Public Goods as Drivers of Private Investment

Ram Singh

February 2nd, 2018

The Model

We begin with a comparison of two localities (l, h) over an infinite time horizon.

- Each locality is owned by a representative individual owning fixed amount of land L
- Welfare of locality i at time t is $a_t^i \in [0, \bar{a}] \subset \mathbb{R}_+$, where $t \in \mathbb{Z}_+$
- a captures the various amenities provided by the locality to it's residents, with a_0 representing the initial amenities of *both* localities at time 0

Land can only be used to provide housing facilities and housing of locality i at time t, given by $H_t^i(L^i, K_t^i)$ is

$$H_t^i(L^i, K_t^i) = (L^i)^\alpha (K_t^i)^{1-\alpha} \text{ where } \alpha \in (0,1) \text{ and } i \in \{l, h\} \quad (0.1)$$

Note that land has no time subscript—it is fixed on a per-locality basis.

Per period profit from housing in each locality is

$$\pi_t^i(a_{t-1}^i) = p_t^i(a_{t-1}^i)H_t^i - rK_t^i \text{ with } a_0^i \text{ given for } i \in \{l, h\} \tag{0.2}$$

Here,

- r is the rental rate of capital
- $p_t^i(a_{t-1}^i)$ is the per unit price of housing services. It is dependant on the amenities in the locality *in the previous period*.
- Agents look back at the state of the locality yesterday in determining their willingness to pay for housing today.
- Specifically, we consider

$$p_t^i(a_{t-1}^i) = \frac{r}{1-\alpha}\left(a_{t-1}^i\right)^\alpha \tag{0.3}$$

Given our assumptions, this is both increasing and concave in a_{t-1}^i

Optimal Capital Investment

Substituting the value of H_t^i from (0.1) in (0.2), the FOC wrt to K_t^i is

$$(1-\alpha)p_t^i(a_{t-1}^i)\left(\frac{L^i}{K_t^i}\right)^{\alpha} - r = 0$$

$$\Rightarrow K_t^i = \left(\frac{(1-\alpha)p_t^i(a_{t-1}^i)}{r}\right)^{\frac{1}{\alpha}} L^i$$

$$\Rightarrow \frac{K_t^i}{L^i} \equiv k_t^i(a_{t-1}^i) = \left(\frac{(1-\alpha)p_t^i(a_{t-1}^i)}{r}\right)^{\frac{1}{\alpha}}$$

Here, k_t^i is the optimal capital-land ratio in locality i at time t. From (0.3), we can see that

$$k_t^i(a_{t-1}^i) = a_{t-1}^i \tag{0.4}$$

Investment in land today is simply the level of amenities yesterday. Clearly, investment is increasing in amenities.

Amenities and Investment

We now model amenities and investment. We have the following *locality-specific* amenity function which depends on the current level of investment per unit of land.

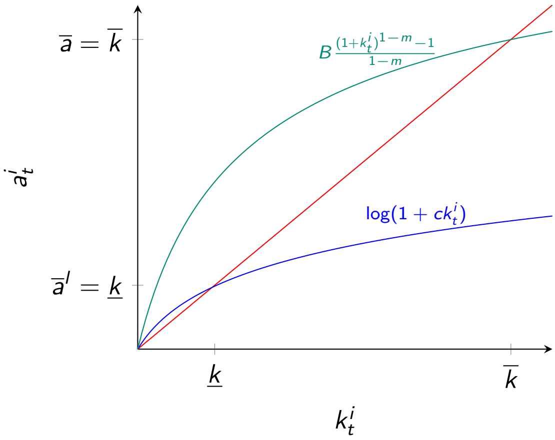
$$a_t^i(k_t^i) = \begin{cases} \log(1 + ck_t^i), & i = l \\ B\frac{(1+k_t^i)^{1-m}-1}{1-m}, & i = h \end{cases} \tag{0.5}$$

Here, c, m and B are positive constants with $B, c, m > 1$ and ${}^B/_c > 1 + \bar{a}$.

- The richer locality has a natural advantage due to higher pre-existing public investment (infrastructure, health, education)
- Because of this, amenities are more responsive to private investment (K_t^i)
- Operation of this mechanism is depicted by the use of log for locality l and iso-elastic (CRRA) for locality h.

Amenities and Investment

Both amenity functions are *increasing* and *concave*, but the log function dampens much quicker and is thus flatter than the iso-elastic functions.



A Steady State

Define $\bar{k}$ as $k : a^h(k) = \bar{a}$, i.e. the level of investment per unit land that results in maximum amenities for the high locality.
Assume that capital employed beyond $\bar{k}$ doesn't translate into higher amenities (as amenities are bounded above by $\bar{a}$), thus capital will be bounded above by $\bar{k}$.

Therefore we have

$$a_t^l : [0, \overline{k}] \to [0, \overline{k}] \text{ defined by}$$
$$a_t^l = \log(1 + ck_t^l) \tag{0.6}$$

Since

- the domain is *compact* and *convex*
- and the amenity function is *continuous* over the domain

We can use *Brouwer's fixed point theorem* to conclude that:

$$\exists\ \underline{k} \in [0, \overline{k}] : a_t^l(\underline{k}) = \underline{k}\ \forall\ t \tag{0.7}$$

A Steady State (cont)

Clearly 0 is one such trivial fixed point.
But we are interested in another. Either

- Choose $c = \frac{e^{\underline{k}}-1}{\underline{k}} > 1$ for some $\underline{k} \in (0, \overline{k})$, which guarantees the existence of such a fixed point
- Alternatively, we can find the steady state by equating $a^l_{t+1} = a^l_t$

$$k^l_{t+1} = \log(1 + ck^l_t)$$
$$\text{At steady state, } k^l_t = k^l_{t+1} = k$$
$$\Rightarrow k = \log(1 + ck)$$
$$\Rightarrow e^k = 1 + ck$$
$$\Rightarrow \frac{e^k - 1}{k} = c$$

Given our assumption on c, this implies that $k = \underline{k}$

Proposition 1

Given (0.4), (0.6), (0.7), the sequence of amenities for locality l will converge to $\underline{k}$ which is a stable level of capital

Proof.

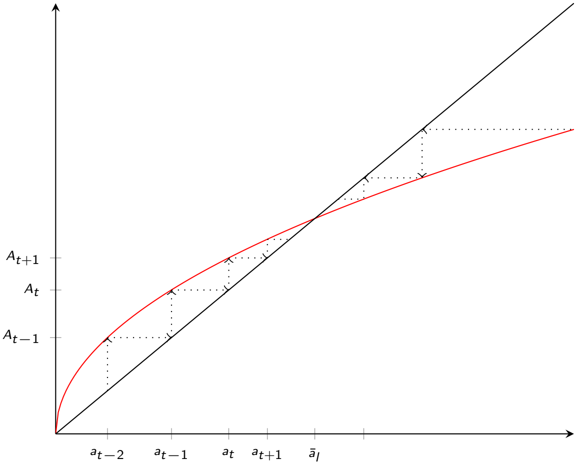
Case 1 : $a_0^l < \underline{k}$

- Define the mapping $A_t^l \equiv a_t^l \circ k_t^l \; \forall \; t$. Thus given previous period's amenity a_{t-1}^l, A_t^l gives the current period's amenity (a_t^l).
- For the given (restricted) domain, A_t^l has the following form:

$$A_t^l(a_{t-1}^l) \equiv a_t^l \circ k_t^l(a_{t-1}^l) = \log(1 + ca_{t-1}^l) \; \forall \; t$$

- Let $A_t^l(a_{t-1}^l)$ be denoted simply as A_t^l. We thus have a sequence $\{A_t^l\}_{t=0}^{\infty}$ of amenities with $A_0^l = a_0^l$ given.

Our task is to show that this sequence converges to $\underline{k}$


A_{t+1}
A_t
A_{t-1}
a_{t-2}
a_{t-1}
a_t
a_{t+1}
$\bar{a}_l$

Proof. (Cont.)

- By construction, if $a_t^l < \underline{k}$ then $\log(1 + ck_t^l) > k_t^l$.

We have thus shown that if the initial amenity level is below $\underline{k}$ then the sequence of amenities is an increasing sequence as long as each member of the sequence is less than $\underline{k}$

Since $\underline{k}$ is a fixed point of the amenity function and the amenity function is increasing and concave, we have that $\underline{k}$ is the *Least Upper Bound* of the sequence $\{A_t^l\}_{t=0}^{\infty}$ i.e. $A_t^l \leq \underline{k} \quad \forall t$.

Proof. (Cont.)

Since every increasing and bounded above sequence converges to it's *supremum*, thus $A_t^l \to \underline{k}$

Case 2 : $\underline{k} < a_0^l$

- Now $\log(1 + ck_t^l) < k_t^l$ i.e. $A_t^l < A_{t-1}^l \ \forall \ t$
- Since $\underline{k}$ is a stable steady state, the decreasing sequence $\{A_t^l\}_{t=0}^{\infty}$ converges to it's *greatest lower bound* $\underline{k}$

Now, let $\gamma(a_t^l) \equiv \frac{a_{t+1}^l - a_t^l}{a_t^l}$ be the rate of growth of amenities for locality l.

We have already shown that $a_{t+1}^l \lesseqgtr a_t^l$ for $a_t^l \gtreqless \overline{a}^l$ and therefore,

$$\gamma(a_t^l) = \frac{a_{t+1}^l}{a_t^l} - 1 \lesseqgtr 0 \quad , a_t^l \gtreqless \overline{a}^l$$

Therefore, $\bar{a}_l$ is the steady state level of amenity for locality l. ■

We now restrict attention to the other locality and look at amenity function in equation (0.5).

$$a_t^h : [0, \overline{k}] \rightarrow [0, \overline{k}] \text{ defined as}$$

$$a_t^h(k_t^h) = B\frac{(1 + k_t^h)^{1-m} - 1}{1 - m} \tag{0.8}$$

Since it is *continuous* on a *compact* and *convex* domain, by *Brouwer's fixed point theorem* we have a fixed point.

$\overline{a}$ is the desirable level of amenity for *any locality* and locality h will achieve it in the long run. To ensure this, we fix B at

$$B = \frac{\overline{a}(1 - m)}{(1 + \overline{a})^{1-m} - 1} \tag{0.9}$$

so that $a_t^h(\overline{a}) = \overline{a}$.

Since $a_t^h(.)$ is *increasing* and *concave*, there exists a unique, non-trivial fixed point at $\overline{k} = \overline{a}$.

Proposition 2

Given (0.4), (0.5), (0.8), (0.9) *and that* $a_0^h \in (0, \overline{a})$, *locality* h *will achieve the best possible amenity in the long run and this steady state is stable.*

Proof.

- Define $A_t^h \equiv a_t^h \circ k_t^h \ \forall \ t$.

$$A_t^h(a_{t-1}^h) = B\frac{(1 + a_{t-1}^h)^{1-m} - 1}{1 - m}$$

- A_t^h is the amenity of the locality at time t and it is a function of the previous period's amenity of the locality.
- Assumption (0.9) and the fact that $\overline{a}$ is the unique fixed point of an *increasing, concave* function guarantee that $\{A_t^h\}_{t=0}^{\infty}$ is an increasing sequence which converges to it's *supremum* $\overline{a}$. ■