetion; they take the market wage rate (w_t) and market the interest rate (r_t) as given.
  - The firms optimally decide about how much labour/capital to employ in every period so as to maximise its period-by-period profit.

General Equilibrium: Decentralized Market Economy (Contd.)

- For expositional simplicity, we shall assume specific functional forms for the utility function and the production function. Accordingly, let

$$u(c) = \log c$$

and

$$Y_t = F(K_t, N_t) = (K_t)^{\alpha}(N_t)^{1-\alpha}; \ 0 < \alpha < 1.$$

General Equilibrium: Decentralized Market Economy (Contd.)

- The optimization problem of a household h with an initial asset holding of a_0^h is given by:

$$\underset{\{c_t^h\}_{t=0}^{\infty}, \{a_{t+1}^h\}_{t=0}^{\infty}}{Max.} \quad \sum_{t=0}^{\infty} \beta^t \log\left(c_t^h\right)$$

subject to

$$a_{t+1}^h = w_t + (1+\hat{r}_t)a_t^h - c_t^h; \; a_t^h \geqq 0 \text{ for all } t \geqq 0; \; a_0^h \text{ given.}$$

- Characterization of the optimal paths:

$$c_{t+1}^h = \beta(1+\hat{r}_{t+1})c_t^h; \tag{14}$$

$$a_{t+1}^h = w_t + (1+\hat{r}_t)a_t^h - c_t^h; \tag{15}$$

$$a_0^h \text{ given; } \lim_{t\to\infty} \frac{a_t^h}{(1+\hat{r}_0)(1+\hat{r}_1)......(1+\hat{r}_t)} = 0 \text{ (NPG/TVC).}$$

General Equilibrium: Decentralized Market Economy (Contd.)

- While one can potentially solve for the optimal paths for each household, in a Macro course we are actually interested in tracking the aggregate economy.
- For this purpose, define per capita consumption and per capita asset holding in this economy as:

$$c_t \equiv \frac{\sum_{h=1}^{H} c_t^h}{H}; \; a_t \equiv \frac{\sum_{h=1}^{H} a_t^h}{H}.$$

- Recall that households hold their assets in the form of either physical capital or financial capital (loans) such that

$$a_t \equiv \frac{\sum_{h=1}^{H} a_t^h}{H} = \frac{\sum_{h=1}^{H} (k_t^h + l_t^h)}{H} = \frac{\sum_{h=1}^{H} k_t^h}{H} + \frac{\sum_{h=1}^{H} l_t^h}{H}.$$

General Equilibrium: Decentralized Market Economy (Contd.)

- Since one household's lending is another household's borrowing, on the aggregate $\frac{\sum_{h=1}^{H} l_t^h}{H} = 0$.
- Thus,

$$a_t \equiv \frac{\sum_{h=1}^{H} a_t^h}{H} = \frac{\sum_{h=1}^{H} k_t^h}{H} \equiv k_t,$$

where k_t denotes the per capita capital stock in the economy.

General Equilibrium: Decentralized Market Economy (Contd.)

- Notice that the individual optimal transition equations (14 & 15) can be used to derive the transition equations for the per capita consumption and per capita capital stock of the economy in the following way:

$$
\begin{aligned}
c_{t+1} &\equiv \frac{\sum_{h=1}^{H} c_{t+1}^{h}}{H} = \frac{\sum_{h=1}^{H} \beta(1+\hat{r}_{t+1})c_t^h}{H} = \beta(1+\hat{r}_{t+1})\frac{\sum_{h=1}^{H} c_t^h}{H} \\
&= \beta(1+\hat{r}_{t+1})c_t
\end{aligned}
$$

General Equilibrium: Decentralized Market Economy (Contd.)

- On the other hand,

$$
\begin{aligned}
k_{t+1} &= a_{t+1} \equiv \frac{\sum_{h=1}^{H} a_{t+1}^h}{H} = \frac{\sum_{h=1}^{H} \left[w_t + (1+\hat{r}_t)a_t^h - c_t^h\right]}{H} \\
&= \frac{\sum_{h=1}^{H} w_t}{H} + (1+\hat{r}_t)\frac{\sum_{h=1}^{H} a_t^h}{H} - \frac{\sum_{h=1}^{H} c_t^h}{H} \\
&= w_t + (1+\hat{r}_t)k_t - c_t.
\end{aligned}
$$

- Finally, the individual boundary conditions can also be aggregated over all H households to get the boundary conditions for k_t as:

$$
k_0 \text{ given; } \lim_{t\to\infty} \frac{k_t}{(1+\hat{r}_0)(1+\hat{r}_1)......(1+\hat{r}_t)} = 0
$$

General Equilibrium: Decentralized Market Economy (Contd.)

- We have now derived the transition equations of the per capita consumption and per capita capital stock for the aggregative economy - except that we still do not know the precise values of the market wage rate (w_t) and the net interest rate ($\hat{r}_t = r_t - \delta$).
- These factor prices are determined in the market by the demand and supply of labour and capital respectively.
- At any time period t, total supply of capital (coming from all the households) is given by:

$$K_t^S = \sum_{h=1}^{H} k_t^h = H.k_t$$

- Likewise, total supply of labour (coming from all the households) is given by:

$$N_t^S = H$$

- The demand for these factors on the other hand comes from the firms.

General Equilibrium: Decentralized Market Economy (Contd.)

- At any point of time t, the profit maximization problem of a firm i is given by:

$$\underset{N_t^i, K_t^i}{Max.} \left[(K_t^i)^{\alpha} (N_t^i)^{1-\alpha} - w_t N_t^i - r_t K_t^i \right].$$

- Corresponding FONCs:

$$\begin{aligned} (1-\alpha)(K_t^i)^{\alpha}(N_t^i)^{-\alpha} &= w_t \\ &\Rightarrow (1-\alpha)\left(\frac{K_t^i}{N_t^i}\right)^{\alpha} = w_t \end{aligned} \tag{16}$$

$$\begin{aligned} \alpha(K_t^i)^{\alpha-1}(N_t^i)^{1-\alpha} &= r_t \\ &\Rightarrow \alpha\left(\frac{K_t^i}{N_t^i}\right)^{\alpha-1} = r_t \end{aligned} \tag{17}$$

General Equilibrium: Decentralized Market Economy (Contd.)

- From the above optimality conditions, we can immediately see that facing any (w_t, r_t) combination, a firm will choose its optimal $\frac{K_t^i}{N_t^i}$ such that the following holds:

$$
\begin{aligned}
\frac{(1-\alpha)\left(\frac{K_t^i}{N_t^i}\right)^{\alpha}}{\alpha\left(\frac{K_t^i}{N_t^i}\right)^{\alpha-1}} &= \frac{w_t}{r_t} \\
\text{i.e., } \left(\frac{K_t^i}{N_t^i}\right) &= \frac{\alpha}{1-\alpha}\left(\frac{w_t}{r_t}\right)
\end{aligned}
$$

- And this would be true for all firms $i = 1, 2, ...M$.

General Equilibrium: Decentralized Market Economy (Contd.)

- Since all firms are endowed with identical technologies and face the same market-determined factor prices, they all employ the same amount of capital and labour, so that the aggregate demand for labour and capital respectively are given by:

$$
\begin{aligned}
K_t^D &= \sum_{i=1}^{M} K_t^i = M.K_t^i \\
N_t^D &= \sum_{i=1}^{M} N_t^i = M.N_t^i
\end{aligned}
$$

General Equilibrium: Decentralized Market Economy (Contd.)

- Equilibrium in the factor market requires that:

$$K_t^S = K_t^D \text{ and } N_t^S = N_t^D \ .$$

- In other words, factor market clearing conditions are given by:

$$\left.\begin{array}{r} H.k_t = M.K_t^i \\ H = M.N_t^i \end{array}\right\} \text{ where } \frac{K_t^i}{N_t^i} = \frac{\alpha}{1-\alpha}\left(\frac{w_t}{r_t}\right)$$

- Writing in ratio terms, factor market clearing condition requires that:

$$k_t = \frac{K_t^i}{N_t^i} = \frac{\alpha}{1-\alpha}\left(\frac{w_t}{r_t}\right) \tag{18}$$

- Thus for any historically given per capita capital stock for the aggregate economy (k_t), we can find out the corresponding market clearing wage-rental ratio $\left(\frac{w_t}{r_t}\right)$ from the above equation.
- But we still don't know the exact values of w_t and r_t.

General Equilibrium: Decentralized Market Economy (Contd.)

- But those are easy to derive. The above market clearing condition also tells us that in the market equilibrium $k_t = \dfrac{K_t^i}{N_t^i}$.
- The only w_t are r_t which are consistent with this reduced form market clearing condition would be the ones that we obtain by solving equations (16) and (17) when we substitute $\frac{K_t^i}{N_t^i}$ by k_t:

$$(1-\alpha)\left(k_t\right)^{\alpha} = w_t; \quad \alpha\left(k_t\right)^{\alpha-1} = r_t. \tag{19}$$

- For any historically given k_t, the w_t and r_t will adjust in every period to maintain the above two equalities. Thus we have precisely identified the market determined values of w_t and r_t in every period as a function of the historically given per capita capita stock (which is also the equilibrium capital-labour ratio employed by each firm).
- We now use these information to completely characterise the dynamic paths for the aggregative economy.

General Equilibrium: Decentralized Market Economy (Contd.)

- Recall the dynamic equations for c_t and k_t :

$$c_{t+1} = \beta(1 + \hat{r}_{t+1})c_t;$$
$$k_{t+1} = w_t + (1 + \hat{r}_t)k_t - c_t.$$

- Noting that $\hat{r}_t = r_t - \delta$, and replacing the market clearing values of w_t and r_t derived above, we get:

$$c_{t+1} = \beta\left[1 + \alpha\left(k_{t+1}\right)^{\alpha-1} - \delta\right] c_t; \qquad \text{(I)}$$

$$\begin{aligned} k_{t+1} &= (1-\alpha)\left(k_t\right)^{\alpha} + \left[1 + \alpha\left(k_t\right)^{\alpha-1} - \delta\right] k_t - c_t \\ \Rightarrow & \quad k_{t+1} = \left(k_t\right)^{\alpha} + (1-\delta)\, k_t - c_t. \qquad \text{(II)} \end{aligned}$$

- These two equations along with the two boundary conditions (initial k_0 and the NPG/TVC) will completely characterize the evolution of per capita consumption and per capita capital stock for this decentralized economy.

- So far, we have analysed the general equilibrium problem from the perspective of a perfectly competitive market economy where 'atomistic' households and firms take optimal decisions in their respective individual spheres treating market variables as given (the de-centralized version).
- Alternatively, we can analyse the problem from the perspective of a social planner, who controls all the resources (capital) and carries out production in a centralized production unit using the services of its citizens and distributes a part of the total produce directly to the citizens at the end of the period for consumption and invests the rest (the centralized version).
- It is assumed that the social planner is **omniscient**, **omnipotent** and **benevolent** who wants to maximise citizens' welfare. Since households have identical preferences, the objective function of the social planner is identical to that of any household:

$$Max.U_0 = \sum^{\infty} \beta^t u(c_t). \tag{20}$$

General Equilibrium: Centralized Version (Contd.)

- The social planner maximises (20) subject to its period by period budget constraint.
- Notice that in a centrally planned economy there are no markets (hence no market wage rate or market rental rate), and there is no private ownership of assets (capital) and no personalized income.
- The social planner employs the existing capital stock in the economy (either collectively owned or owned by the planner/government) and the existing labour force to produce the final output -using the aggregate production technology.
- After production it distributes a part of the total output among its citizens for consumption puoposes and invests the rest.
- Thus the budget constraint faced by the planner in any period t is nothing but the aggregate resource constraint:

$$C_t + I_t = Y_t = F(K_t, N_t).$$

General Equilibrium: Centralized Version (Contd.)

- Investment augments next period's capital stock: $K_{t+1} = I_t + (1-\delta)K_t$.
- Thus the budget constraint faced by the planner in period t can be written as:

$$C_t + K_{t+1} = F(K_t, N_t) + (1-\delta)K_t.$$

- Writing in per capita terms (since there is no population growth):

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t.$$

- Thus the dynamic optimization problem of the social planner is:

$$\underset{\{c_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}}{Max.} \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} \text{(i) } c_t &\leqq f(k_t) + (1-\delta)k_t \text{ for all } t \geqq 0; \\ \text{(ii) } k_{t+1} &= f(k_t) + (1-\delta)k_t - c_t;\ k_t \geqq 0 \text{ for all } t \geqq 0;\ k_0 \text{ given.} \end{aligned}$$

- In the earlier part of the lecture, we discussed various theorems related to a stationary dynamic programming problem.
- Indeed the social planner's problem in a centralized economy is a stationary dynamic programming problem.
- Recall that the social planner's problem is given by:

$$\underset{\{c_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}}{Max.} \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} \text{(i) } c_t &\leqq f(k_t) + (1-\delta)k_t \text{ for all } t \geqq 0; \\ \text{(ii) } k_{t+1} &= f(k_t) + (1-\delta)k_t - c_t;\ k_t \geqq 0 \text{ for all } t \geqq 0;\ k_0 \text{ given.} \end{aligned}$$

- Here c_t is the control variable; k_t is the state variable, and the corresponding state space is given by $\Re^+$.

- As before, we can use constraint (ii) to eliminate the control variable and write the dynamic programming problem in terms of the state variable alone:

$$\underset{\{k_{t+1}\}_{t=0}^{\infty}}{Max.} \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) + (1-\delta)k_t - k_{t+1}\right)$$

subject to

$$\text{(i) } k_{t+1} \geqq 0 \text{ for all } t \geqq 0; \; k_0 \text{ given.}$$

- This now looks exactly like the canonical stationary dynamic programming problem that we had seen earlier.
- We write the corresponding Bellman equation relating the two value functions $V(k_0)$ and $V(k_1)$ as:

$$V(k_0) = \underset{\{k_1\}}{Max}\left[u\left(\{f(k_0) + (1-\delta)k_0 - k_1\}\right) + \beta V(k_1)\right].$$

- More generally, we can write the Bellman equation for any two time periods t and $t+1$ as:

$$V(k_t) = \underset{\{k_{t+1}\}}{Max} \left[u\left(\{f(k_t) + (1-\delta)k_t - k_{t+1}\}\right) + \beta V(k_{t+1}) \right].$$

- Maximising the RHS above with respect to k_{t+1}, we get the FONC as:

$$u'\left(\{f(k_t) + (1-\delta)k_t - k_{t+1}\}\right) = \beta V'(k_{t+1}) \quad (21)$$

- Noting that $V(k_{t+1})$ and $V(k_{t+2})$ would be related through a similar Bellman equation and applying Envelope Theorem on the latter:

$$\begin{aligned} V'(k_{t+1}) \quad &= \quad u'\left(\{f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}\}\right) \\ &\qquad \left[f'(k_{t+1}) + (1-\delta)\right] \quad (22) \end{aligned}$$

- Combining (26) and (27):

$$\begin{aligned} & u'\left(\{f(k_t) + (1-\delta)k_t - k_{t+1}\}\right) \\ = \quad & \beta u'\left(\{f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}\}\right) \left[f'(k_{t+1}) + (1-\delta)\right] \end{aligned}$$

- Now bringing back the control variable (using the constraint (ii) again), we get the FONC of the social planner's optimization problem as:

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1 - \delta) \right]. \tag{23}$$

- We also have the constraint function:

$$k_{t+1} = f(k_t) + (1 - \delta) k_t - c_t; \;\; k_0 \text{ given.} \tag{24}$$

- Equations (28) and (29) represent a 2×2 system of difference equations which (along with the two boundary conditions) implicitly define the 'optimal' trajectories of c_t and k_t in this centralized economy.

- For expositional simplicity, let us assume specific functional forms for the utility function and the production function (just as we did for the de-centralized market economy).
- Accordingly, let

$$u(c) = \log c$$

  and

$$Y_t = F(K_t, N_t) = (K_t)^{\alpha}(N_t)^{1-\alpha};\ 0 < \alpha < 1.$$

- It is easy to show that then the above 2×2 system of difference equations reduces to:

$$c_{t+1} = \beta \left[1 + \alpha \left(k_{t+1}\right)^{\alpha-1} - \delta \right] c_t; \qquad (\mathrm{I}')$$

$$k_{t+1} = \left(k_t\right)^{\alpha} + (1-\delta)\, k_t - c_t. \qquad (\mathrm{II}')$$

- These two equations along with the two boundary conditions (initial k_0 and the **TVC**) will completely characterize the evolution of per capita consumption and per capita capital stock for this centralized economy.

General Equilibrium: Centralized Version (Contd.)

- Comparing the dynamic equations charcaterizing the optimal paths of c_t and k_t for the decetralized market economy (equations (I) and (II)) with that of the central planner (equations (I′) and (II′)), we find that they are exactly identical.
- Thus **the market economy's equilibrium path would be idential to that attained by the social planner**.
- This is a very strong result: it says that the market economy and the centrally planned economy are equivalent in terms of outcomes!
- But this strong result is based on a number of assumptions (each of which is questionable in the context of the real world):
  - That there is perfect competition in the market economy and no externalities;
  - That households are endowed with perfect foresight in the market economy;
  - That the social planner is benevolent in the planned economy;
  - That the social planner is omnipotent in the planned economy.

General Equilibrium (Centralized Version): Optimal Paths

- Let us now solve these difference equations to characterise the optimal paths of average/per capita consumption (c_t) and average/per capital capital holding (k_t) for this centralized economy economy.
- For analytical convenience, we shall assume that rate of depreciation be 100%, i.e., $\delta = 1$.
- Our dynamic system is then represented by the following two equations:

$$\frac{c_{t+1}}{c_t} = \beta\left[\alpha(k_{t+1})^{\alpha-1}\right]; \qquad (I')$$

$$k_{t+1} = (k_t)^{\alpha} - c_t. \qquad (II')$$

- The associated boundary conditions are:

$$k_0 \text{ given; } \lim_{t\to\infty} \beta^t \alpha(k_t)^{\alpha-1}.\frac{k_t}{c_t} = 0$$

Characterization of the Optimal Paths:

- We are now all set to characterise the dynamic paths of c_t and k_t as charted out by the above simplified system.
- We could use the standard methods of solving difference equations (note however that these are non-linear difference equations) to characterise the solution paths. But that requires more work and we postpone that analysis for the time being. (We shall come back to it later in the module).
- Instead, we use here the direct method of '*guess and verify*' (also known as the method of undetermined coefficients).
- Under the guess and verify method, we begin with a conjecture about a trial solution path.
- If the conjectured solution is indeed a solution, then it has to obey the dynamic equations for all t.
- We then verify under what conditon (if at all) our conjectured solution can indeed be a solution to the dynamic system.

Characterization of the Optimal Paths (Contd.):

- Now let us make a conjecture that the optimal path of per capita stock looks as follows:

$$k_{t+1} = M\left(k_t\right)^{\alpha} \text{ for all } t, \tag{C}$$

  where M is a **yet unknown** constant.

- If (C) is indeed the solution path for k_{t+1} for all t, then (from (II')) the corresponding solution path for c_t would be given by:

$$c_t = \left(k_t\right)^{\alpha} - k_{t+1} = [1 - M]\left(k_t\right)^{\alpha}.$$

- Likewise,

$$c_{t+1} = [1 - M]\left(k_{t+1}\right)^{\alpha}.$$

Characterization of the Optimal Paths (Contd.):

- Now from equation (I') :

$$\begin{aligned}
\frac{c_{t+1}}{c_t} &= \beta\left[\alpha(k_{t+1})^{\alpha-1}\right] \\
\text{i.e., } \frac{(k_{t+1})^{\alpha}}{(k_t)^{\alpha}} &= \alpha\beta(k_{t+1})^{\alpha-1} \\
\text{i.e., } k_{t+1} &= \alpha\beta(k_t)^{\alpha} \\
&\Rightarrow \quad M(k_t)^{\alpha} = \alpha\beta(k_t)^{\alpha} \text{ (given our conjecture).}
\end{aligned}$$

- Thus our conjecture would indeed be true (and satisfy all the relevant equations) iff

$$M \equiv \alpha\beta.$$

- Hence by the guess and verify method we have indeed identifed the optimal solution paths of c_t and k_t for this simplified problem.

Characterization of the Optimal Paths (Contd.):

- These optimal paths are:

$$\begin{aligned} c_t &= (1-\alpha\beta)\,(k_t)^{\alpha} \text{ for all } t \geqq 0; \\ k_{t+1} &= \alpha\beta(k_t)^{\alpha} \text{ for all } t \geqq 0. \end{aligned}$$

- It is easy to see that the the steady state value of k_t is given by $k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$.
- The corresponding steady state value of c_t is given by $c^* = (1-\alpha\beta)\,(k^*)^{\alpha}$.
- Notice that above equation also charts out a growth path for per capita capital stock k_t : $\dfrac{k_{t+1}-k_t}{k_t} = \alpha\beta(k_t)^{\alpha-1} - k_t$.
- Hence there will be a concomitant growth path for the per capita consumption c_t.
- How do these growth paths look?
- Starting from any given k_0, and choosing the corresponding optimal $c_0 = (1-\alpha\beta)\,(k_0)^{\alpha}$ - does the economy approches its steady state (k^*, c^*)?

General Equilibrium with Perfect Markets: What have we learnt so far?

- Moreover note that given that total population/labour force is constant at H, this equations will also govern the evolution of the per capita output (y_t) as well as aggregate output (Y_t) in this economy.
- In other words, through this dynamic general equilibrium (DGE) analysis, we have actually characterized the **growth path** for the economy, which brings us directly to the realm of economic growth.
- Notice however that such growth path would be relevant only for a perfectly competetive market economy populated by rational agents with complete information and **neo-classical technology**.
- Hence the growth model associated with the DGE framework also falls within the ambit of neo-classical growth models.

- How does this growth path of output for the economy evolve over time in the neoclassical growth model?
- Does aggregate output increase perpetually along such a growth path, or does it go to a steady state in the long run?
- Could there be alternative growth paths associated with alternative specification of the macroeconomy (say, with non-neoclassical technology or imperfect markets)?
- To answer these questions, we shall have to get into a detailed discussion of various theories of economic growth, which we take up as our the next topic.

DGE Approach: Reference

- Reference for Dynamic Programming Technique:
  - **Daron Acemoglu (2009): Introduction to Modern Economic Growth**; Princeton University Press, chapter 6.
- Reference for DGE approach to Macroeconomics (Perfect Market Version):
  - **Michael Wickens (2008): Macroeconomic Theory: A Dynamic General Equilibrium Approach**, Princeton University Press, chapters 1& 2.
- Reference for Methods of Solving Difference Equations:
  - **Oded Galor (2007): Discrete Dynamical Systems**, Springer.
- **Statutory Warning: I do not** follow any particular textbook word by word. The references are to be treated only as broad guidebooks, complementary to the lecture notes.