Output Dynamics In the Long Run: Issues of Economic Growth

Mausumi Das

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Output Dynamics: Short Run to Long Run

- In the first part of the course, we have seen how output and employment are determined in the short run.
- In the Classical/Neoclassical system (and its various extensions), these are determined by the supply side factors (production conditions); role of demand is limited to the detemination of the equilibrium price level.
- In the Keynesian system (and its various extensions), aggregate demand plays a direct role in determining equilibrium output and employment in the short run.
- As discussed before, both these systems are based on aggregative behavioural equations.

Output Dynamics: Short Run to Long Run (Contd.)

- Alternatively, one could build an internally consistent, dynamic general equilibrium (DGE) framework, where forward-looking agents make their optimal decisions, taking both current and future variables into account.
- But this would entail dynamic equations involving today's and tomorrow's consumption; today's and tomorrow's capital stock etc.
- Indeed, as we have seen in the previous topic, the DGE framework brought us directly into the realm of output dyamics over time, i.e., economic growth.
- In the present module, we are going to explore some issues pertaining to this output dynamics over time.
- In particular, we shall examine how the output dynamics under the DGE framework generate a growth path for the economy and what happens to this growth trajectory in the long run.

Output Dynamics: Short Run to Long Run (Contd.)

- Must growth dynamics be necessarily based on the DGE framework?
- In other words, can the aggregative behavioural equations of the Classical/Neoclassical or the Keynesian system also throw up some growth trajectories for the economy?
  - If yes, how would they differ from the growth trajectories predicted by the DGE framework?
- Indeed, one can develop growth models based on the short run (static) characterization of the macroeconomy in either the Classical system or the Keynesian system.
- And the long run charateristics of these growth models may differ substantially from that of the DGE framework.
- We start our discussion by analysing the dynamic version of one of these aggregative models - namely the Classical/Neoclassical system.

Dynamic Extension of the Classical System: the Neoclassical Growth Model

- The reason why we focus on the dynamic extension of the Classical/Neoclassical system only and not the Keynesian one is because we want to discuss **long run** growth, not short/medium run business fluctuations.
- It is generally believed that in the long run, it is the supply side factors which are crucial in determining the output dynamics, not the demand side factors.
- We are also going to abstract away from price dynamics and indeed abstract from all nominal variables, including money itself.
- Moreover, unless stated otherwise, we shall generally assume that government is passive in the sense that it does not intervene in the functioning of the market economy via fiscal policies.

Definition of Long Run: Steady State vis-a-vis Balanced Growth Path

- Before we proceed further, it is important to define the concept of 'long run equilibrium' in the context of growth models.
- Long run equilibrium in a growth model is typically defined as a **balanced growth path**, where all endogeneous variables grow at some *constant* rate.
- This constant growth rate may differ from variable to variable.
- More importantly, this constant growth rate could even be zero for some variables.
- The latter case is typically identified as the **steady state** in the conventional dynamic analysis (which is a special case of a balanced growth path).

Neoclassical Growth Model: Solow

- Solow (QJE, 1956) extended the static Classical/Neoclassical system to a dynamic 'growth' framework where both factors of production - capital and labour - grow over time.
- As before, we consider a closed economy producing a single final commodity which is used for consumption as well as for investment purposes (i.e, as capital.)
- The single final commodity is produced using two factors of production - capital and labour.
- Capital stock grows due to the savings-cum-investment decisions of the households (all capital is owned by the households; there is no independent investment function coming from the firms).
- Labour supply grows due to the population growth (at an exogenous rate n).
- Growth of these two factors generates a growth rate of output in this economy.

Neoclassical Growth Model: Solow (Contd.)

- The questions that we are interested in are as follows:
  - What is the rate of growth of aggregate as well as per capita output in this economy?
  - Does the economy attain a balanced growth path in the long run?
  - What is the welfare level attained by the households in the long run?
  - Is there any role of the government in the growth process - either in terms of augmenting growth or in terms of ensuring maximum welfare?

Solow Growth Model: The Economic Environment

- The timing of events at any time period t is as follows:
- There are H households in the economy (not necessarily identical). These households together own the entire labour and capital stock in eceonomy - although ownership of these factors are not necessarily equal across all the households.
- At the beginning of the time period t, the all households begin with some endowment of capital (k_t^h) and labour (n_t^h) - which have been carried forward from the previous period. Thus the economy starts with a historically given total endowment of labour ($N_t \equiv \sum n_t^h$) and a historically given aggregate capital stock ($K_t \equiv \sum k_t^h$). All households offer their labour and capital (inelastically) to the firms.
- The competitive firms then carry out production and the total output produced is distributed as factor incomes to the households.
- Upon receiving their respective factor incomes, the households then decide how much to consume and how much to save.
- Period t ends here.

Solow Growth Model: The Economic Environment (Contd.)

- From the sequence of events described above, it is clear how the next period is related to the previous period:
  - The households' savings decisions at the end of period t augments their capital stock *in the next period* (i.e., k^h_{t+1}). At the same time new members are born to every household at the end of period t, which enhances the labour endowment of the household *in the next period* (i.e., n^h_{t+1}).
  - Therefore in the next period $(t+1)$, the economy again starts with a (new) total endowment of labour ($N_{t+1} \equiv \sum n^h_{t+1}$) and a (new) aggregate capital stock ($K_{t+1} \equiv \sum k^h_{t+1}$) and the process repeats itself.
- Note that the sequence of events in the Solow model is exactly analogous to the sequence of events that was described in the DGE model previously. What is different between the two frameworks is *how* households arrive at their consumption/savings decisions.

Solow Growth Model: Savings Behaviour of Households

- Unlike the DGE models, consumption/savings decisions of households in the Solow model are *not optimally determined* from any utility maximization exercise.
- They are determined by some norm/convention which are exogenously given.
- Accordingly, it is assumed that in each period the households consume a constant fraction of their total income and save the rest.
- **A Crucial Assumption:** The savings propensity is exogenously fixed, denoted by $s \in (0, 1)$, and is *same* for all households irrespective of their factor ownership.

Solow Growth Model: Savings Behaviour of Households (Contd.)

- **Another Crucial Assumption:** All savings are automatically invested in capital formation, which augments the capital stock in the next period.
- As we had noted earlier, this assumption implies that:
  - it is the households who make the investment decisions; not firms.
  - Firms simply rent in the capital from the households for production and distributes the output as wage and rental income to the households at the end of the period.
- In other words, this assumption rules out the existence of an independent investment function - as we had assumed earlier in the static Neoclassical Model (recall the IS relationship)!
- In fact, rate of retun of capital (r_t) is determined here by the market clearing condition in the factor market; not in the goods market.
- Indeed in Solow, savings is always equal to investment (by assumption); hence the goods market is always in equilibrium.

Solow Growth Model: Production Side Story

- The production side story in the Solow model is identical to the production side story in the DGE model (discussed earlier).
- The economy is characterized by M firms, with identical technology.
- Each firm i is endowed with a standard '**Neoclassical**' production technology

$$Y_{it} = F(N_{it}, K_{it})$$

  which satisfies all the standard properties e.g., diminishing marginal product of each factor (or law of diminishing returns), CRS and the Inada conditions.
- In addition, $F(0, K_{it}) = F(N_{it}, 0) = 0$, i.e., both inputs are essential in the production process.
- The firms operate in a competitive market structure and take the market wage rate (w_t) and rental rate for capital (r_t) in real terms as given.
- The firm maximises its current profit:
  $\Pi_{it} = F(N_{it}, K_{it}) - wN_{it} - rK_{it}$.

Production Side Story (Contd.)

- Static (period by period) optimization by the firm yields the following FONCs:

$$\begin{aligned} \text{(i) } F_N(N_{it}, K_{it}) &= w. \\ \text{(ii) } F_K(N_{it}, K_{it}) &= r. \end{aligned}$$

- Recall (from our previous analysis) that identical production function and CRS technology imply that firm-specific marginal products and economy-wide (social) marginal products (derived from the corresponding aggregate production function) of both labour and capital would be the same. Thus

$$\begin{aligned} F_N(N_{it}, K_{it}) &= F_N(N_t, K_t); \\ F_K(N_{it}, K_{it}) &= F_K(N_t, K_t). \end{aligned}$$

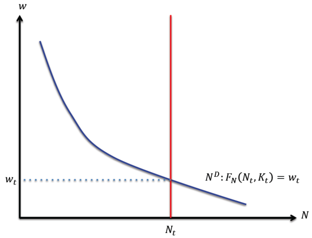
Production Side Story (Contd.)

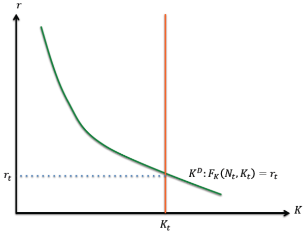
- Thus we get the familiar demand for labour schedule for the aggreagte economy at time t, and a similarly defined demand for capital schedule at time t as:

$$\begin{aligned} N^D &: \quad F_N(N_t, K_t) = w_t; \\ K^D &: \quad F_K(N_t, K_t) = r_t. \end{aligned}$$

- Recall that the supply of labour and that of capital at any point of time t is historically given at N_t and K_t respectively.
- **Assumption:** The market wage rate and the rental rate for capital, w_t and r_t, are fully flexible and they adjust so that the labour market and the capital market clear in every time period.

Determination of Market Wage Rate & Rental Rate of Capital at time t:





Distribution of Aggregate Output:

- Recall that the firm-specific production function is CRS; hence so is the aggregate production function.
- We know that for any constant returns to scale (i.e., linearly homogeneous) function, by Euler's theorem:

$$\begin{aligned} F(N_t, K_t) &= F_N(N_t, K_t)N_t + F_K(N_t, K_t)K_t \\ &= w_t N_t + r_t K_t. \end{aligned}$$

- This implies that after paying all the factors their respective marginal products, the entire output gets exhausted, confirming that firms indeed earn zero profit.
- Thus the total output currently produced goes to the households as income (Y_t) - of which they consume a fixed proportion and save the rest.

Dynamics of Capital and Labour:

- Recall that the capital stock over time gets augmented by the savings/investment made by the households.
- Also recall that all households invest a fixed proportion (s) of their income (which adds up to the aggregate output - as we have just seen).
- Hence aggegate savings (& investment) in the economy is given by :

$$S_t \equiv I_t = sY_t;\ 0 < s < 1.$$

- Let the existing capital depreciate at a constant rate $\delta : 0 \leqq \delta \leqq 1$.
- Thus the capital accumulation equation in this economy is given by:

$$\begin{aligned} K_{t+1} &= I_t + (1-\delta)K_t = sY_t + (1-\delta)K_t \\ \text{i.e., } K_{t+1} &= sF(N_t, K_t) + (1-\delta)K_t, \end{aligned} \quad (1)$$

- Labour stock increases due to population growth at a constant rate n (which is same across all households):

$$N_{t+1} = (1+n)N_t. \quad (2)$$

Dynamics of Capital and Labour (Contd):

- Equations (1) and (2) represent a 2×2 system of difference equations, which we can directly analyse to determine the time paths of N_t and K_t, and therefore the corresponding dynamics of Y_t.
- However, given the properties of the production function, we can transform the 2×2 system into a single-variable difference equation - which is easier to analyse.
- We shall follow the latter method here.

Capital-Labour Ratio & Per Capita Production Function:

- Using the CRS property, we can write:

$$y_t \equiv \frac{Y_t}{N_t} = \frac{F(N_t, K_t)}{N_t} = F\left(1, \frac{K_t}{N_t}\right) \equiv f(k_t),$$

  where y_t represents per capita output, and k_t represents the capital-labour ratio (or the per capita capital stock) in the economy at time t.
- The function $f(k_t)$ is often referred to as the *per capita* production function.
- Notice that using the relationship that $F(N_t, K_t) = N_t f(k_t)$, we can easily show that:

$$\begin{aligned} F_N(N_t, K_t) &= f(k_t) - k_t f'(k_t); \\ F_K(N_t, K_t) &= f'(k_t). \end{aligned}$$

[Derive these two expressions yourselves].

Properties of Per Capita Production Function:

- Given the properties of the aggregate production function, one can derive the following properties of the per capita production function:

$$\begin{aligned}
\text{(i) } f(0) &= 0; \\
\text{(ii) } f'(k) &> 0;\ \ f''(k) < 0; \\
\text{(iii) } \underset{k \to 0}{Lim} f'(k) &= \infty;\ \ \underset{k \to \infty}{Lim} f'(k) = 0.
\end{aligned}$$

- Condition (i) indicates that capital is an essential input of production;
- Condition (ii) indicates diminishing marginal product of capital;
- Condition (iii) indicates the Inada conditions with respect to capital .
- Finally, using the definition that $k_t \equiv \frac{K_t}{N_t}$, we can write

$$\begin{aligned}
k_{t+1} &\equiv \frac{K_{t+1}}{N_{t+1}} = \frac{sF(N_t, K_t) + (1-\delta)K_t}{(1+n)N_t} \\
&\Rightarrow \quad k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{(1+n)} \equiv g(k_t). \qquad (3)
\end{aligned}$$

- Equation (3) represents the basic dynamic equation in the discrete time Solow model.
- Notice that Equation (3) represents a single non-linear difference equation in k_t. Once again we use the phase diagram technique to analyse the dynamic behaviour of k_t.
- We now use the phase diagram technique to analyse this non-linear difference equation. By this technique, we first plot the $g(k_t)$ function with respect to k_t. Then we identify its possible points of intersection with the 45° line - which denote the steady state points of the system.
- In plotting the $g(k_t)$ function, note:

$$
\begin{aligned}
g(0) &= \frac{sf(0) + (1-\delta).0}{(1+n)} = 0; \\
g'(k) &= \frac{1}{(1+n)} \left[sf'(k) + (1-\delta) \right] > 0; \\
g''(k) &= \frac{1}{(1+n)} sf''(k) < 0.
\end{aligned}
$$

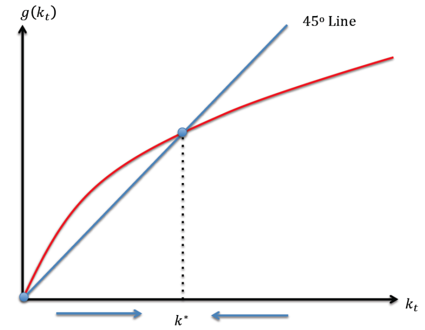
Dynamics of Capital-Labour Ratio (Contd.):

- Moreover,

$$\lim_{k\to 0} g'(k) = \frac{1}{(1+n)}\left[s\lim_{k\to 0} f'(k) + (1-\delta)\right] = \infty;$$

$$\lim_{k\to\infty} g'(k) = \frac{1}{(1+n)}\left[s\lim_{k\to\infty} f'(k) + (1-\delta)\right] = \frac{(1-\delta)}{(1+n)} < 1.$$

- We can now draw the phase diagram for k_t :



Dynamics of Capital-Labour Ratio (Contd.):

- From the phase diagram we can identify two possible steady states:

$$\begin{aligned} \text{(i) } k &= 0 \text{ (Trivial Steady State);} \\ \text{(ii) } k &= k^* > 0 \text{ (Non-trivial Steady State).} \end{aligned}$$

- Since an economy is always assumed to start with some positive capital-labour ratio (however small), we shall ignore the non-trivial steady state.
- From the diagram, it is clear that the economy has a unique non-trivial steady state, given by k^*.
- Is this steady state stable?
- The phase diagram technique also allows us to comment on the global/locan stability of a dynamic system by analysing the movement of the variable when it is **not** on a steady state.
- This movement is captured by the following expression:

$$\Delta k \equiv k_{t+1} - k_t$$

Dynamics of Capital-Labour Ratio (Contd.):

- Notice that

$$\begin{aligned} \Delta k &> 0 \Rightarrow k_t \text{ increases over time;} \\ \Delta k &< 0 \Rightarrow k_t \text{ decreases over time.} \end{aligned}$$

- Thus by evaluating the sign of the Δk expression, we can comment on the direction of movement of k_t when it is not on a steady state.
- Now, from the dynamic equation of the Solow model:

$$\Delta k \equiv k_{t+1} - k_t = \frac{sf(k_t) - (\delta + n)k_t}{(1+n)}$$

- From the phase diagram, it is easy to verify that for all $k_t \in (0, k^*)$, $\Delta k > 0$, while for all $k_t \in (k^*, \infty)$, $\Delta k < 0$.
- This allows us to conclude that the non-trivial steady state of the Solow model (k^*) is globally asymptotically stable: starting from any initial capital-labour ratio $k_0 > 0$, the economy would always move to k^* in the long run.

Solow Model: Steady State & its Implication

- Thus we find that the capital labour ratio in the Solow model goes to a constant (k^*) in the long run.
- Implications:
  - In the long run, per capita capital stock does not grow, or equivalently grows at zero rate;
  - in the long run, per capita output: $y_t \equiv f(k_t)$ becomes constant at the level $f(k^*)$, or equivalently grows at zero rate;
  - in the long run, aggregate output $Y_t \equiv N_t f(k_t)$ grows at a constant rate n (which is the same as the rate of growth of population/labour force N_t).
- Notice that the Solovian economy in the long run will indeed be on a "balanced growth path", where all its variables are growing at *some* constant rate (not necessarily the same across all variables).

Verification of Stability Property: Linearization

- One can veify the stability property of the non-trivial steady state (k^*) via another methos called the linearization technique. (Note though that the lineraization technique will only tell us about the local stability; not global stability).
- Recall that the difference equation is given by:

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{(1+n)} \equiv g(k_t)$$

- Linearizing around the non-trivial steady state:

$$\begin{aligned} k_{t+1} &= g(k^*) + g'(k^*)(k_t - k^*) \\ &= g'(k^*)k_t + [g(k^*) - g'(k^*)k^*] \end{aligned}$$

- The non-trivial steady state would be locally stable iff $g'(k^*) = \left[\frac{sf'(k^*) + (1-\delta)}{(1+n)}\right] < 1.$

Verification of Stability Property: Linearization (Contd.)

- To see how the linearization works, let us take a specific production function of the Cobb-Douglas variety:

$$f(k) = k^{\alpha};\ 0 < \alpha < 1.$$

- Then

$$g(k_t) = \frac{s(k_t)^{\alpha} + (1-\delta)k_t}{(1+n)}$$

- And the nontrivial steady state is:

$$k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- Verify that at $k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$, $g'(k^*) = \left[\frac{s\alpha(k^*)^{\alpha-1} + (1-\delta)}{(1+n)}\right] < 1.$

Some Long Run Implictions of Solow Growth Model:

- Notice that the non-trivial steady state k^* can be written as:

$$
\begin{aligned}
k^* &= \frac{sf(k^*) + (1-\delta)k^*}{(1+n)} \\
&\Rightarrow (1+n)\,k^* = sf(k^*) + (1-\delta)k^* \\
&\Rightarrow \frac{f(k^*)}{k^*} = \frac{n+\delta}{s}.
\end{aligned}
$$

- Total differentiating and using the properties of the $f(k)$ function, it is easy to show that,

$$\frac{dk^*}{ds} > 0; \frac{dk^*}{dn} < 0; \frac{dk^*}{d\delta} < 0.$$

  - A higher savings ratio generates a *higher* **level** of per capita output in the long run;
  - A higher rate of growth of population generates a *lower* **level** of per capita output in the long run;
  - A higher rate of depreciation generates a *lower* **level** of per capita output in the long run.

Some Long Run Implictions of Solow Growth Model (Contd.):

- But these are all **level** effects. What would be the impact on the long run **growth**?
- We have already seen that in this Solovian economy the per capita income does not grow in the long run ($f(k^*)$ is a constant).
- In fact, **the long run growth rate of aggregate income in the Solovian economy is always equal to** n (and is independent of other parameters, e.g., s or δ)
- Notice there is no role for the government in influencing the long run growth rate here. In particular, if the government tries to manipulate the savings ratio (by imposing an appropriate tax on households' income and investing the tax revenue in capital formation), then such a policy will have no long run **growth** effect.

Solow Model: Long Run vis-a-vis Short/Medium Run

- To summarise:
  - The per capita income does not grow in the long run; it remains constatnt at $f(k^*)$ - the exact level being determined by various parameters (s, n, δ).
  - The aggregate income grows at a constant rate - given by the exogenous rate of growth of population (n).
- But all these happen only in the long run, i.e., as $t \longrightarrow \infty$.
- Starting from a given initial capital-labour ratio k_0 $(\neq k^*)$, it will obviously take the economy some time before it reaches k^*.
- What happens during these transitional periods?
- In particular, what would be the rate of growth of per capita income and that of aggregate income in the short run - when the economy is yet to reach its steady state?

Transitional Dynamics in Solow Growth Model:

- When the economy is out of steady state, the rate of growth of capital-labour ratio is given by:

$$\begin{aligned}
\gamma_k &\equiv \frac{k_{t+1} - k_t}{k_t} = \frac{\frac{sf(k_t)+(1-\delta)k_t}{(1+n)} - k_t}{k_t} \\
&= \frac{sf(k_t) - (n+\delta)k_t}{(1+n)k_t} \gtreqless 0 \text{ according as } k_t \lesseqgtr k^*.
\end{aligned}$$

- Moreover,

$$\frac{d\gamma_k}{dk} = \frac{-s(1+n)\left[f(k) - kf'(k)\right]}{\left[(1+n)k\right]^2} < 0.$$

- In other words, during transition, the higher is the capital-labour ratio of the economy, the lower is its (short run) growth rate.
- This last result has important implications for cross country growth comparisons.

Further Implictions of Solow Growth Model: Absolute vis-a-vis Conditional Convergence

- The above result implies that the transitional growth rate of per capita income in the poorer countries (with low k_0) will be higher than that of the rich countries (with high k_0); and eventually they will converge to the same level of per capita income (**Absolute Convergence**).
- This proposition however has been strongly rejected by data. In fact empirical studies show the opposite: richers countries have remained rich and poorer countries have remained poor and there is no significant tendencies towards convergence - even when one looks at long run time series data.
- The proposition of absolute convergence of course pre-supposes that the underlying parameters for all economies (rich and poor alike) are the same.

Absolute vis-a-vis Conditional Convergence (Contd.)

- If we allow rich and poor countries to have different values of s, δ, n etc. (which is plausible), then the Solow model generates a much weaker prediction of **Conditional Convergence.**
- Conditional Convergence states that a country grows faster - the further away it is from its own steady state.
- An alternative (and more useful) statement of Conditional Convergence runs as follows: Among a group of countries which are similar (similar values of s, δ, n etc.), the relatively poorer ones will grow faster and eventually the per capita income of all these countries will converge.
- This weaker hypothesis in generally supported by data.
- However, Conditional Convergence Hypothesis is not very helpful in explaining the persistent differences in per capita income amongst the rich and the poor countries.

Solow Model: Golden Rule & Dynamic Inefficiency

- Let us now go back from short run to long run (steady state).
- Recall that for given values of δ and n, the savings rate in the economy uniquely pins down the corresponding steady state capital-labour ratio:

$$k^*(s) : \frac{f(k^*)}{k^*} = \frac{n+\delta}{s}.$$

- We have already seen that a higher value of s is associated with a higher k^*, and therefore, a higher **level** of steady state per capita income $(f(k^*))$.
- So it seems that a higher savings ratio - though has no growth effect - may still be welcome because it generates higher standard of living (as reflected by a higher per capita income) at the steady state.
- But welfare of agents do not depend on just income; it depends on the level of consumption. So what is the corresponding level of consumption associated with the steady state $k^*(s)$?

Solow Model: Golden Rule & Dynamic Inefficiency (Contd.)

- Notice that in this model, per capita consumption is defined as:

$$\frac{C_t}{N_t} \equiv \frac{Y_t - S_t}{N_t}$$
$$\Rightarrow \quad c_t = f(k_t) - sf(k_t)$$

- Accordingly, for given values of δ and n, steady state level of per capita consumption is related to the savings ratio of the economy in the following way:

$$\begin{aligned} c^*(s) &= f(k^*(s)) - sf(k^*(s)) \\ &= f(k^*(s)) - (n+\delta)\, k^*(s). \quad \text{[Using the definition of } k^*\text{]} \end{aligned}$$

- We have already noted that if the government tries to manipulate the savings ratio (by imposing an appropriate tax on household's income and using the tax proceeds for capital formation), then such a policy would have no long run **growth** effect.

Golden Rule & Dynamic Inefficiency in Solow Model (Contd.)

- But can such a policy still generate a higher **level** of steady state per capita consumption at least?
- If yes, then such a policy would still be desirable, even if it does not impact on growth.
- Taking derivative of $c^*(s)$ with respect to s :

$$\frac{dc^*(s)}{ds} = \left[f'(k^*(s)) - (n+\delta) \right] \frac{dk^*(s)}{ds}.$$

- Since $\dfrac{dk^*}{ds} > 0$,

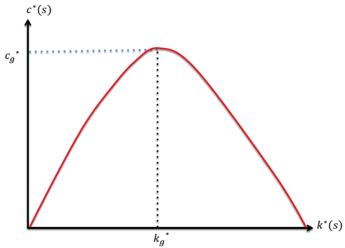
$$\frac{dc^*(s)}{ds} \gtreqless 0 \text{ according as } f'(k^*(s)) \gtreqless (n+\delta).$$

- In other words, steady state value of per capita consumption, $c^*(s)$, is maximised at that level of savings ratio and associated $k^*(s)$ where

$$f'(k^*) = (n+\delta).$$

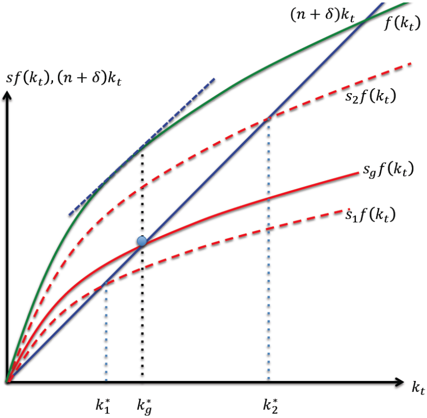
Digrammatic Representation of the Golden Rule Steady State:

- We shall denote this savings ratio as s_g and the corresponding **steady state** capital-labour ratio as k_g^* - where the subscript 'g' stand for **golden rule.**



- The point (k_g^*, c_g^*) in some sense represents the 'best' or the 'most desirable' steady state point (although in the absence of an explicit utility function, such qualifications remain somewhat vague).

Alternative Digrammatic Representation of the Golden Rule Steady State:



- There are many possible steady states to the left and to the right of k_g^* - associated with various other savings ratios.

- Importantly, all the steady states to the **right** of k_g^* are called '**dynamically inefficient**' steady states.
- From any such point one can '**costlessly**' move to the left - to a lower steady state point - and in the process enjoy a higher level of current consumption as well as higher levels of future consumption at all subsequent dates. (**How?**)
- Notice however that the steady states to the **left** of k_g^* are **not** 'dynamically inefficient'. (**Why not?**)

Cause of 'Dynamic Inefficiency' in Solow Model

- Dynamic inefficiency occurs because people oversave.
- This possibility arises in the Solow model because **the savings ratio is exogenously given - it is not chosen through households' optimization process.**
- Note that if indeed the steady state of the economy happens to lie in the dynamically inefficient region, then that in itself would justify a pro-active, interventionist role of the government in the Solow model - even though government cannot influence the long run rate of growth of the economy.

Limitations of the Solow Growth Model:

- There are three major criticisms of the Solow model.

1. It does not take into account the demand side of the economy. In fact the model *assumes* that demand is always equal to supply. (In fact there exists some post-Keynesian demand-led growth models which address this issue. However, due to paucity of time, we shall not consider those 'Keynesian' growth models in this course).
2. The steady state in the Solow model might be dynamically inefficient, because people may oversave. If one allows households to choose their savings ratio optimally, then this inefficinency should disappear. But this latter possibility is simply not allowed in the Solow model.
3. Even though the Solow model is supposed to be a growth model - it cannot really explain long run growth:

-
    - The per capita income does not grow at all in the long run;
    - The aggregate income grows at an exogenously given rate n, which the model does not attempt to explain.

Extensions of the Solow Growth Model:

- The basic Solow growth model has subsequently been extended to counter some of these critisisms.
- We shall look at one such extension:
  1. Neoclassical Growth Model with Optimizing Households: This extension allows the households to choose their consumption/savings behaviour optimally (this corresponds to the micro-founded DGE framework developed earlier)

Neoclassical Growth with Optimizing Agents:

- We shall now discuss the extension which allows for optimizing consumption/savings behaviour by households over infinite horizon: **The Ramsey-Cass-Koopmans Inifinite Horizon Framework** (henceforth **R-C-K**)
- This framework retains all the production side assuption of the Solovian economy; but households now choose their consumption and savings decision optimally be maximising their utility defined over an infinite horizon.
- This latter statement should immediately tell you that the underlying macro structure would be very similar to the DGE framework that we have constructed earlier.
- Let us revisit the underlying macro framework.

Neoclassical Growth with Optimizing Agents: The R-C-K Model

- The R-C-K model is considered Neoclassical - because it retains **all** the assumptions of the Neoclassical production function (including the diminishing returns property and the Inada conditions.)
- In fact the production side story is exactly identical to Solow.
- As before, the economy starts with a given stock of capital (K_t) and a given level of population (L_t) at time t.
- These factors are supplied **inelastically** to the market in every period. This implies that households do not care for leisure.
- Population grows at a constant rate n.
- Capital stocks grows due to optimal savings (and investment) decisions by the households.
- Notice that once again savings and investment are always identical. So just like Solow, this is a supply driven growth model.

- There are H **identical** households indexed by h.
- Each household consists of a single **infinitely lived** member to begin with (at $t = 0$). However **population within a household increases over time at a constant rate** n. (And each newly born member is infinitely lived too!)
- At any point of time t, the total capital stock and the total labour force in the economy are equally distributed across all the households, which they offer inelastically to the market at the market wage rate w_t and the market rental rate r_t.
- Thus total earning of a household at time t: $w_t N_t^h + r_t K_t^h$.
- Corresponding **per member** earning: $y_t^h = w_t + r_t k_t^h$,
  where k_t^h is the **per member capital stock** in household h, which is also the **per capita capital stock** (or the capital-labour ratio, k_t) in the economy.

The Household Side Story (Contd.):

- In every time period, the instantaneous utility of the household depends on its **per member** consumption:

$$u_t = u\left(c_t^h\right); \; u' > 0; \; u'' < 0; \; \lim_{c^h \to 0} u'(c^h) = \infty; \; \lim_{c^h \to \infty} u'(c^h) = 0.$$

- The household at time 0 chooses its entire consumption profile $\left\{c_t^h\right\}_{t=0}^{\infty}$ so as to maximise the discounted sum of its life-time utility:

$$U_0^h = \sum_{t=0}^{\infty} \beta^t u\left(c_t^h\right)$$

  subject to its period by period budget constraint.

- Notice once again that identical households implied that **per member** consumption (c_t^h) of any household is also equal to the per capita consumption (c_t) in the economy at time t.

The R-C-K Model: Centralized Version (Optimal Growth)

- There are two version of the R-C-K model:
  - A centralized version - which analyses the problem from the perspective of a social planner (who is **omniscient**, **omnipotent** and **benevolent**).
  - A decentralized version - which analyses the problem from the perspective of a perfectly competitive market economy where 'atomistic' households and firms take optimal decisions in their respective individual spheres.
- The centralized version was developed by Ramsey (way back in 1928) and is often referred to as the 'optimal growth' problem.
- In the DGE framework (discussed in Module 1) we have solved the problem both from the perspective of the social planner as well as for the perfectly competitive market economy.
- We have shown that **under rational expectations and perfect information on the part of the households**, the solution paths for the economy under the two institutional structures are identical.
- We shall therefore characterise the solution paths for only one of them, namely that of the social planner.

- From our analysis of the DGE frameowrk, we know that the dynamic optimization problem of the social planner is:

$$\underset{\{c_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}}{Max.} \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} \text{(i) } c_t &\leqq f(k_t) + (1-\delta)k_t \text{ for all } t \geqq 0; \\ \text{(ii) } k_{t+1} &= \frac{f(k_t) + (1-\delta)k_t - c_t}{1+n};\; k_t \geqq 0 \text{ for all } t \geqq 0;\; k_0 \text{ given.} \end{aligned}$$

- In Module 1, we have seen how to solve a dynamic programming problem (using the Bellman equation) to arrive at the dynamic equations characterizing the optimal paths for the control and state variable.
- Using this method, we can derive the following two dynamic equations characterizing the solution to the social planner's optimization problem as:

$$u'(c_t)[1+n] = \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right]. \tag{4}$$

$$k_{t+1} = \frac{f(k_t) + (1-\delta)k_t - c_t}{1+n}; \; k_0 \text{ given.} \tag{5}$$

- These two equations represent a $2X2$ system of difference equations which implicitly defines the 'optimal' trajectories of c_t and k_t.
- Of course, we still need two boundary conditons to precisely characterise the solution paths for this $2X2$ system.

- One boundary condition is given by the initial condition: k_0.
- The other boundary condition is provided by the Transversality condition:

$$\lim_{t\to\infty} \beta^t \left[f'(k_t) + (1-\delta)\right].u'(c_t)k_t = 0$$

- However, the dynamic equations given above are very involved and chracterizing the optimal path is not very easy.
- To simplify, let us begin by assuming specific functional forms for $u(c)$ and $f(k)$.
- Let

$$u(c_t) = \log c_t;$$

$$f(k_t) = (k_t)^{\alpha};\ 0 < \alpha < 1.$$

R-C-K Model (Centralized Version): Optimal Paths (Contd.)

- Given these specific functional forms, the dynamic equations for the centralized R-C-K model are represented by the following two equations:

$$c_{t+1} = \frac{\beta\left[\alpha(k_{t+1})^{\alpha-1} + (1-\delta)\right]}{1+n} c_t; \tag{6}$$

$$k_{t+1} = \frac{(k_t)^{\alpha} + (1-\delta)k_t - c_t}{1+n}. \tag{7}$$

- The associated boundary conditions are:

$$k_0 \text{ given; } \lim_{t\to\infty} \beta^t \left[\alpha(k_t)^{\alpha-1} + (1-\delta)\right] . \frac{k_t}{c_t} = 0$$

Characterization of the Optimal Paths:

- We are now all set to characterise the dynamic paths of c_t and k_t as charted out by the above dynamic system.
- Before that let us quickly characterize the steady state.
- Using the steady state condition that $c_t = c_{t+1} = c^*$ and $k_t = k_{t+1} = k^*$ in equations (6) and (7), the steady state values are given by:

$$\begin{aligned} k^* &= \left[\frac{\alpha\beta}{1+n-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}; \\ c^* &= (k^*)^\alpha - (n+\delta)k^*. \end{aligned}$$

- Since we have identified the nontrivial steady state, we can linearize the dynamic equations around the steady state to comment about its local stability property.
- Try that as a homework. (Messy but you can do it!)

Characterization of the Optimal Paths:

- Now let us chart out the optimal paths of c_t and k_t - starting from any given initial value of k_0. (Note that the initial value of c_0 is **not** given - it is to be optimally determined).
- We now construct the phase diagram to qualitatively characterise the solution paths for these specific functional forms.

R-C-K Model (Centralized Version): Phase Diagram

- To construct the phase diagram for this 2×2 system we have to plot the two level curves $\triangle c = 0$ and $\triangle k = 0$ in the (k_t, c_t) plane.
- The equations of these two level curves are as follows:

$$c_{t+1} - c_t \equiv \triangle c = \left[\frac{\beta \left[\alpha (k_{t+1})^{\alpha - 1} + (1 - \delta) \right]}{1 + n} - 1 \right] c_t = 0; \quad (8)$$

$$k_{t+1} - k_t \equiv \triangle k = \frac{(k_t)^{\alpha} - (n + \delta) k_t - c_t}{1 + n} = 0. \quad (9)$$

- From (8):

$$\triangle c = 0 \Rightarrow \text{either} \left[\frac{\beta \left[\alpha (k_{t+1})^{\alpha - 1} + (1 - \delta) \right]}{1 + n} - 1 \right] = 0;$$

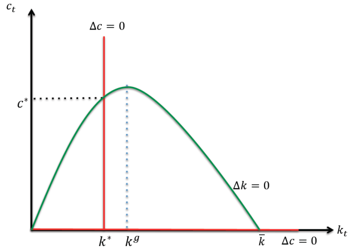
$$\text{i.e., either } k_{t+1} = \left(\frac{\alpha \beta}{(1 + n) - \beta(1 - \delta)} \right)^{\frac{1}{1 - \alpha}} = k^* = k_t; \quad \text{or } c_t = 0$$

R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- From (9):
$$\triangle k = 0 \Rightarrow c_t = (k_t)^{\alpha} - (n+\delta)k_t$$
- Thus along the $\triangle k = 0$ locus:
$$\left.\frac{dc_t}{dk_t}\right|_{\triangle k=0} \gtreqless 0 \text{ according as } \alpha(k_t)^{\alpha-1} \gtreqless (n+\delta)$$
$$\text{i.e., } k_t \lesseqgtr \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}} \equiv k_g$$
$$\text{Also, when } k_t = 0,\ c_t = 0;\ \&\ \text{when } k_t \geqq \left(\frac{1}{n+\delta}\right)^{\frac{1}{1-\alpha}} \equiv \bar{k},\ c_t = 0.$$
- A Comment: Earlier (in the context of the Solow model), we had defined the 'golden rule' capital-labour ratio as $k_g : f'(k) = n + \delta$. Now with the Cobb-Douglas production function, $f'(k) = \alpha(k_t)^{\alpha-1}$. Therefore 'golden rule' is suitably defined as: $k_g = \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}$.

R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- All these information can be put together in constructing the following phase diagram:



- This phase diagram is not complete yet. We have to put in the direction of movements of c and k, which we do next.

R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- In order to determine the direction of movements of k and c, we proceed in the following way:
  - First consider the movement of c which is captured by the $\triangle c$ function, as given by

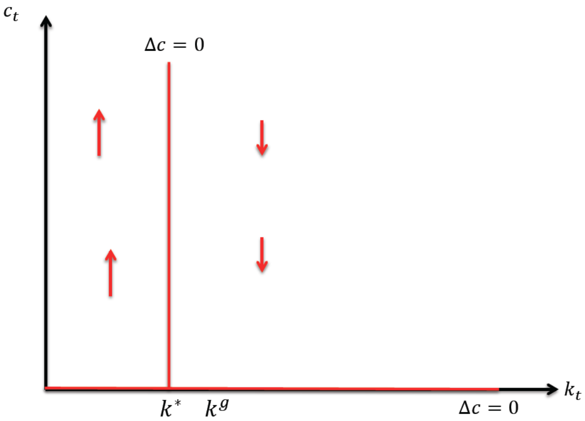
$$\triangle c = \left[\frac{\beta \left[\alpha (k_{t+1})^{\alpha - 1} + (1 - \delta) \right]}{1 + n} - 1 \right] c_t$$

  - It is easy to see that for any $c_t > 0$,

$$\begin{aligned} \triangle c &\gtreqless 0 \\ \text{according as } \frac{\beta \left[\alpha (k_{t+1})^{\alpha - 1} + (1 - \delta) \right]}{1 + n} - 1 &\gtreqless 0 \\ \Rightarrow k_{t+1} \lesseqgtr \left(\frac{\alpha \beta}{(1 + n) - \beta (1 - \delta)} \right)^{\frac{1}{1 - \alpha}} &= k^* \end{aligned}$$

R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- Accordingly, the movement of c is shown in the diagram below:



R-C-K Model (Centralized Version): Phase Diagram (Contd.)

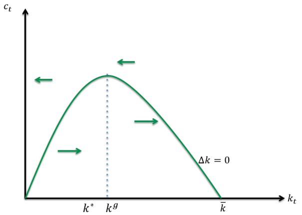
- Next consider the movement of k.
  - The movement of k is captured by the $\triangle k$ function, as given by

$$\triangle k = \frac{(k_t)^{\alpha} - (n+\delta)k_t - c_t}{1+n}$$

  - It is easy to see that

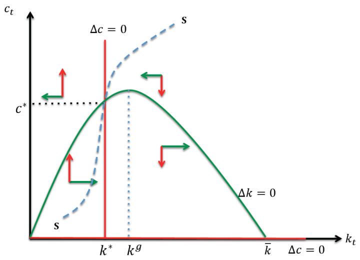
$$\begin{aligned} \triangle k &\gtreqless 0 \\ \text{according as } c_t &\lesseqgtr (k_t)^{\alpha} - (n+\delta)k_t \end{aligned}$$

  - Accordingly, the movement of k is shown in the diagram below:



R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- Combining the two, we get the complete phase diagram for this dynamic system as follows:



- There exists a unique path (denoted by SS) which will take us to the non-trivial steady state. This is indeed the optimal path, which staisfies the TVC. All other paths violate the TVC (**try proving that**).

R-C-K Model (Centralized Version): Phase Diagram (Contd.)

- Although we have drawn the phase diagram here for a specific example where the instantaneous utility function is logarithmic, the diagrammatic analysis will go through for any other utility function which belongs to the CRRA family, i.e.,

$$u(c_t) = \frac{(c_t)^{1-\sigma}}{1-\sigma}; \quad \sigma \neq 1.$$

(Verify this yourself)

R-C-K Model (Centralized Version): Growth Conclusions

- In the R-C-K model, for any given k_0, the optimal c_0 will be chosen such that the economy is on the SS path.
- Along this path, the average capital stock and average consumption approach their steady state values $(k^*.c^*)$ in the long run (i.e, as $t \longrightarrow \infty$).
- Thus the long run growth conclusions of the Solow model prevails:
  - The per capita income does not grow in the long run; it remains constatnt at $f(k^*)$ - the exact level being determined by various parameters (s, n, δ).
  - The aggregate income grows at a constant rate - given by the exogenous rate of growth of population (n).
- BUT, is the steady state now dynamically efficient? For that we have to compare this steady state with the 'golden rule'.

R-C-K Model (Centralized Version): Dynamic Efficiency

- It is easy to verify that for our specific example (with a Cobb-Douglas production function):

$$k^* = \left(\frac{\alpha\beta}{(1+n) - \beta(1-\delta)} \right)^{\frac{1}{1-\alpha}} < \left(\frac{\alpha}{n+\delta} \right)^{\frac{1}{1-\alpha}} = k_g$$

- But is it true in general, or is this an artifact of the Cobb-Douglas production function?
- To answer this question, let us go back to the generalized R-C-K model and examine whether its steady state is dynamically efficient or not.

R-C-K Model (Centralized Version): Dynamic Efficiency (Contd.)

- Recall that the dynamic equations characterizing the optimal paths of the generalized R-C-K model were given by:

$$u'(c_t)[1+n] = \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right] \quad (10)$$

$$k_{t+1} = \frac{f(k_t) + (1-\delta)k_t - c_t}{1+n} \quad (11)$$

- We know that at steady state, by definition:

$$c_t = c_{t+1} = c^*;$$

$$k_t = k_{t+1} = k^*.$$

- Using this steady state definition in the dyamic equations, the steady state for this system is given by:

$$k^* \quad : \quad f'(k^*) = \frac{(1+n) - \beta(1-\delta)}{\beta}; \quad (12)$$

$$c^* \quad = \quad f(k^*) - (n+\delta)k^* \quad (13)$$

R-C-K Model (Centralized Version): Golden Rule & Dynamic Efficiency

- Recall that the 'golden rule' capital-labour ratio is defined as:

$$k_g : f'(k_g) = (n + \delta)$$

- Given that $0 < \delta, \beta < 1$, one can again easily verify that

$$\begin{aligned} f'(k^*) &= \frac{(1+n) - \beta(1-\delta)}{\beta} > (n+\delta) = f'(k_g) \\ &\Rightarrow \quad k^* < k_g . \end{aligned}$$

- In other words, the steady state of the R-C-K model necessarily lies in the dynamically efficient region.
- This is true for any n and δ, as long as $0 < \beta < 1$ (i.e., households prefer curren consumption to future consumption). Indeed it is only when $\beta = 1$ that household will optimally reach the golden rule such that:

$$k^* = k_g .$$

R-C-K Model (Centralized Version): Golden Rule & Dynamic Efficiency (Contd.)

- Thus we see that in the centralized version of the R-C-K model, when the social planner decides on how much to save and how much to leave for households' consumption in order to maximise households' utility over infinite horizon, it ensures that the corresponding steady state is **always dynamically efficient**.
- Finally, will all these results hold for a decentralized market economy as well?
- The answer is: "yes" - because (under rational expectation, complete information and no externality) the solution path for the social planner is exactly identical to the solution path for the competitive market economy.
- Thus the growth path of the decentralized market economy will be exactly identical to the growth path of the planned economy; moreover the steady state of the market economy will be dynamically efficient too.

Solow vis-a-vis R-C-K Model: Summary

- We have now seen that all the growth conclusions of the aggregative Solow model go through in the micro-founded R-C-K model.
- Moreover, the R-C-K model also removes the possibility of dynamic inefficiency.
- Dynamic inefficiency does not arise in the R-C-K model because now households base their savings decisions on explicit optimization; under perfect information and perfect markets the scope for an inefficient solution does not arise.

Long Run Growth of Per Capita Income?

- Recall that in all the growth models discussed so far, the capital-labour ratio (k_t) in the long run becomes a constant. Hence none of them can explain the steady rise in per capita income that is historically observed in almost all economies since the industrial revolution in Europe in the late 18th century.
- Such steady growth is often explained by appealing to exogenous technological shocks.
- Can we have long run growth of per capita income in a Solow or R-C-K economy even without such exogenous shocks?
- The answer is: yes, but only if you allow some of the Neoclassical properties of the production function to be relaxed.

Long Run Growth of Per Capita Income?

- The long run constancy of the per capita income in the Solow and R-C-K model arises due to the strong uniqueness and stability property of the steady state - which in turn depends on two key assumptions: the property of diminishing returns and the Inada Conditions. One can generate long run growth of per capita income in these models if we relax one of these conditions.
- As we have seen in an earlier discussion (in the Frankel-Romer example), under certain circumstances, the aggregate production technology may not retain its neoclassical properties,even when individual firms face neoclassical technologies with diminishing returns.
- Many of the latter growth models (e.g., endogenous growth models) exploit this feature to generate perpetual long run growth.
- Due to lack of time, we shall not cover these models here. Many of them will be discussed in various optional courses later.

References:

- Reference for Solow Model & Ramsey-Cass-Koopmans Growth Model :
  - D. Acemoglu: Introduction to Modern Economic Growth, Chapter 2, Sections 2.1,2.2, 2.3 & Chapter 5, Sections 5.1,5.2, 5.3, 5.4, 5.5, 5.9.