Dynamic Techniques in Macroeconomics

Methods of Solving Difference Equations

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7-28 Feb, 2019

Discrete Dynamical System: Some Definitions

- **State Vector:** At any given point of time t, a dynamic system is typically described by a dated n-vector of real numbers, $\mathbf{x}_t$, which is called the state vector and the elements of this vector are called state variables. In other words,

$$\mathbf{x}_t \equiv \begin{pmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{pmatrix}$$

- As we have already seen, in dynamic optimization problems, there could be other type of variables, which are called control variables. The definition of a state variable vis-a-vis a control variable is not always exact; they change from context to context. For the time being however we shall refer to all the variables whose changes we wish to study as state variables.

- **State Space:** The state space X is a subset of $\Re^n$ that contains all feasible state vectors of the system.
- A **difference equation of order** m in a time dependent variable x_t is an equation of the form:

$$F(t, x_t, x_{t-1}, ..., x_{t-m}; \boldsymbol{\alpha}) = 0$$

  where for each t and $\boldsymbol{\alpha}$, the function F maps points in $\Re^m \times I$ to $\Re$. Here $\boldsymbol{\alpha}$ represents the set of parameters (which are not time-dependent).
- In general the above equation can be written in an explicit form as

$$x_t = f(t, x_{t-1}, ..., x_{t-m}; \boldsymbol{\alpha})$$

  where for each t and $\boldsymbol{\alpha}$, the function f maps points in $\Re^{m-1} \times I$ to $\Re$.
- In other words, the f function relates the state variable x at time t to its m number of previous values.

Discrete Dynamical System: Some Definitions (Contd.)

- The **order** of a difference equation is the difference between the largest and the smallest time subscript appearing in the equation.
- A difference equation is said to be **linear** if f is a linear function of the state variables.
- A difference equation is said to be **autonomous** if the time variable t does not enter as a separate argument in the f function.
- A difference equation is said to be **homogeneous** if f is a homogeneous function of the state variables.

Discrete Dynamical System: Some Definitions (Contd.)

- A **system of difference equations of first order** in an n-dimensional vector of time dependent variables $\mathbf{x}_t$ is defined as

$$\mathbf{x}_t = f(t, \mathbf{x}_{t-1}; \boldsymbol{\alpha})$$

where for each t and $\boldsymbol{\alpha}$, the function f now maps points in $\Re^n$ to $\Re^n$.

- An alternative representation of the above system of difference equations in $\mathbf{x}_t$ is given by

$$\begin{pmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{pmatrix} = \begin{pmatrix} f^1(t, x_{1t-1}, x_{2t-1}, \ldots x_{nt-1}; \boldsymbol{\alpha}) \\ f^2(t, x_{1t-1}, x_{2t-1}, \ldots x_{nt-1}; \boldsymbol{\alpha}) \\ \vdots \\ f^n(t, x_{1t-1}, x_{2t-1}, \ldots x_{nt-1}; \boldsymbol{\alpha}) \end{pmatrix}$$

where for each t and $\boldsymbol{\alpha}$, the functions f^i, $i = 1, 2, \ldots, n$ map points in $\Re^n$ to $\Re$.

Discrete Dynamical System: A Lemma

- **Lemma:** Any difference equation of higher order can be reduced to a system of difference equations of first order by introducing additional equations and variables.
- For example, consider the following difference equation of order 2:

$$x_t = f(t, x_{t-1}, x_{t-2}; \boldsymbol{\alpha})$$

  - Let us define a new variable $y_{t-1} \equiv x_{t-2}$ for all t. By this definition: $y_t \equiv x_{t-1}$.
  - Hence the above second order difference equation can be expressed as system of first order difference equations in variables x_t and y_t in the following way:

$$\begin{aligned} x_t &= f(t, x_{t-1}, y_{t-1}; \boldsymbol{\alpha}) \\ y_t &= x_{t-1} \equiv \hat{f}(x_{t-1}, y_{t-1}) \end{aligned}$$

- Given this Lemma, in the discussion that follows, we shall only focus on difference equations which are of first order.

Discrete Dynamical System: A Theorem

- **Superposition Principle:** The general solution to any linear system of difference equation of the form

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + \mathbf{b} \tag{1}$$

can be written as

$$\mathbf{x}_t^g = \mathbf{x}_t^c + \mathbf{x}_t^p$$

where $\mathbf{x}_t^c$ is the *general solution* to the corresponding homogeneous equation $\mathbf{x}_t = A\mathbf{x}_{t-1}$; and $\mathbf{x}_t^p$ is any *particular solution* to (1).

- $\mathbf{x}_t^c$ is called the **complementary function** and $\mathbf{x}_t^p$ is called a **particular solution**.

Solving a First Order Difference Equation: Linear & Autonomous

- We shall start with an autonomous, linear, first order difference equation of the form
$$x_t = ax_{t-1} + b \tag{2}$$
where the initial value of the variable at time 0 (i.e., x_0) is given.
- First let us look at the corresponding homogeneous equation which we solve to get the complementary function:
$$x_t = ax_{t-1} \tag{2'}$$
- We shall use the method of iteration to find a solution to $(2')$.
- Note that using $(2')$, we can write $x_t = ax_{t-1}$; $x_{t-1} = ax_{t-2}$; $x_{t-2} = ax_{t-3}$ and so on.
- Hence iterating backward (until we reach time $t = 0$), we find:
$$x_t = ax_{t-1} = a^2 x_{t-2} = a^3 x_{t-3} = ... = a^n x_{t-n} = ... = a^t x_0$$

First Order Difference Equation: Linear & Autonomous (Contd.)

- If we knew the exact value of x_0 for the homogenous equation $(2')$, we could have used that initial condition to get a precise solution to the above homogeneous equation as:

$$x_t = a^t x_0. \tag{S}$$

- Note however that we do not necessarily need an *initial* condition to solve the above homogeneous equation given by $(2')$.
- If instead the value of x at some other point of time (say $t = s$) was known to us, then we could have derived the exact solution in terms of that particular value (x_s).
- For example, suppose we knew the value of x at time $t = 5$ is given by $x_5 = 55$ (say). Then we could have interated the above equation backward until we reached $t = 5$ such that

$$x_t = ax_{t-1} = a^2 x_{t-2} = a^3 x_{t-3} = ... = a^{t-5} x_5$$

$$\Rightarrow x_t = 55a^{t-5}. \tag{S'}$$

First Order Difference Equation: Linear & Autonomous (Contd.)

- Indeed both (S) and (S′) constitute a solution to the homogenous equation (2′).
- In fact we can generate as many solutions as we want using different boundary values x_s for any other time period $t = s$.
- Also note that these solution are not very different from one another. In fact we can re-write solution (S′) as

$$x_t = \frac{55}{a^5} a^t.$$

- Since the value of a is known (it is a parameter), the only difference between (S) and (S′) is in terms of the constant term that is affixed to a^t.
- This tells us that we can write the *general solution* to the homogeneous equation (2′) in a generic form as

$$x_t = Ca^t \text{ where } C \text{ is an arbitrary constant.}$$

First Order Difference Equation: Linear & Autonomous (Contd.)

- Thus we have now found the complementary function to the non-homogenous equation (2):

$$x_t^c = Ca^t; \; C \text{ is an arbitrary constant.} \tag{3}$$

- The next step is to look for a particular solution to (2).
- Here we are going to use the 'guess and verify' method (also known as the method of undetermined coefficients).
- Let us make a conjecture that the particular solution would look as follows:

$$x_t = K \tag{C}$$

  where K is a **yet unknown** constant.
- If (C) is indeed a solution to (2) that it has to satisfy the equation for all t.

First Order Difference Equation: Linear & Autonomous (Contd.)

- Hence substituting $x_t = K$ and $x_{t-1} = K$ in equation (2), we get

$$K = aK + b \Rightarrow K = \frac{b}{1-a}.$$

- Thus our conjectured solution $x_t = K$ would indeed be a solution if and only if $K = \frac{b}{1-a}$.
- Thus we have now found a particular solution to the non-homogenous equation (2):

$$x_t^p = \frac{b}{1-a}. \tag{4}$$

- Hence by superposition principle, the general solution to the linear and autonomous first order difference equation in (2) is given by

$$x_t = Ca^t + \frac{b}{1-a}$$

where C is an arbitrary constant whose value is to be determined by the given initial or any other boundary condition.

First Order Difference Equation: Boundary Conditions

- Incidentally such boundary conditions could entail terminal conditions as well.
- For example, for some economic problems the boundary condition could be given by a limiting condition such that

$$\lim_{t \to \infty} x_t = \bar{x} \text{ (given)}.$$

- Then we shall have to find the appropriate value of the arbitrary constant C such that this terminal condition is satisfied.
- In Economics, if the dynamic equation of a varible is tethered to its initial value (a given x_0) then the variable is called a "stock" variable (or 'pre-determined' variable or 'backward looking').
- On the other hand, if the dynamic equation of a varible is tethered to a terminal condition (as specified above) then the variable is called a "jump" variable (or 'forward looking'). Its current value is not tethered to the past and therefore can adjust immediately.

First Order Difference Equation: Linear & Autonomous (Contd.)

- Observe that while the complementary function given in (3) is defined for any value of a, the particular solution given in (4) is not defined when $a = 1$.
- Thus if $a = 1$, this particular solution will not work; we have to find some other particular solution.
- Notice however that when $a = 1$, equation (2) reduces to

$$x_t = x_{t-1} + b \tag{5}$$

- The complementary function is still given by (3), although when $a = 1$, it reduces to:

$$x_t^c = C; \; C \text{ is an arbitrary constant.} \tag{6}$$

- We now have to find a particular solution for this case.

First Order Difference Equation: Linear & Autonomous (Contd.)

- Let us make a conjecture that the particular solution in this case would look as follows:
$$x_t = Kt \tag{C'}$$
where K is a **yet unknown** constant.
- If (C′) is indeed a solution to (5) that it has to satisfy the equation for all t.
- Hence substituting $x_t = Kt$ and $x_{t-1} = K(t-1)$ in equation (5), we get
$$Kt = K(t-1) + b \Rightarrow K = b.$$
- Thus our conjectured solution for this case $x_t = Kt$ would indeed be a solution if and only if $K = b$.
- Thus we have now found a particular solution for this case, which is given by :
$$x_t^p = bt. \tag{7}$$

First Order Difference Equation: Linear & Autonomous (Contd.)

- Once again by superposition principle, he general solution to the linear and autonomous first order difference equation in (5) is given by

$$x_t = C + bt; \;\; C \text{ is an arbitrary constant.}$$

- Let me now summarise the results derived so far in terms of the following proposition.
- **Proposition 1:** Consider a linear and autonomous first order difference equation of the form

$$x_t = ax_{t-1} + b.$$

The general solution to this equation is given by:

$$x_t = \begin{cases} Ca^t + \dfrac{b}{1-a} & \text{for } a \neq 1 \\ C + bt & \text{for } a = 1 \end{cases} \tag{P1}$$

where C is an arbitrary constant.

Solving a First Order Difference Equation: Linear & Non-Autonomous

- Let us now consider a non-autonomous, linear first-order difference equation of the form

$$x_t = ax_{t-1} + b_t \tag{8}$$

- Once again the superposition principal holds.
- Note that the homogeneous component of (8) is the same as that of (2); hence it will the same complementary function given by

$$x_t^c = Ca^t; \; C \text{ is an arbitrary constant.} \tag{9}$$

- Thus we just have to find a particular solution to (2).
- Also note that our earlier guess of trying out a constant value of x_t as a solution will not work here because the term b_t is changing over time; so no x_t and x_{t-1} could be the same (unless b_t is a constant, but that would make the difference equation autonomous).
- If we know the exact time path of b_t, then we could proceed with some conjecture, based on the functional form of b_t.

First Order Difference Equation: Linear & Non-Autonomous (Contd.)

- For example, suppose the non-autonomous equation given to us looks as follows:
$$x_t = ax_{t-1} + B\left(b\right)^t \tag{10}$$
- Here the non-autonomous terms has a specific form given by $B\left(b\right)^t$, where B and b are given parameters.
- In order to find the particular solution for this case, let us make the following conjecture:
$$x_t = K\left(b\right)^t \tag{C$''$}$$
where K is a **yet unknown** constant.
- If (C$''$) is indeed a solution to (10), then it has to satisfy the equation for all t.

First Order Difference Equation: Linear & Non-Autonomous (Contd.)

- Hence substituting $x_t = K\left(b\right)^t$ and $x_{t-1} = K\left(b\right)^{t-1}$ in equation (10), we get

$$\begin{aligned} K\left(b\right)^t &= aK\left(b\right)^{t-1} + B\left(b\right)^t \\ &\Rightarrow K\left(b\right)^t = \frac{a}{b}K\left(b\right)^t + B\left(b\right)^t \\ &\Rightarrow K = \frac{bB}{b-a}. \end{aligned}$$

- Thus our conjectured solution for this case $x_t = K\left(b\right)^t$ would indeed be a solution if and only if $K = \frac{bB}{b-a}$.
- Thus we have now found a particular solution for this case, which is given by :

$$x_t^p = \frac{bB}{b-a}\left(b\right)^t \tag{11}$$

First Order Difference Equation: Linear & Non-Autonomous (Contd.)

- Therefore, the general solution to (10) is given by

$$x_t = Ca^t + \frac{bB}{b-a}\left(b\right)^t$$

  where C is an arbitrary constant.
- Note once again that this slution is defined only when $b \neq a$.
- If $b = a$, then we shall have to find some other particular solution.
- Notice that when $b = a$, equation (10) reduces to

$$x_t = ax_{t-1} + B\left(a\right)^t \tag{12}$$

- In order to find the particular solution for this case, let us make the following conjecture:

$$x_t = Kt\left(a\right)^t \tag{C'''}$$

  where K is a **yet unknown** constant.

First Order Difference Equation: Linear & Non-Autonomous (Contd.)

- If (C$'''$) is indeed a solution to (12), then it has to satisfy the equation for all t.
- Hence substituting $x_t = Kt\left(a\right)^t$ and $x_{t-1} = K\left(t-1\right)\left(a\right)^{t-1}$ in equation (12), we get

$$\begin{aligned} Kt\left(a\right)^t &= aK\left(t-1\right)\left(a\right)^{t-1} + B\left(a\right)^t \\ &\Rightarrow Kt\left(a\right)^t = K\left(t-1\right)\left(a\right)^t + B\left(a\right)^t \\ &\Rightarrow K = B \end{aligned}$$

- Thus our conjectured solution for this case $x_t = Kt\left(a\right)^t$ would indeed be a solution if and only if $K = B$.
- Thus we have now found a particular solution for this case, which is given by :

$$x_t^p = Bt\left(a\right)^t \tag{13}$$

First Order Difference Equation: Linear & Non-Autonomous (Contd.)

- Therefore, the general solution to (12) is given by

$$x_t = Ca^t + Bt\left(a\right)^t = (C + Bt)a^t$$

where C is an arbitrary constant.

- Once again let me summarise these results in terms of the following proposition.
- **Proposition 2:** Consider a linear and non-autonomous first order difference equation of the form

$$x_t = ax_{t-1} + B\left(b\right)^t .$$

The general solution to this equation is given by:

$$x_t = \begin{cases} Ca^t + \frac{bB}{b-a}\left(b\right)^t & \text{for } b \neq a \\ (C + Bt)a^t & \text{for } b = a \end{cases} \tag{P2}$$

where C is an arbitrary constant.

First Order Difference Equation: Linear & Non-Autonomous (Contd.)

- Often the exact functional form of the non-autonomous term $b(t)$ is not known; hence we cannot derive a particular solution using this kind of guess and verify method. In these cases, a particular solution is arrived at by iterating the equation backward and then using an initial condition to arrive at the exact solution.
- The particular solutions thus obtained are called the **backward-looking solutions**. (In Economics backward-looking solutions are typically used in static or adaptive expectation models.)
- There however another method which entails iterating the equation forward, and applying of some terminal condition.
- The particular solutions thus obtained are called the **forward-looking solutions**. (In Economics backward-looking solutions are typically used in rational expectation models.)
- For brevity, I shall not discuss the details of these iterated solutions here.

First Order Difference Equation: Linear & Non-Autonomous (Contd.)

- Alternatively, sometime a non-automous difference equation is converted into a *system* of autonomous difference equations by suitably defining new variables.
- For example, consider a generic non-autonomous linear first order difference equation of the form:

$$x_t = a(t)x_{t-1} + b(t) \tag{14}$$

- Define

$$y_{t-1} \equiv t$$

- By this definition

$$y_t = t + 1 = y_{t-1} + 1$$

Solving a First Order Difference Equation: Linear & Non-Autonomous (Contd.)

- Thus we can convert the single non-autonomous difference equation given by (14) to the following 2×2 system of difference equations :

$$\begin{aligned} x_t &= a(y_{t-1})x_{t-1} + b(y_t); \\ y_t &= y_{t-1} + 1. \end{aligned}$$

- Of course this system in no longer linear, since the RHS of the first equation is non-linear in x_{t-1} and y_{t-1}.
- Hence we **cannot** apply the standard methods of solving linear difference equations here.
  (We shall discuss the methods of solving non-linear difference equations later in this lecture.)

Autonomous First Order Difference Equation: Steady States and Stability

- Consider an autonomous first order difference equation of the form
$$x_t = f(t, x_{t-1}; \boldsymbol{\alpha}) \tag{15}$$
- **Steady state:** A point $\bar{x}$ is a steady state of the difference equation given in (15) if it is a fixed point of the map f, that is, if
$$\bar{x} = f(t, \bar{x}; \boldsymbol{\alpha}).$$
- A more conventional (and equivalent) way of characterising the steady state is:
$$x_t = x_{t-1} = \bar{x} \quad \text{for all } t$$
- As is clear from either definition, stationary or steady state values of autonomous dynamical systems are those values, which will be preserved in perpetuity if they are attained once. These are also called rest points or long run equilibrium points.
- **Comment:** Note that we are defining the steady state only in the context of *autonomous* equations. (Why?)

Autonomous First Order Difference Equation: Steady States and Stability

- **Stability of a dynamical system:** A dynamical system with a steady state $\bar{x} \in X$ is said to be *asymptotically stable* if

$$\lim_{t \to \infty} x_t = \bar{x}.$$

- In other words, a dynamical system is asymptotically stable if the all its trajectories approach the steady state value over time, irrespective of the initial position.
- **Example:** Let us consider a first order linear autonomous difference equation of the form:

$$x_t = a x_{t-1} + b; \;\; a \neq 1 \qquad (16)$$

- We had earlier derived a general solution to this equation as

$$x_t = Ca^t + \frac{b}{1-a}$$

Autonomous First Order Difference Equation: Steady States and Stability (Contd.)

- Let us now derive its steady state value and also examine its stability property.
- Note that here $f(t, x_{t-1}; \boldsymbol{\alpha}) \equiv ax_{t-1} + b$. Therefore, by definition, the steady state solves the equation is identified by the following equation:

$$\bar{x} = a\bar{x} + b$$

- Solving, we get the steady state value as

$$\bar{x} = \frac{b}{1-a}$$

  which is nothing but the particular solution that we had derived earlier.
- For stability we require x_t to tend $\bar{x}$ to as t approaches infinity.

Autonomous First Order Difference Equation: Steady States and Stability (Contd.)

- Writing the general solution as $x_t = Ca^t + \bar{x}$, we see that $\lim_{t\to\infty} x_t = \bar{x}$ if and only if
$$|a| < 1.$$
- In this case the Ca^t term vanishes as t gets larger and the system converges asymptotically to the steady state $\bar{x}$ for any value of C (i.e., for any initial condition).
- If $0 < a < 1$, then x_t approaches $\bar{x}$ monotonically.
- If $-1 < a < 0$, then the Ca^t term becomes positive and negative for alternate (even and odd) values of t, and x_t approaches $\bar{x}$ in an oscillating manner.
- Finally if $|a| > 1$, then the term Ca^t explodes as t goes to infinity, unless $C = 0$.
- In other words, the system will be unstable, *unless we start with a very precise boundary condition which ensures that* $C = 0$. We shall discuss one such boundary condition later.

Linear First Order Difference Equation: An Economic Example

- Where in Economics are you likely to see this kind of linear (autonomous or non-autonoumous) difference equation involving a single variable?
- A typical example is given by the price dynamics underlying the Lucas Imperfect Information Model.
- Recall (from topic1 (Keynes & the Classics)) that in the Lucas Imperfect Information model the equilibrium price level at time t is determined by the following equation:

$$P_t = \frac{1}{\alpha + \mu}\left[\gamma \bar{G} + \mu M_t - \bar{Y}^* + \alpha P_t^e\right] \tag{III}$$

- Assuming that $\frac{\gamma \bar{G}}{\alpha+\mu} = \bar{Y}^*$ such that the constant term drops out, we can write the above equation as:

$$P_t = aP_t^e + bM_t$$

where a and b are both positive fractions.

Linear First Order Difference Equation: An Economic Example (Contd.)

- Recall that under static expecation $P_t^e = P_{t-1}$, which makes the current price level in the economy (P_t) tethered to its past value (P_{t-1}). If money supply remains constant over time such that $M_t = \bar{M}$, then the following equation will capture the price dynamics of this economy:

$$P_t = aP_{t-1} + b\bar{M} \tag{A}$$

  where the parameter a captures the degree of price persistence and the parameter b captures the responsiveness of current price level to the current money supply.
- This is an example of a linear autonomous difference equation of first order.
- The general solution is given by,

$$P_t = Ca^t + \frac{b\bar{M}}{1-a}.$$

Linear First Order Difference Equation: An Economic Example (Contd.)

- Recall that under static expecation $P^e_t = P_{t-1}$.
- If we know the initial price level P_0, then we can calculate the exact value of C as:
$$P_0 = C + \frac{b\bar{M}}{1-a}$$
- The time path of the equilibrium price level in this economy is then given by:
$$P_t = (P_0 - P^*)a^t + P^*.$$
where $P^* \equiv \dfrac{b\bar{M}}{1-a}$ is the steady state value of P_t.
- From the solution path, you can immediately see that since a is a positive fraction, starting from *any* initial price level P_0, the economy will gradually approach the steady state price level P^* in the long run.
- Moreover, the price level will exactly follow the path of money supply (i.e., P_t will be proportional to $M_t = \bar{M}$) *only in the long run*.

Linear First Order Difference Equation: An Economic Example (Contd.)

- Now suppose there is a sudden increase in money supply from $\bar{M}$ to $\bar{M}'$ at some future date T.
- Following the earlier dynamic path (with $\bar{M}$), the economy had by now reached a specific price level $P_T = (P_0 - P^*)a^T + P^* = \tilde{P}$ (say).
- Now at time T, as $\bar{M}$ increases to $\bar{M}'$, the new dynamic equation is specified by

$$P_t = aP_{t-1} + b\bar{M}';\ t \geqq T,\ P_T = \tilde{P} \text{ (given)}.$$

- The general solution to this new system is given by

$$P_t = Ca^t + \frac{b\bar{M}'}{1-a},\ t \geqq T$$

where $\frac{b\bar{M}'}{1-a} \equiv P^{*'}$ is the new steady state value of P_t.

Linear First Order Difference Equation: An Economic Example (Contd.)

- Note that if we treat time T as the initial point for another time subsrcipt $\tau = t - T$ such that $P_{\tau=0} = P_{t=T} = \tilde{P}$, then we can write the above solution in terms of τ as
$$P_\tau = Ca^\tau + \frac{b\bar{M}'}{1-a}, \tau \geqq 0;\ \ P_{\tau=0} = \tilde{P} \text{ (given)}.$$
- Using the above "initial" condition for the τ system, we can write the exact time path of the price level for the for the τ system as
$$P_\tau = (\tilde{P} - P^{*'})a^\tau + P^{*'}.$$
- Now writing the complete time path for the equilibrium price level in the economy in terms of the original time subscript t, we get
$$P_t = \begin{cases} (P_0 - P^*)a^t + P^*;\ \ t \leqq T;\ (P_T = \tilde{P}) \\ (\tilde{P} - P^{*'})a^{t-T} + P^{*'},\ \ t \geqq T. \end{cases}$$
- Notice that the time path for P_t is smooth even at time T when the new policy is implemented; there is no discontinous jump at time T.

Linear First Order Difference Equation: An Economic Example (Contd.)

- So far we have considered the case of constant money supply.
- Of course in real world money supply would change over time. That brings us to the realm of non-autonomous difference equations.
- Suppose monetary authorities change the money supply at constant rate γ such that

$$M_t = \bar{M}\left(b\right)^t$$

  where $b = 1 + \gamma$.
- Then the following equation will capture the price dynamics of this economy:

$$P_t = aP_{t-1} + \bar{M}\left(b\right)^t \tag{B}$$

- This is exactly analogous to the specific form of non-autonomous equation that we had considered earlier. So the general solution is given by:

$$P_t = Ca^t + \frac{b\bar{M}}{b-a}\left(b\right)^t$$

Linear First Order Difference Equation: An Economic Example (Contd.)

- Once again If we know the initial price level P_0, then we can calculate the exact value of C as:

$$P_0 = C + \frac{b\bar{M}}{b-a}$$

- The time path of the equilibrium price level in this economy is then given by:

$$P_t = \left[P_0 - \frac{b\bar{M}}{b-a}\right] a^t + \frac{b\bar{M}}{b-a} (b)^t .$$

- Once again the price level in the economy will exactly follow the growth path of money supply (i.e., P_t will be proportional to M_t) *only in the long run*.
- Once again if there is any sudden change in the money supply at some future date T such that $\bar{M}$ rises to $\bar{M}'$, the equilibrum price level would smoothly adjust to that change when the new policy is implemented without any discountinuous jump.

Linear First Order Difference Equation: An Economic Example (Contd.)

- How would these equations look if people had perfect foresight/rational expectations?
- Recall that with a constant money supply at $\bar{M}$, we already calculated the equilibrium price level under perfect foresight/rational expectations from the AS-AD relationship for the Lucas Imperfect Information Model. And those solutions did not entail any difference equation.
- However, there was a basic conceptual problem in calculating that rational expectation solution in the sense that the AS-AD relationship was not derived from households' optimization exercise.
- The "Rational Expectation" school is based on two premises: (1) Households base their decisions on explicit optimization exercises; (2) In those optimizations decisions, they form their expectations rationally based on all available information.
- The AS-AD formulation discussed earlier ignored the first premise.

Linear First Order Difference Equation: An Economic Example (Contd.)

- Recall that in topic 2 (Microfoundations) when we solved the 2-period optimization problem of the household to solve for their current consumption (and therefore, current savings) path, it depended on the expected value of future price level (P^e_{t+1}), the expected value of future nominal income (y^e_{t+1}) and the expected value of future interest rate (r^e_{t+1}).
- So if consumption is influenced by these expected values then they should show up in the AD equation and therefore in the equilibrium price equation.
- Accordingly, a micro-founded AS-AD relationship will write the equilibrium price equation as:

$$P_t = aP^e_t + bM_t + cP^e_{t+1} + dy^e_{t+1} + er^e_{t+1}$$

where a, b, c, d, e are all constant terms.

Linear First Order Difference Equation: An Economic Example (Contd.)

- Let us assume that $y_{t+1} = \bar{y}$ and $r_{t+1} = \bar{r}$ for all t and normalize $d\bar{y} + e\bar{r}$ to zero so that we can focus only on the relationship between current price and its expected values. (This is just to simplify the analysis).
- Thus the reduce form micro-founded AS-AD relationship would look as follows:
$$P_t = aP_t^e + bM_t + cP_{t+1}^e$$
- Now assume that agents have rational expectations. Then $P_t^e = P_t$ and $P_{t+1}^e = P_{t+1}$.
- Simplifying we get the rational expectation price determination equation as
$$P_t = \hat{b}M_t + \hat{c}P_{t+1} \tag{C}$$
- Notice however that now the initial P_0^e is **not** given.

Linear First Order Difference Equation: An Economic Example (Contd.)

- Indeed under rational expectations, the current price level is determined by the future price level (due to forward looking expecations). So the price equation is to be read from right to left.
- If there is no initial value which is predetermined (i.e., the current price under rational expecation is a jump variable) how do we apply any boundary condition?
- Typically rational expecation school specify a **terminal condition** by postulating that *in the long run the economy converges to its steady state:*

$$\lim_{t \to \infty} P_t = \bar{P}.$$

- How do we apply this terminal condition here?
- And what does it imply for the initial value of P_t? More importantly, how does this initial value responds to changes in the money supply at some future date T?

Linear First Order Difference Equation: An Economic Example (Contd.)

- Suppose money supply is constant at $M_t = \bar{M}$.
- We can still apply the earlier method of solving a linear autonomus difference equation. Rewite the price equation under rational expectaions (equation (C)) as

$$\begin{aligned} P_{t+1} &= \frac{1}{\hat{c}} P_t + \frac{\hat{b}}{\hat{c}} \bar{M} \\ &\equiv \tilde{a} P_t + \tilde{b} \bar{M} \end{aligned} \qquad \text{(D)}$$

- As before, the general solution is given by,

$$P_t = C \left(\tilde{a}\right)^t + \frac{\tilde{b}\bar{M}}{1 - \tilde{a}}. \qquad \text{(i)}$$

- Of course now there is no *initial condition* to peg down the value of the arbitrary constant C. Instead, the rational expectation school postulates that the price level converges to its steady state value $\bar{P}$.

Linear First Order Difference Equation: An Economic Example (Contd.)

- What kind of time paths will satisfy the rational expectation solution gievn in (i) along with the associated terminal condition that $\lim_{t \to \infty} P_t = \bar{P} \equiv \frac{\tilde{b}\bar{M}}{1-\tilde{a}}$? That depends on the value of $\tilde{a}$.
- If $0 < \tilde{a} < 1$, then **any** arbitary initial P_0 will satisfy the above solution path.
- For example, take $P_0 = 10$. Then $C = 10 - \frac{\tilde{b}\bar{M}}{1-\tilde{a}}$. Correspondingly one rational expectation solution path will be given by $P_t = \left[10 - \frac{\tilde{b}\bar{M}}{1-\tilde{a}}\right](\tilde{a})^t + \frac{\tilde{b}\bar{M}}{1-\tilde{a}}$.
- Again, take $P_0 = 50$. Then $C = 50 - \frac{\tilde{b}\bar{M}}{1-\tilde{a}}$. Correspondingly another rational expectation solution path will be given by $P_t = \left[50 - \frac{\tilde{b}\bar{M}}{1-\tilde{a}}\right](\tilde{a})^t + \frac{\tilde{b}\bar{M}}{1-\tilde{a}}$.

Linear First Order Difference Equation: An Economic Example (Contd.)

- Thus there are now multiple rational expectation paths - all approaching the steady state value as $t \longrightarrow \infty$.
- In other words, the rational expecation path is no longer unique. This particular problem of rational expecation solution is called '**multiple equilibria'** or **'indeterminacy'** problem (since the exact equilibrium path is not fully specified/determinate; any path can be an equilibrium path).
- On the other hand if $\tilde{a} > 1$, then the rational expectation solution given by (i) along with terminal condition will be satisfied *if and only if* $C = 0$.
- In this case, the equilibrium price level will be its steady state value from time 0 onwards.

Linear First Order Difference Equation: An Economic Example (Contd.)

- Now suppose there is a sudden increase in money supply from $\bar{M}$ to $\bar{M}'$ at some future date T.
- Following the earlier dynamic path (with $\bar{M}$), the economy was at $P_t = \bar{P} \equiv \frac{\tilde{b}\bar{M}}{1 - \tilde{a}}$ until time T.
- Now at time T, as $\bar{M}$ increases to $\bar{M}'$, the new dynamic equation is specified by
$$P_{t+1} = \tilde{a}P_{t-1} + \tilde{b}\bar{M}'; \; t \geqq T.$$
- The new steady state value for $t > T$ is given by $\frac{b\bar{M}'}{1 - a} \equiv \bar{P}'$.
- Once again if $\tilde{a} > 1$, then the rational expecation solution path jumps from its previous value to its new steady state value $\bar{P}'$ at $t = T$.
- Thus there is now a discontinuous jump in the price level at time $t = T$ when the new policy is introduced.

Solving a System First Order Difference Equations: Linear & Autonomous

- So far we have looked at methods for solving a single linear difference equation.
- Next consider an $n \times n$ *system* of linear and autonomous equations of the form:
$$\mathbf{x}_t = A\mathbf{x}_{t-1} + \mathbf{b} \tag{17}$$
where A is an $n \times n$ matrix of constant coefficients; $\mathbf{x}$ is a n dimensional column vector of dated state variables, and $\mathbf{b}$ is a n dimensional column vector of constant terms.
- Since the system is linear, the superposition principle holds.
- Thus we can write the general solution of the system as can be written as
$$\mathbf{x}_t^g = \mathbf{x}_t^c + \mathbf{x}_t^p.$$
- Let us now try to identify the complementary function and a particular solution for the above $n \times n$ non-homogeneous system.

System of First Order Difference Equations: Linear & Autonomous (Contd.)

- First let us try to find a particular solution $\mathbf{x}_t^p$.
- As before, we use the guess and verify method to identify a particular solution.
- Recall that our conjectured solution of x_t taking a constant value worked earlier for the single equation case.
- Moreover, we now know that such a constant solution, it exists, would also define the steady state of the equation.
- Knowing this, we can now directly derive the particular solution by using the steady state condition. (Indeed the steady state is *a* particular solution to any difference equation).
- The steady state for the system of equation is defined by a vector $\bar{\mathbf{x}}$ such that

$$\bar{\mathbf{x}} = A\bar{\mathbf{x}} + \mathbf{b}.$$

- From this, we can directly derive the particular solution as

$$\mathbf{x}_t^p = \bar{\mathbf{x}} = (I - A)^{-1}\mathbf{b}.$$

System of First Order Difference Equations: Linear & Autonomous (Contd.)

- Notice that the above particular solution will exists iff the matrix $(I - A)$ is invertible.
- Next let us try to identify the complementary function by deriving the general solution to the corresponding $n \times n$ homogeneous system given by:

$$\mathbf{x}_t = A\mathbf{x}_{t-1} \tag{18}$$

- The matrix $A_{n\times n}$ is called the **coefficient matrix** which in general will have the following form:

$$A_{n\times n} \equiv \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ \vdots & \vdots & .. & \vdots \\ a_{n1} & a_{n2} & .. & a_{22} \end{bmatrix}$$

System of First Order Difference Equations: Linear & Autonomous (Contd.)

- Suppose for some reason (a special case), the coefficient matrix was diagonal:

$$A_{n\times n} \equiv \begin{bmatrix} a_{11} & 0 & .. & 0 \\ 0 & a_{22} & .. & 0 \\ \vdots & \vdots & .. & \vdots \\ 0 & 0 & .. & a_{nn} \end{bmatrix}$$

- Then the homogeneous system in (18) would have the following special character:

$$\begin{pmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{pmatrix}_{n\times 1} = \begin{bmatrix} a_{11} & 0 & .. & 0 \\ 0 & a_{22} & .. & 0 \\ \vdots & \vdots & .. & \vdots \\ 0 & 0 & .. & a_{nn} \end{bmatrix}_{n\times n} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \\ \vdots \\ x_{nt-1} \end{pmatrix}_{n\times 1}$$

System of First Order Difference Equations: Linear & Autonomous (Contd.)

- In other words, we would then be able to completely un-couple the system to derive n single equations for each of n variables, such that

$$\begin{aligned} x_{1t} &= a_{11}x_{1t-1} \\ x_{2t} &= a_{22}x_{2t-1} \\ &\ldots \\ x_{nt} &= a_{nn}x_{nt-1} \end{aligned}$$

- We now know that their general solutions will be given as follows:

$$\begin{aligned} x_{1t} &= C_1\left(a_{11}\right)^t \\ x_{2t} &= C_2\left(a_{22}\right)^t \\ &\ldots \\ x_{nt} &= C_n\left(a_{nn}\right)^t \end{aligned}$$

where $C_1, C_2, \ldots, C_n$ are all arbitrary constants (one for each equation).

System of First Order Difference Equations: Linear & Autonomous (Contd.)

- Hence had A been a diagonal matrix, then we could have immediately identified the complementary function as

$$\mathbf{x}_t^c = \begin{pmatrix} C_1 \left(a_{11}\right)^t \\ C_2 \left(a_{22}\right)^t \\ \vdots \\ C_n \left(a_{nn}\right)^t \end{pmatrix}$$

- This, along with the particular solution $\mathbf{x}_t^p = \bar{\mathbf{x}}$ would have immediately given us the general solution for this special case as:

$$\mathbf{x}_t^g = \begin{pmatrix} C_1 \left(a_{11}\right)^t \\ C_2 \left(a_{22}\right)^t \\ \vdots \\ C_n \left(a_{nn}\right)^t \end{pmatrix} + \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix} = \begin{pmatrix} C_1 \left(a_{11}\right)^t + \bar{x}_1 \\ C_2 \left(a_{22}\right)^t + \bar{x}_2 \\ \vdots \\ C_n \left(a_{nn}\right)^t + \bar{x}_n \end{pmatrix}$$

System of First Order Difference Equations: Linear & Autonomous (Contd.)

- Could we have said something about its stability?
- Turns out that we could!
- As before, each variable would approeach its steady state value the corresponding coefficient a_{ii} is such that $|a_{ii}| < 1$.
- Likewise, each variable move away from its steady state value the corresponding coefficient a_{ii} is such that $|a_{ii}| > 1$.
- What happnes if some of the a_{ii} s are less than unity, other are greater than unity?
- This case is know as saddle-point stable; the system will approach its steady state **if and only if the initial conditions are such that the arbitrary constants associated with all the (unstable)** a_{ii} terms such that $a_{ii} > 1$ **are equal to zero.**
- For any other initial condition, the system would move away from its steady state.

System of First Order Difference Equations: Linear & Autonomous (Contd.)

- Having a diagonal coefficient matrix is of course a special case.
- Most of the time we are not that lucky!
- However, even when the coefficient matrix is **not** diagonal, there is a way to convert the original dynamic system into another system with a diagonal or near-diagonal coefficient matrix by using some results of linear algebra.
- Let us discuss some of these results now.

A Digression: Diagonalization of a Square Matrix:

- We start with some definitions.
- **Square matrix** is a matrix that has same number of rows and columns.
- **Singular matrix** is a matrix which has at least one linearly dependent row/column. The determinant of a singular matrix is zero.
- Consider a square matrix $A_{n \times n}$.
- An **eigenvalue** of A is a number (scaler) λ, which when subtracted from each of the diagonal entries of A converts A into a singular matrix.
- Subtracting a scaler from each of the diagonal elements of A is equivalent to subtracting λ times the identity matrix I from A.Therefore λ is an eigenvalue of A if and only if $A - \lambda I$ is a singular matrix.

Diagonalization of a Square Matrix: (Contd.)

- The matrix $A - \lambda I$ will be singular if and only if

$$\det(A - \lambda I) = 0 \tag{19}$$

- The left side of the above equation is an n-th order polynomial in λ, which is called the **characteristic polynomial** of A.
- The equation itself is known as the **characteristic equation** of the matrix A.
- By constuction, each of the n roots of this characteristic equation constitute an eigenvalue of A.
- Recall that a square matrix is non-singular if all its rows (and columns) are linearly independent. This implies that for a non-singular matrix B, the only solution to $B\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
- Conversely matrix B is singular if and only is $B\mathbf{x} = \mathbf{0}$ has a non-zero solution.
- The fact that matrix $A - \lambda I$ is singular means that the system of equations $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has a solution other than $\mathbf{v} = \mathbf{0}$.

Diagonalization of a Square Matrix: (Contd.)

- For every λ which is an eigenvalue of the square matrix A, a *non-zero* vector $\mathbf{v}$ such that
$$(A - \lambda I)\, \mathbf{v} = \mathbf{0}$$
is called an **eigenvector** of A corresponding to that particular eigenvalue λ.
- **An Example:** Let
$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$
  - Then it's characteristic polynomial is:
$$\det \begin{bmatrix} -1-\lambda & 3 \\ 2 & 0-\lambda \end{bmatrix}$$
$$= \lambda\,(1+\lambda) - 6 = \lambda^2 + \lambda - 6$$

Diagonalization of a Square Matrix: (Contd.)

- Hence from the characteristic equation
$$\lambda^2 + \lambda - 6 = 0 \Rightarrow (\lambda + 3)((\lambda - 2) = 0$$
the eigenvalues of this matrix are given by -3 and 2.
- To get an eigenvector corresponding to the eigenvalue -3, we look for a *nonzero* vector $\mathbf{v}$ which solves the following equation:
$$(A - (-3)I)\,\mathbf{v} = \mathbf{0}$$
$$\Rightarrow \begin{bmatrix} -1+3 & 3 \\ 2 & 0+3 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
- Thus any (v_1, v_2) that solves the equation $2v_1 + 3v_2 = 0$ will constitute an eigenvector for the eigenvalue -3. For example, $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is one such eigenvector.
- Likewise we can construct an eigenvector corresponding to the eigenvalue 2 by solving the equation $(A - 2I)\,\mathbf{v} = \mathbf{0}$.

Diagonalization of a Square Matrix: (Contd.)

- Note that eigenvector to a particular eigenvalue is not unique. (Why?)
- I am now going to state some important results of linear algebra (without proof) which I am going to use later.
- **Theorem 1:** Consider a square matrix $A_{n\times n}$ whose eigenvalues are given by λ_1, λ_2, ...λ_n. Let $\mathbf{v_1}$, $\mathbf{v_2}$, ...,$\mathbf{v_n}$ represent the corresponding set of eigenvetors.
  (i) If the eigenvalues are all distinct, then the corresponding eigenvectors are all linearly independent.
  (ii) Hence we can construct a nonsingular matrix

$$M \equiv \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & .. & \mathbf{v_n} \end{pmatrix}$$

such that

$$M^{-1}AM = \begin{bmatrix} \lambda_1 & 0 & .. & 0 \\ 0 & \lambda_2 & .. & 0 \\ : & : & .. & : \\ 0 & 0 & .. & \lambda_n \end{bmatrix} \equiv \Lambda$$

- **Theorem 2:** Even if a matrix is not diagonalizable, it is always possible to find a nonsingular matrix P such that

$$P^{-1}AP = J$$

where J is a matrix of *Jordan canonical form* with the property that
(i) it has the eigenvalues of A on the leading diagonal, and
(ii) either ones or zeroes on the superdiagonal;
(iii) rest of the elements are all zeros.
(The superdiagonal of a square matrix is the set of elements directly above the elements comprising the diagonal.)

- Armed with this knowledge, we are now going to go back to our problem of solving a system of linear and autonomous difference equations.

Back to a System of First Order Difference Equations: Linear & Autonomous

- Recall that we were trying to solve an $n \times n$ *system* of linear and autonomous equations of the form:

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + \mathbf{b} \tag{20}$$

- We have already identified a particular solution as the steady state solution (under the assumption that $(I - A)$ is invertible) such that

$$\mathbf{x}_t^p = \bar{\mathbf{x}} = (I - A)^{-1}\,\mathbf{b}.$$

- We are now looking for a general solution to the corresponding $n \times n$ homogeneous system given by:

$$\mathbf{x}_t = A\mathbf{x}_{t-1} \tag{21}$$

A System of First Order Difference Equations: Linear & Autonomous (Contd.)

- Let us first assume that A has eigenvalues which are *real* and *distinct*; hence it is diagonizable.
- Let these eigenvalues be denoted by by λ_1, λ_2, ...λ_n.
- Let M be the corresponding matrix of eigenvectors that diagonalizes A.
- Then $M^{-1}AM = \Lambda$, where is a Λ diagonal matrix with all the eigenvalues of A as its diagonal elements.
- Now consider the homogeneous system of difference equations given in (21):

$$\mathbf{x}_t = A\mathbf{x}_{t-1}$$

- Let us define a new set of variables $\mathbf{y}$ such that

$$\mathbf{y} = M^{-1}\mathbf{x} \quad \text{for all } t$$

- Then by this definition, $\mathbf{x}_t = M\mathbf{y}_t$ and $\mathbf{x}_{t-1} = M\mathbf{y}_{t-1}$.

- Substituting these values to (21):

$$\begin{aligned} M\mathbf{y}_t &= AM\mathbf{y}_{t-1} \\ &\Rightarrow M^{-1}M\mathbf{y}_t = M^{-1}AM\mathbf{y}_{t-1} \\ &\Rightarrow \mathbf{y}_t = \Lambda\mathbf{y}_{t-1} \qquad (22) \end{aligned}$$

where Λ is a diagonal matrix which has all the eigenvalus of A as its diagonal elements.

- Equation (22) represents a system of n independent difference equations of the form:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{nt} \end{pmatrix}_{n\times 1} = \begin{bmatrix} \lambda_1 & 0 & .. & 0 \\ 0 & \lambda_2 & .. & 0 \\ \vdots & \vdots & .. & \vdots \\ 0 & 0 & .. & \lambda_n \end{bmatrix}_{n\times n} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \\ \vdots \\ y_{nt-1} \end{pmatrix}_{n\times 1}$$

A System of First Order Difference Equations: Linear & Autonomous (Contd.)

- We already know that the general solution to this **y**-system would be given by:

$$\begin{aligned} y_{1t} &= C_1 (\lambda_1)^t \\ y_{2t} &= C_2 (\lambda_2)^t \\ &\dots \\ y_{nt} &= C_n (\lambda_n)^t \end{aligned}$$

- From this we can easily derive the corresponding solution to the **x**-system using the relationship defined earlier, namely $\mathbf{x}_t = M\mathbf{y}_t$.
- Thus we have now been able to derive the complementary function of the given system as

$$\mathbf{x}_t^c = M \begin{pmatrix} C_1 (\lambda_1)^t \\ C_2 (\lambda_2)^t \\ \vdots \\ C_n (\lambda_n)^t \end{pmatrix}$$

A System of First Order Difference Equations: Linear & Autonomous (Contd.)

- Thus the general solution to the given system can be written as follows:

$$\mathbf{x}_t = M \begin{pmatrix} C_1 (\lambda_1)^t \\ C_2 (\lambda_2)^t \\ \vdots \\ C_n (\lambda_n)^t \end{pmatrix} + \bar{\mathbf{x}}$$

- Note that we also know the elements of the M matrix.
- By construction, each of the n columns of the M matrix corresponds to an eigenvector of the n eigenvalues of matrix A.
- Let these eigenvalues be denoted by $\mathbf{v_1} = \begin{pmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{pmatrix}$,

  .....and so on.

A System of First Order Difference Equations: Linear & Autonomous (Contd.)

- Therefore we can now completely characterise the general solution to the given system as follows:

$$\begin{pmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{pmatrix} = \begin{bmatrix} v_{11} & v_{12} & .. & v_{1n} \\ v_{21} & v_{22} & .. & v_{2n} \\ \vdots & \vdots & .. & \vdots \\ v_{n1} & v_{n2} & .. & v_{nn} \end{bmatrix} \begin{pmatrix} C_1 (\lambda_1)^t \\ C_2 (\lambda_2)^t \\ \vdots \\ C_n (\lambda_n)^t \end{pmatrix} + \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}$$

- Expanding, these solutions can be written as

$$\begin{aligned} x_{1t} &= v_{11} C_1 (\lambda_1)^t + v_{12} C_2 (\lambda_2)^t + ... + v_{1n} C_n (\lambda_n)^t + \bar{x}_1 \\ x_{2t} &= v_{21} C_1 (\lambda_1)^t + v_{22} C_2 (\lambda_2)^t + ... + v_{2n} C_n (\lambda_n)^t + \bar{x}_2 \\ &\vdots \\ x_{nt} &= v_{n1} C_1 (\lambda_1)^t + v_{n2} C_2 (\lambda_2)^t + ... + v_{nn} C_n (\lambda_n)^t + \bar{x}_m \end{aligned}$$

where $C_1,\ C_2, ...,\ C_n$ are all arbitrary constants.

A System of First Order Difference Equations: Linear & Autonomous (Contd.)

- The stability property of this system is similar to what we had discussed earlier:
  - all the variables will approach their steady state values if $|\lambda_i| < 1$ for all i. In this case the steady state will be stable.
  - The variables will explode away from the steady state values if $|\lambda_i| > 1$ for all i. In this case the steady state is unstable.
  - If some of the $|\lambda_i|$s are < 1 and some of the $|\lambda_i|$s are > 1, then the stability of the system depends crucially on the boundary conditions. The system will approach the steady state for some initial values, and will move away from the steady state for all other initial values. In this case the steady state is said to be a saddle point and the system is saddle point stable.

A System of First Order Difference Equations: Linear & Autonomous (Contd.)

- Recall that we had assumed that all the eigenvalues of A matrix are distinct.
- If some of them are repeated then A may not be diagonizable; hence deriving the solution would not be that easy.
- However, as we have already noted any square matrix A can be converted into a Jordan normal form by a similarity transformation.
- If we could transform A to a Jordan form, then once again solving the system would become relatively easy, because now all the equation in the system will be at least partially uncoupled.
- But the procedure to identify the similarity transformation that converts a square matrix to a Jordan form could be rather complex and we shall not attempt to do so except for the 2×2 case.
- We present below a *complete* analysis of the 2×2 case.

Solving a Two-Dimensional System:

- Consider a two dimensional system of the form

$$\begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A_{2\times 2}} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad (23)$$

- We already know that a particular solution to this system will be given by the corrsponding steady state: $\mathbf{x}_t^p = \bar{\mathbf{x}}$.
- Hence we focus only on the homogenous part to find the complementary function.
- The first step is to find the eigenvalues and the corresponding eigenvectors of A.

Case A: Eigenvalues are real and distinct:

- We have analysed a similar case for the $n \times n$ system. Hence I do not repeat the argument here.
- From our analysis before, we know that the solution to the system will be given by:

$$x_{1t} = v_{11} C_1 (\lambda_1)^t + v_{12} C_2 (\lambda_2)^t + \bar{x}_1$$
$$x_{2t} = v_{21} C_1 (\lambda_1)^t + v_{22} C_2 (\lambda_2)^t + \bar{x}_2$$

- The system will be asymptotically statble if $|\lambda_1|, |\lambda_2| < 1$; will be unstable if $|\lambda_1|, |\lambda_2| > 1$.
- The system will be saddle-point stable (stable for some boundary values, not stable otherwise) iff one of the λ value is greater than unity and the other one is not.

Solving a Two-Dimensional System: (Contd.)

- Notice that when the system is saddle-point stable, it will approach its steady state point $(\overline{x}_1, \overline{x}_2)$ if and only if the boundary conditions are such that arbitrary constant associated with the unstable root is equal to zero.
- For example, suppose in the above solutions, we find that $|\lambda_1| < 1$ while $|\lambda_2| > 1$. Then the systme will be stable if and only if the boundary conditions are such that $C_2 = 0$.
- Indeed, using this knowledge, we can chart out a precise path of (x_{1t}, x_{2t}) such that the system will approach its steady state point $(\overline{x}_1, \overline{x}_2)$ if and only if the initial values happen to lie on this path. This path is called the "**stable arm**" *of the saddle path*.

Solving a Two-Dimensional System: (Contd.)

- We can characterise this path for the 2×2 case by using backward induction.
- We want $C_2 = 0$. So setting $C_2 = 0$ in the above solutions, we get:

$$x_{1t} = v_{11} C_1 (\lambda_1)^t + \bar{x}_1$$
$$x_{2t} = v_{21} C_1 (\lambda_1)^t + \bar{x}_2$$

- Substituting for the common term $C_1 (\lambda_1)^t$ from one equation to the other, we get:

$$(x_{1t} - \bar{x}_1) = \frac{v_{11}}{v_{21}} (x_{2t} - \bar{x}_2) \qquad \text{(SS)}$$

- Any (x_{1t}, x_{2t}) configuration that saisfies the SS property above will, **by construction**, have $C_2 = 0$.
- Thus if the system starts anywhere on the SS path, it will always converge to its steady state point $(\bar{x}_1, \bar{x}_2)$; it will diverge otherwise.
- Hence for the 2×2 case, the SS path represents the "**stable arm**" *of the saddle path*.

Case B: Eigenvalues are real and repeated:

- Let the repeated eigenvalue be denoted by λ.
- In this case the matrix A is not diagonaizable because it does not have enough linearly independent eigenvectors.
- However, as we have mentioned before, by a similarity transformation, we can still transform A to a Jordan caconical form J where

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

- How do we do that?
- For that we use something called a 'generalised eigenvector'.

- If A is a $n \times n$ matrix, a **generalised eigenvector** of A corresponding to the eigenvalue λ is a nonzero vector $\mathbf{x}$ satisfying the following property:

$$(A - \lambda I)^p \, \mathbf{x} = \mathbf{0}$$

  where p is some positive integer.
- Let $\mathbf{v}$ be a eigenvector of the repeated eigenvalue λ. Then we can construct a 'generalised eigenvector' $\mathbf{e}$ in the following way:

$$(A - \lambda I) \, \mathbf{e} = \mathbf{v}$$

- Notice that the 'generalised eigenvector' $\mathbf{e}$ thus constructed would satisfy the above property since

$$(A - \lambda I)^2 \, \mathbf{e} = (A - \lambda I) \, \mathbf{v} = \mathbf{0}$$

- Indeed for any $n \times n$ matrix 'defective' matrix (not directly diagonalizable) with an eigenvalue λ repeated k times ($k \leqslant n$) one can find a set of exactly k 'linearly independent' generalised eigenvectors (the set is not unique though).

Solving a Two-Dimensional System: (Contd.)

- From the eigenvector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and generalised eigenvector $\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ we can now construct a nonsingular matrix $M = \begin{bmatrix} v_1 & e_1 \\ v_2 & e_2 \end{bmatrix}$
- This nonsingular matrix M has the following property:

$$M^{-1}AM = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

- Now let us go back to our homogenous system:

$$\mathbf{x}_t = A\mathbf{x}_{t-1}$$

- Once again we define a new set of variables $\mathbf{y}$ such that

$$\mathbf{y} = M^{-1}\mathbf{x} \quad \text{for all } t$$

- Then by this definition, $\mathbf{x}_t = M\mathbf{y}_t$ and $\mathbf{x}_{t-1} = M\mathbf{y}_{t-1}$.

Solving a Two-Dimensional System: (Contd.)

- Proceeding exactly as before, we can show that:

$$\begin{aligned} M\mathbf{y}_t &= AM\mathbf{y}_{t-1} \Rightarrow M^{-1}M\mathbf{y}_t = M^{-1}AM\mathbf{y}_{t-1} \\ \Rightarrow & \quad \mathbf{y}_t = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \mathbf{y}_{t-1} \end{aligned} \qquad (24)$$

- Expanding the system, we can write it as

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix}$$

$$\Rightarrow \left. \begin{array}{r} y_{1t} = \lambda y_{1t-1} + y_{2t-1} \\ y_{2t} = \lambda y_{2t-1} \end{array} \right\}$$

- Note that by this transformation we have been able to partially uncouple the system.
- We can now solve the latter equation independently, which will give us the following general solution:

$$y_{2t} = C_2 (\lambda)^t ; \ C_2 \text{ is an arbitrary constant.}$$

Solving a Two-Dimensional System: (Contd.)

- Substitution the solution for y_{2t} back into the first equation we get the following linear *non-automous* difference equation:

$$y_{1t} = \lambda y_{1t-1} + C_2 (\lambda)^t$$

- This equation can now be solved independently. Moreover this equation exactly corresponds to a linear nonautonmous equation of the form: $x_t = ax_{t-1} + B(b)^t$ with $a = b$.
- We have solved this form of non-autonomous equation before and we know that the general solution to this form is given by $(C + Bt)a^t$.
- Given this knowledge, the general solution for y_{1t} is given by

$$y_{1t} = (C_1 + C_2 t)(\lambda)^t; \; C_1 \text{ is an arbitrary constant.}$$

- So we have now found the general solution to this **y**-system which is given by:

$$\begin{aligned} y_{1t} &= (C_1 + C_2 t)(\lambda)^t \\ y_{2t} &= C_2 (\lambda)^t \end{aligned}$$

Solving a Two-Dimensional System: (Contd.)

- From this we can easily derive the corresponding solution to the $\mathbf{x}$-system using the relationship defined earlier, namely $\mathbf{x}_t = M\mathbf{y}_t$, where $M = \begin{bmatrix} v_1 & e_1 \\ v_2 & e_2 \end{bmatrix}$
- Then using the steay state values as the particular solution, we can completely characterise the general solution of this 2×2 system with a repeated eigenvalue as follows:

$$\mathbf{x}_t = \begin{bmatrix} v_1 & e_1 \\ v_2 & e_2 \end{bmatrix} \begin{pmatrix} (C_1 + C_2 t)(\lambda)^t \\ C_2 (\lambda)^t \end{pmatrix} + \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

- The system will be asymptotically statble if $|\lambda| < 1$; will be unstable if $|\lambda| > 1$.

Case C: eigenvalues are complex conjugate

- When the eigenvalues of A are complex conjugate, they are *necessarily* distinct.
- So we can diagonalize A matrix by using their corresponding eigenvectors and then proceed to solve the simplified system just as we did when the eigenvalues were real and distinct.
- The corresponding solution would be given by

$$\begin{aligned} x_{1t} &= v_{11} C_1 (\lambda_1)^t + v_{12} C_2 (\lambda_2)^t + \bar{x}_1 \\ x_{2t} &= v_{21} C_1 (\lambda_1)^t + v_{22} C_2 (\lambda_2)^t + \bar{x}_2 \end{aligned} \tag{25}$$

- However here now λ_1 and λ_1 as well as some of the coefficients are imaginary numbers. So it it difficult to visualize the solution and comment on their stability property.
- We can however write the solutions in a more meaningful way by applying various theorems of complex numbers (in particular **De Moivre's Theorem**). However since we shall rarely come across complex eigen values in our economic examples, I shall not discuss

First Order Difference Equations: Non-Linear & Autonomous

- So far we have discussed methods of solving linear difference equations
- Let us now discuss the case of nonlinear difference equations.
- The first point to be noted here is that it is extremely difficult to derive an exact solution (i.e., a precise time path) to a generic non-linear difference equation (except for a few well-known examples).
- However, even when an exact solution is not found, two techniques are often employed to draw some inference about the behaviour of the dynamic system: one is the **linearization technique**, and the other is the **phase diagram technique**.
- Here we shall first discuss the linearization technique and then move on to phase diagram technique.

  (Phase diagram technique entails a *qualitative* analysis; it is often used along with linearization technique to provide an more complete characterization).

Linearization of non-linear difference equations and local stability analysis:

- Consider any nonlinear function of a single variable x

$$f(x) : D \rightarrow R$$

  where D and R are the domain and range of the function respectively.
- Let $\hat{x} \in D$ be some given value of the variable.
- Then by Taylor's Theorem, the function can be expanded around $\hat{x}$ in the following way:

$$f(x) \approx f(\hat{x}) + \frac{f'(\hat{x})}{1!}(x-\hat{x}) + \frac{f''(\hat{x})}{2!}(x-\hat{x})^2 + \frac{f'''(\hat{x})}{3!}(x-\hat{x})^3 + \ldots\ldots$$

  (Note: Taylor's theorem is not applicable to all functions. There are domain restrictions.)
- Now a **linear approximation** of the non-linear function around the point $\hat{x}$ is given by:

$$f(x) \approx f(\hat{x}) + \frac{f'(\hat{x})}{1!}(x-\hat{x})$$

Linearization and local stability analysis: (Contd.)

- We can expand the RHS above to write the relationship as:

$$
\begin{aligned}
f(x) &\approx \underbrace{f(\hat{x}) - f'(\hat{x})\hat{x}}_{A \text{ (constant)}} + \underbrace{f'(\hat{x})}_{B \text{ (constant)}} \; x \\
&= A + Bx
\end{aligned}
$$

- Note the linear function given above is only an approximation of the function around $\hat{x}$, i.e., it resembles the function only in a small neighbourhood of $\hat{x}$.
- In general this linear function does not closely approximate the function for all values of x.
- Thus *whatever conclusion we draw on the basis of this linear approximation will only be valid locally around* $\hat{x}$.

- Linearization technique is often used to convert a non-linear difference equation into a linear form.
- Generally the non-linear equation is linearly approximated around its steady state value.
- This allows us to derive some conclusions about the time path of the variable in the neighbourhood of the steady state and thus its *local stability property*.
- Consider the following non-linear autonomous difference equation of the from:

$$x_t = f(x_{t-1}).$$

- We know that the steady state of the above difference equation is defined as $x_t = x_{t-1} = \bar{x}$, or equivalently:

$$\bar{x} = f(\bar{x}).$$

- Suppose there exsist an $\bar{x}$ that solves this equation. In other words, suppose a steady state indeed exists and we can somehow find it.

Linearization and local stability analysis: (Contd.)

- Then linearizing the $f(x_{t-1})$ equation around $\bar{x}$, we get a linear differential equation of the following form:

$$\begin{aligned} x_t &= f(x_{t-1}) \\ &\approx f(\bar{x}) + \frac{f'(\bar{x})}{1!}(x_{t-1} - \bar{x}) \\ &= f'(\bar{x})x_{t-1} + \left[f(\bar{x}) - f'(\bar{x})\hat{x}\right] \\ &= ax_{t-1} + b. \end{aligned}$$

- We now know that solution to this equation is given by:

$$x_t = C\,(a)^t + \bar{x};\;\; C \text{ an arbitrary constant.}$$

- We also know that stability of the dynamic system depends on the term 'a', i.e., on the value of $f'(\bar{x})$:
  - If $|f'(\bar{x})| < 1$, the system is *locally* stable;
  - if $|f'(\bar{x})| > 1$, the system is *locally* unstable.

- To illustrate how the linearization technique works, let us consider the following non-linear and autonomous first order difference equation:

$$x_t = \underbrace{(x_{t-1})^3 - 3(x_{t-1})^2 + 3x_{t-1}}_{f(x_{t-1})}$$

- The steady state(s) of this system are defined by the follwing equation:

$$\begin{aligned} \bar{x} &= (\bar{x})^3 - 3(\bar{x})^2 + 3\bar{x} \\ &\Rightarrow (\bar{x})^3 - 3(\bar{x})^2 + 2\bar{x} = 0 \\ &\Rightarrow \bar{x}(\bar{x}-1)(\bar{x}-2) = 0 \end{aligned}$$

- Thus the above system admits three steady states: $\bar{x}_1 = 0$; $\bar{x}_2 = 1$; $\bar{x}_3 = 2$.

Linearization and local stability analysis: An example (Contd.)

- Now suppose we want to check the *local stability* property of the third steady state point: $\bar{x}_3 = 2$.
- Linearizing $f(x_{t-1})$ around $\bar{x}_3 = 2$ and noting that $f'(x_{t-1}) = 3(x_{t-1})^2 - 6(x_{t-1}) + 3$, we get:

$$\begin{aligned} f(x_{t-1}) &\approx f(2) + \frac{f'(2)}{1!}(x_{t-1} - 2) \\ &= \left[(2)^3 - 3(2)^2 + 3.2\right] + \frac{\left[3(2)^2 - 6(2) + 3\right]}{1!}(x_{t-1} - 2) \\ &= 3x_{t-1} - 4 \end{aligned}$$

- Thus the linearizing around the third steady state $\bar{x}_3 = 2$, we get the following linear difference equation:

$$x_t = 3x_{t-1} - 4$$

which will have a general solution $x_t = C\,(3)^t + 2$. It is easy to conclude that the third steady state $\bar{x}_3 = 2$ will be *locally unstable*.

2X2 Nonlinear System: Linearization and local stability analysis:

- Consider now a 2×2 autonomous system non-linear system of difference equations, given by:

$$\left.\begin{aligned} x_t &= f(x_{t-1}, y_{t-1}) \\ y_t &= g(x_{t-1}, y_{t-1}) \end{aligned}\right\}$$

- The steady state of this system of difference equations is defined as $(\bar{x}, \bar{y})$ such that

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{y}); \\ \bar{y} &= g(\bar{x}, \bar{y}). \end{aligned}$$

- Suppose such a steady state $(\bar{x}, \bar{y})$ exists.
- Then we can derive the linear approximations of the two functions f and g around $(\bar{x}, \bar{y})$ as:

$$\begin{aligned} f(x_{t-1}, y_{t-1}) &\approx f(\bar{x}, \bar{y}) + f_x(\bar{x}, \bar{y})(x_{t-1} - \bar{x}) + f_y(\bar{x}, \bar{y})(y_{t-1} - \bar{y}); \\ g(x_{t-1}, y_{t-1}) &\approx g(\bar{x}, \bar{y}) + g_x(\bar{x}, \bar{y})(x_{t-1} - \bar{x}) + g_y(\bar{x}, \bar{y})(y_{t-1} - \bar{y}). \end{aligned}$$

Linearization and local stability analysis: (Contd.)

- Rearranging terms, we can write the linearized version of the above system of difference equations as:

$$
\begin{aligned}
x_t &= \underbrace{f_x(\bar{x},\bar{y})}_{a_{11}} x_{t-1} + \underbrace{f_y(\bar{x},\bar{y})}_{a_{12}} y_{t-1} + \underbrace{\left[f(\bar{x},\bar{y}) - \bar{x} f_x(\bar{x},\bar{y}) - \bar{y} f_y(\bar{x},\bar{y})\right]}_{b_1}; \\
y_t &= \underbrace{g_x(\bar{x},\bar{y})}_{a_{21}} x_{t-1} + \underbrace{g_y(\bar{x},\bar{y})}_{a_{22}} y_{t-1} + \underbrace{\left[g(\bar{x},\bar{y}) - \bar{x} g_x(\bar{x},\bar{y}) - \bar{y} g_y(\bar{x},\bar{y}\right.}_{b_2}
\end{aligned}
$$

- The system is then represented in matrix form as follows:

$$
\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
$$

- The coefficient matrix of the corresponding homogeneous system is given by the following Jacobian matrix:

$$
A = \begin{bmatrix} f_x(\bar{x},\bar{y}) & f_y(\bar{x},\bar{y}) \\ g_x(\bar{x},\bar{y}) & g_y(\bar{x},\bar{y}) \end{bmatrix}
$$

Linearization and local stability analysis: (Contd.)

- From our previous analysis, we once again know that the stability of the system once again depends on the eigenvalues of this co-efficient matrix.
- If the eigenvalues are real, stability condition now requires that the absolute values of both these be less than unity.
- If the eigenvalues are real and have absolute values greater than unity, the system in unstable.
- If of the two real eigenvalues, one has absolute value greater than unity and the other one has absolute value less than unity, then equilibrium is a saddle point.

- If we could calculate the precise steady state values in numerical terms and evaluate all the partial derivatives at those values, then we could directly derive the corresponding eigenvalues and comment about local stability property of the dynamic system.
- However, sometimes, calculation of precise steady state values is not feasible. At most we might have some charasteristic propoperties of the partial derivative (e.g., positive/negative, interger or fraction etc.). Can we still infer something about the stability property of the system?
- It turns out that we can - provided we have sufficient information about the trace and determinant of the coefficient matrix A.
- To see how, first note that the trace and the determinant of the co-efficient matrix of this linearized system are given by:

$$\begin{aligned} TraceA &= f_x(\bar{x},\bar{y}) + g_y(\bar{x},\bar{y}); \\ DetA &= f_x(\bar{x},\bar{y})g_y(\bar{x},\bar{y}) - f_y(\bar{x},\bar{y})g_x(\bar{x},\bar{y}) \end{aligned}$$

Linearization and local stability analysis: (Contd.)

- Also, the characteristic equation of matrix A is given by:

$$Det \left[\begin{array}{cc} f_x(\bar{x},\bar{y}) - \lambda & f_y(\bar{x},\bar{y}) \\ g_x(\bar{x},\bar{y}) & g_y(\bar{x},\bar{y}) - \lambda \end{array} \right] = 0$$

$$\begin{aligned} &\Rightarrow \quad (f_x(\bar{x},\bar{y}) - \lambda)\,(g_y(\bar{x},\bar{y}) - \lambda) - f_y(\bar{x},\bar{y})g_x(\bar{x},\bar{y}) = 0 \\ &\Rightarrow \quad \lambda^2 - [f_x(\bar{x},\bar{y}) + g_y(\bar{x},\bar{y})]\,\lambda + [f_x(\bar{x},\bar{y})g_y(\bar{x},\bar{y}) - f_y(\bar{x},\bar{y})g_x(\bar{x},\bar{y}) \\ &\Rightarrow \quad \lambda^2 - [TraceA]\,\lambda + [DetA] = 0 \end{aligned}$$

- Hence the eigenvalues of matrix A can be represented by λ_1, $\lambda_2 = \frac{TraceA \pm \sqrt{(TraceA)^2 - 4DetA}}{2}$ such that

$$\lambda_1 + \lambda_2 = TraceA; \qquad \text{(i)}$$

$$\lambda_1 \lambda_2 = DetA. \qquad \text{(ii)}$$

- We can use these relationships between the eigenvalues and *TraceA* and *DetA* to infer about the stability property of the system (provided we have enough information about *TraceA* and *DetA*.)

Linearization and local stability analysis: (Contd.)

- The above analysis tells us that for the linearized system sometimes one can deduce the stability property of the system (even without explicitly solving for the exact eigenvalues, which can be messy) if one has sufficient information about the trace and the determinant of the coefficient matrix.
- For example, suppose you find that for a linearized coefficient matrix, both *TraceA* and *DetA* lie strictly between 0 and 1, you can immediately infer that in this case both λ_1 and λ_2 must lie between 0 and 1; so the system would be stable.
- However sometimes even linearization may become too messy and/or fail to tell us anything definitive about the nature of the characteristic roots (whether > 1 or < 1).
- In those cases, we often use a diagrammatic technique - called phase diagram - which allows us to duduce some '*qualitative*' results about the time path of the variables and their stability property - even though we cannot explicitly solve them.
- We now describe the phase diagram technique.

Non-linear difference equation & Phase Diagram Technique:

- Consider the following non-linear autonomous difference equation of the from:
$$x_t = f(x_{t-1}).$$
- It we have enough information about the $f(x_{t-1})$ function then we can use a diagrammatic technique to comment about the time path of the variable and its stability.
- These conclusions are qualitative. One cannot precisely pin down the exact solution path/time path of the variable.

Phase Diagram Technique: (Contd.)

- Phase diagram technique for a single equation entails the following steps:

1. Plot the RHS of the above equation with x_{t-1} on the horizontal axis;
2. Plot another function: $x_t = x_{t-1}$ with x_{t-1} on the horizontal axis. This latter function essentially is a map unto itself and is identifies by the 45° line;
3. Identify the steady states which are the intesection points of the two plots;
4. Define $\Delta x \equiv x_t - x_{t-1} = f(x_{t-1}) - x_{t-1}$. This difference essetially compares the difference between the two plotted curves/lines.
5. Identify the sign of the Δx expression for the entire map of x_{t-1} values. $\Delta x > 0 \Rightarrow x$ is increasing over time;$\Delta x < 0 \Rightarrow x$ is increasing over time; (of course $\Delta x = 0 \Rightarrow x$ is at steady state)
6. The sign of Δx allows us to draw conclusions about the direction of movement of the variable x in any region and thus allows us to draw conclusions about its (global) stability.

Phase Diagram Technique: (Contd.)

- Consider now a 2×2 autonomous non-linear system of difference equations, given by:

$$\left.\begin{array}{l} x_t = f(x_{t-1}, y_{t-1}) \\ y_t = g(x_{t-1}, y_{t-1}) \end{array}\right\}$$

- Since these equations involve two variables, we can no longer plot the RHS of the functions in the two dimensional (x_{t-1}, y_{t-1}) space. But we can always plot the level curves of two other functions.
- But instead of plotting any level curve of the functions themselves, we plot the level curves of following functions which we define from the difference equations above:

Define

$$\begin{aligned} \Delta x &\equiv x_t - x_{t-1} = f(x_{t-1}, y_{t-1}) - x_{t-1} \equiv \hat{f}(x_{t-1}, y_{t-1}); \\ \Delta y &\equiv y_t - y_{t-1} = g(x_{t-1}, y_{t-1}) - y_{t-1} \equiv \hat{g}(x_{t-1}, y_{t-1}). \end{aligned}$$

Phase Diagram Technique: (Contd.)

- Phase diagram technique for a 2×2 autonomous system entails the following steps:

1. Plot the level curves $\Delta x = 0$ and $\Delta y = 0$ in the (x_{t-1}, y_{t-1}) plane. By definition the intersection points of the two curves are the steady state points of the 2×2 system;
2. Identify the sign of the Δx expression for the entire map of (x_{t-1}, y_{t-1}) values. $\Delta x > 0 \Rightarrow x$ is increasing over time;$\Delta x < 0 \Rightarrow x$ is increasing over time; $\Delta x = 0 \Rightarrow x$ is not changing;
3. Identify the sign of the Δy expression for the entire map of (x_{t-1}, y_{t-1}) values. $\Delta y > 0 \Rightarrow y$ is increasing over time;$\Delta y < 0 \Rightarrow y$ is increasing over time; $\Delta y = 0 \Rightarrow y$ is not changing;
4. The signs of Δx and Δy allow us to draw conclusions about the direction of movement of the two variables x and y in any region of the (x_{t-1}, y_{t-1}) plane and thus allows us to draw conclusions about its (global) stability.