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Allan Feldman
E-mail: allan_feldman@brown.edu
Department of Economics,
Brown University

Ram Singh
E-mail: ramsingh@econdse.org
Department of Economics,
Delhi School of Economics

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CENTRE FOR DEVELOPMENT ECONOMICS
DELHI SCHOOL OF ECONOMICS
DELHI - 110007

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EQUILIBRIA UNDER NEGLIGENCE LIABILITY: *HOW THE STANDARD CLAIMS FALL APART*

ALLAN FELDMAN[†] AND RAM SINGH[‡]

Abstract. In many accident contexts, the expected accident harm depends on the care levels and activity levels chosen by the parties involved. In an important and seminal contribution Shavell (1980) extended the scope of economic analysis of liability rules by providing a model that allows for the care and activity level choices. The subsequent works have extended the model to predict outcomes under various liability rules and also to compare their efficiency properties. These works make several claims about the existence and efficiency of equilibria under different liability rules, without providing formal proof. In this paper, examine the claims in the literature using the standard models as such. Yet, contrary to the existing claims, we show that none of the negligence based liability rules induces an equilibrium in most accident contexts admissible under the standard models. Moreover, we show that the rules of strict liability for injurer, and no-liability for injurer, generally are much more efficient than the standard negligence rules as well as the rules that require sharing of accident loss between the parties. We explain why the standard models are inherently flawed. The social optimization problem induced by it generally does not have a solution, or has solutions not discoverable by the first order conditions. Consequently, predictions emanating from the model do not gel with the real world use of liability rules. Finally, we explore the possibility of equilibria by extending the framework of analysis beyond the standard models. We provide a necessary and sufficient condition for existence of a unique Nash equilibrium under the negligence liability.

1. Introduction

In many tort settings, the probability of an accident and the harm caused by the accident depend on the care exercised by the parties involved as well as their activity levels. In a pioneering contribution, Shavell (1980) extended the scope of economic analysis of liability rules by providing a model that allows for the care and activity level choices by the parties. Shavell's model has served as a basis for much of the subsequent works, including Shavell (1987), Endres (1989), Miceli (1997 p. 29), Cooter and Ulen (2004, pp. 332-33), a Dari-Mattiacci (2002), Delhaye (2002), Goerke (2002), Parisi and Fon (2004), Singh (2006), Singh (2009), Parisi and Singh (2010), Dari-Mattiacci, Lovat and Parisi (2014), Guerra (2015), Carbonara, Guerra and Parisi (2016), Miceli (2017, Ch 2) among others. These works have used or developed or extended the main model in Shavell (1980).[1]

For the most part, the existing literature on the subject has examined the efficiency properties of strict liability, no-liability, and the standard negligence criterion-based rules;

‡ Department of Economics, Brown University. Email: allan_feldman@brown.edu.

‡ Department of Economics, Delhi School of Economics, University of Delhi. Email: ramsingh@econdse.org.

[1]For works dealing with the doctrinal foundations of liability and the related issues See Polinsky (1980), Landes and Posner (1987), Arlen (1990), Miceli (1997), Hylton (2001), Jain and Singh (2002), Feldman and Singh (2009), Cooter and Ulen (2004), BarGil and Ben Shahar (2003), Singh (2003, 2007).

namely, the rule of simple negligence, the rule of negligence with a defense of contributory negligence, the rule of negligence with a defense of comparative negligence, and the rule of strict liability with a defense of contributory negligence. In an important contribution, Dari-Mattiacci, Lovat and Parisi (2014) and Carbonara, Guerra and Parisi (2016) have extended the analysis to examine efficiency of rules that permit sharing of liability between non-negligent parties.

In this paper, we reexamine the existing claims about the equilibria under liability rules and their efficiency properties. See Shavell (1980), Shavell (1987), Endres (1989), Miceli (1997 p. 29), Cooter and Ulen (2004, pp. 332-33), a Dari-Mattiacci (2002), Delhaye (2002), Goerke (2002), Parisi and Fon (2004), Singh (2006), Parisi and Singh (2010). For discussion see Dari-Mattiacci, Lovat and Parisi (2014, page 572), Carbonara, Guerra and Parisi (2016), and Miceli (2017, Ch 2).

Specifically, we use the standard model along with its assumptions.[2] Yet, our findings dispute the following interdependent claims in the existing literature.

First, under the rule of negligence (with or without defense of contributory negligence), the injurer's activity level will be excessive, i.e., greater than the first best level of his activity. However, the victim will make efficient choices, given the inefficient activity choice by the injurer. Analogously, under the rule of strict liability with defense of contributory negligence, the victim's activity level will be inefficiently high, but the injurer's choices will be efficient.

Second, the rule of strict liability with a defense of contributory negligence is more efficient than the rule of negligence, if reduction in the injurer's activity level is relatively important for reducing the accident loss. And, the rule of negligence is more efficient than the rule of strict liability with defense of contributory negligence if reduction in the victim's activity level is relatively important for reducing accident loss.

Third, equilibria exist under the standard liability rules. This claim follows from the first two claims - otherwise, it will be pointless to talk about the outcome under liability rules. Some works have argued that the negligence criterion based liability rules induce equilibria in which the injurer and the victim opt for care levels that are appropriate from the view point of first-best efficiency.[3]

On their face, the above claims seem plausible and perhaps that is why the literature has not care to verify them mathematically. In this paper, we examine the above claims rigorously. Their intuitive appeal notwithstanding, we show that the above claims do not hold even if we strictly follow the standard model including for identification of the first best, setting of due care standards for the parties, etc.[4]

In particular, we show that for a large class of accident contexts consistent with the standard models, there do not exist Nash equilibria under any of the negligence based rules. Moreover, under the standard models, in many contexts the the rule of strict

[2]Several do not rely on standard approach, see, e.g., Emons (1990), Singh (2004, 2007b), Dari-Mattiacci et al (2014) and Carbonara, Guerra and Parisi (2016).

[3]See Miceli (1997 p. 29), Cooter and Ulen (2004, pp. 332-33), and research papers Dari Mattiacci (2002), Parisi and Fon 2004, also see Delhaye (2002) and Miceli (2017, Ch 2). For an exhaustive review of the literature on this and related issues, see Guerra (2015).

[4]These problems do no arise under the assumption of constant activity levels. For analysis with different contexts with constant activity levels See Polinsky (1980), Landes and Posner (1987), Arlen (1990), Miceli (1997), Hylton (2001), Jain and Singh (2002), Feldman and Singh (2009), Cooter and Ulen (2004), BarGil and Ben Shahar (2003), Singh (2003, 2005, 2007a and 2007b).

liability and/or the rule of no-liability are significantly more efficient than any of the standard negligence based rules.

Another strand of literature relates to the sharing of liability between non-negligent parties. For instance, in an significant contribution Dari-Mattiacci, Lovat, and Parisi (2014) have shown that the sharing of liability between non-negligent parties enhances efficiency of negligence liability rules under certain conditions. This work has been extended further in interesting contribution by Carbonara, Guerra and Parisi (2016). The results in these two works are not stated for the standard models as such. For the standard models, we show that the splitting the liability does not improve efficiency of liability rules, in general. In fact, in many accident settings, the rule of strict liability and/or the rule of no-liability are more efficient than any negligence based rules, including the rules that split liability between vigilant parties.

To investigate the above issues further, we examine the nature of the social objective function under the standard models. We show that there is a serious problem with the standard models. Even when the individual optimization problems are well defined, under the existing models the social optimization problem does not possess properties assumed in the literature. Besides, the existing models do not gel with the prevalence of negligence liability rules in the real world. As is shown below, for many accident contexts, under the standard models the rules of strict liability and no liability are significantly more efficient than the negligence liability based rules. In the real world, however, we observe that negligence based rules are much more prevalent than the all-or-nothing rules of strict liability and no-liability of injurer.

Finally, we revisit the question of existence of Nash equilibrium using a general model that by assumption does not suffer from the above problems with the existing models. Specifically, we address the following question: If we assume that social benefit function is 'well behaved', would a Nash equilibrium necessarily exist under negligence based rules? As our analysis shows, the answer is 'No.' Ensuring existence of an equilibrium requires conditions more stringent than discussed in the literature. We provide a necessary and sufficient condition for existence of a unique Nash equilibrium under the negligence based rules.

Section 2 summarizes the standard models used in the existing literature including the standard approach towards the first best and the due care levels for the parties. Section 3 shows how the claims in the existing literature do not hold, and we prove our results disputing the existing claims. This section also has some illustrative examples. In Section 4 we explain the nature of underlying problems with the standard models. In Section 5, we explore the possibility of equilibria under liability rules by extending the framework of analysis beyond the standard models. Section 6 we explain why the standard model does not gel with the real-world use of liability rules and discuss the questions that need to be addressed by the future research on the subject. Technical details and analysis are provided in the Appendix.

2. The Standard Model

2.1 The Basics

Here we described the standard model that is used in the subsequent sections. There are two people, X and Y; both are private benefit maximizing and risk-neutral. They engage in activities that creates a risk of accidents. If an accident takes place there is one injurer, person X, and one victim, person Y. All the accident costs fall initially on the victim, Y. After an accident has taken place, a court adjudicates any dispute between X and Y.

Each person chooses two things: a level of care and an activity level. The care levels for X and Y are x and y respectively. These variables are observable by X and Y, and by the court. The activity levels for X and Y are s and t, respectively. These variables are observable by X and Y. However, *the court cannot observe either activity level.* All 4 variables, s, x, t and y, are non-negative. Care levels x and y are measured in dollars. Activity levels are measured in some other units, depending on the nature of the activity.

For example, X may be the driver of a large old truck in rough condition, and Y may be the driver of an expensive new BMW. If they collide, all the damages will fall on Y. They share the same roads. Each party can vary its level of care (controlling their speed, obeying traffic signals, remaining sober, etc, all of which are translated into x or y, measured in dollars.) The levels of care are observable by both parties, and by the court. Each party can also vary its activity level, s and t for X and Y, respectively. The activity level might be miles driven, for example. The severity of any one accident might depend on x and y, but the probability of an accident in any time interval might depend on s and t, as well as on x and y.

The injurer X has a benefit function which depends on his activity level s and his care level x. His benefit is measured in dollars. In general this may be written as $u(s, x)$. Following the literature, u is a decreasing function of care level x. The benefit function u is often assumed to be an always increasing and strictly concave function of s.[5] However, some works have assumed that u starts as an increasing function of s but becomes a declining function after some point. Formally, We analyze both scenarios. Similarly, the victim Y has a benefit function $v(t, y)$ that depends on his activity level t and his care level y. This is measured in dollars; it is a strictly concave function of activity level t, and either always increasing in t or increasing at first but eventually declining. $v(t, y)$ is a decreasing function of care level y. Both benefit functions have the usual smoothness properties and are public knowledge.

If an accident occurs, the victim suffers a loss, denoted by D. Let $l(s, x, t, y)$ denotes the expected value of the accident loss. $l(s, x, t, y)$ is deceasing and convex in x and y but increasing and weakly convex in s and t. Formally, $l_x(.) < 0$, $l_y(.) < 0$, $l_s(.) > 0$, $l_{ss}(.) \geq 0$, $l_t(.) > 0$, $l_{tt}(.) \geq 0$, and $l_{st}(.) \geq 0$.[6]

The *net social benefit* (NSB) from the activities of X and Y equals the sum of their benefit functions $u(s, x)$ and $v(t, y)$, minus expected loss. This gives

$$NSB = u(s, x) + v(t, y) - l(s, x, t, y).$$

[5]E.g., see Miceli (1997), Parisi and Fon 2004, Shavell (1980) and (1987) Shavell (2007 a and b), Parisi and Singh 2009, Dari-Mattiacci, Lovat, and Parisi (2014), and Miceli (2017).

[6]That is, the parties cares as well as activities are substitutes in reducing/increasing the accident loss.

2.2 Individual Choices and Payoffs

Initially, when an accident occurs the victim Y initially incurs a loss. Afterward, a court adjudicates the dispute between X and Y. That is, the court determines what part of the loss will fall on each of the two parties. It is standard to assume that the court cannot observe s and t, but it can observe x and y, and it knows how all the variables enter the benefit functions and the $l(s,x,t,y)$ function. Accordingly, the court determines shares of damages to fall on each of the two parties, contingent on the chosen care levels x and y, the liability rule in force and the due care standard. More formally, the court determines weights (w_X, w_Y), where w_X is the fraction of the loss, D, to fall on the injurer X, and w_Y is the fraction to fall on the victim Y; $w_X = w_X(x,y)$, $w_Y = w_Y(x,y)$. For a normal rule the weights satisfy $0 \leq w_X, w_Y \leq 1$, and $w_X + w_Y = 1$.[7].

Given that the shares of accident losses are determined by the court in view of the liability rule in force, the injurer and victim act accordingly. A liability rule generates a normal form game with X and Y as players. Each wants to choose an activity level and a care level to maximize his benefit function *net of the expected damages placed on him by the liability rule.* Specifically, given t and y opted by Y, the injurer wants to choose s and x to maximize:

$$u(s,x) - w_X(x,y)l(s,x,t,y).$$

Similarly, given s and x opted by X, the victim wants to choose t and y to maximize:

$$v(t,y) - w_Y(x,y)l(s,x,t,y).$$

Note that the sum of the net benefit functions of the injurer and the victim (in the general case) is

$$u(s,x) + v(t,y) - (w_X + w_Y)l(s,x,t,y) = u(s,x) + v(t,y) - l(s,x,t,y) = NSB,$$

since the liability weights sum to 1, by assumption.

2.3 The Standard Approach

Under the *standard* model, it is common to assume that the benefit functions of the injurer and victim have the following simple form:[8]

$$u(s,x) = u_0(s) - xs \qquad \text{and} \qquad v(t,y) = v_0(t) - yt.$$

With these functions, the injurer's care x can be interpreted as money spent on care per unit of his activity level s. Similar comments apply to the victim's care. Moreover, $l(s,x,t,y)$ taken to be of the form: $l(s,x,t,y) = sth(x,y)$, where $h(x,y)$ is decreasing in both x and y.[9] Therefore, under the standard models and assumptions, and assumptions

[7]Note that w_X depends on only care levels x and y, and not on activity levels, which are unobservable to the court. See Shavell (1980, 1987), Miceli (1997), Cooter and Ulen (2004), Parisi and Fon (2004), and Dari-Mattiacci, Lovat, and Parisi (2014) and Miceli (2017)

[8]See Shavell 1980 and 1987, Miceli 1997, Parisi and Fon 2004, Shavell (2007), Parisi and Singh (2009) and Miceli (2017). These forms are a special case of the model Dari-Mattiacci at el (2014), see their page number 575 and 580.

[9]A few works have also worked with the form $l(s,x,t,y) = (s+t)h(x,y)$. (See Parisi and Fon 2004 and Singh (2007)). We will discuss this form as well.

discussed above, the net social benefit becomes[10]

$$NSB = u_0(s) - xs + v_0(t) - yt - sth(x, y).$$

It is *standard* in the literature is to assume that the social goal, and goal of the court, is to maximize net social benefit. That is, the social objective is to solve:

$$\max_{s,x,t,y} \{u_0(s) - xs + v_0(t) - yt - sth(x, y), \}.$$

The two versions of the standard models: There are two different approaches to the benefit functions $u(s, x)$ and $v(t, y)$ in the literature; in both approaches $u(0, 0) = v(0, 0) = 0$. In approach 1, $u(s, x)$ and $v(t, y)$ are increasing and concave in s and t, respectively. But in approach 2, $u(s, x)$ is increasing in s up to a level, say $\hat{s}$; beyond which it decreases in s. Similarly, $v(t, y)$ is increasing in t up to a point say $\hat{t}$. E.g., see Shavell (1980, 1987), Miceli (1997), Cooter and Ulen (2004), and Parisi and Fon (2004), Shavell (2007 a and b), Parisi and Singh (2010), Dari-Mattiacci, Lovat, and Parisi (2014), Carbonara, Guerra and Parisi (2016) and Miceli (2017).

More specifically, the standard model has following two versions:

Version 1 assumes $u_s(s, x) > 0$, $u_{ss}(s, x) < 0$ for all $s, x \geq 0$. When $u(s, x) = u_0(s) - xs$, this means $u_0'(s) > 0$, and $u_0''(s) < 0$ for all $s \geq 0$. Similarly for $v(t, y)$ and $v_0(t)$, when $v(t, y) = v_0(t) - yt$.

Version 2 assumes $u_{ss}(s, x) < 0$ for all $s, x \geq 0$. But, $u_s(s, x) > 0$ holds only up to some $\hat{s} > 0$; $u_s(s, x) < 0$ for $s > \hat{s}$. When $u(s, x) = u_0(s) - xs$, this means $u''{}_0(s) < 0$ but $u_0'(s) > 0$ up to some $\hat{s} > 0$ and $u_0'(s) < 0$ for $s > \hat{s}$. Similarly $v_{tt}(t, y) < 0$ but $v_t(t, y) > 0$ only up to a point $\hat{t}$; when $v(t, y) = v_0(t) - yt$, $v_0'(t) > 0$ only up to some $\hat{t}$.[11]

For each version, it is assumed that the social optimization problem has a unique solution, say (s^*, x^*, t^*, y^*), identifiable by solving the first-order conditions (FOCs). This approach towards the social objective and identification of the socially efficient solution is standard in the literature. The solution to the system of FOCs, (s^*, x^*, t^*, y^*), is called the *first best* profile of activity and care levels.

In this paper, we want to check validity of the claims in the literature by solving for the first best and possible Nash equilibria under liability rules, using the standard approach. To get specific results, we will work with specific functional forms consistent with the standard model. To keep things simple we take $D = 50$, but we have checked results for different values of D.

Corresponding to Version 1 of the standard model, we use $u(s, x) = u_0(s) - xs = s^{1/2} - xs$ and $v(t, y) = v_0(t) - yt = t^{1/2} - yt$. Specifically, we work with the following NSB:

Specification 1: $$NSB = s^{1/2} - xs + t^{1/2} - yt - \frac{50st}{1 + x + y}.$$

[10]This form of NSB function emerges as a special cases in Goerke (2002) and Dari-Mattiacci, Lovat, and Parisi (2014, p. 575) where focus is on general form of NSB function. Also see Hindley and Bishop (1983) or de Meza (1986).

[11]Generally, $\hat{s}$ and $\hat{t}$ depend on the given level of x and y, respectively.

It is easy to see that these $u_0(s)$ and $v_0(t)$ functions are in line with version 1 of the standard model described above. A slight complication of these benefit functions is that they are unbounded in the activity levels s and t, respectively. We take care of this problem by putting constraints $s \leq 1,000,000$ and $t \leq 1,000,000$. As is shown in the subsequent sections, our results do not depend on these constraints.

Consistent with Version 2 of the standard model, we use $u_0(s) = s^{1/2} - \delta_1 s$ and $v_0(t) = t^{1/2} - \delta_2 t$, where $\delta_1, \delta_2 > 0$. Specifically, we work with the following NSB:

Specification 2: $$NSB = s^{1/2} - xs - \delta_1 s + t^{1/2} - yt - \delta_2 t - \frac{50st}{1+x+y}.$$

Note that the benefit functions $u_0(s)$ and $v_0(t)$ used for specification 2 are not unbounded. So, it is no longer be necessary to constrain s and t. For ease of illustration for the most part we will work with $\delta_1 = \delta_2 = 0.01$. However, we have checked our results for $\delta_1 \neq \delta_2$.

For Specification 1, the first-order conditions for maximizing NSB, corresponding to equations (1), (2), (3), and (4) above, are as follows:

$$(1') \qquad (1/2)s^{-1/2} - x - \frac{50t}{1+x+y} = 0,$$

$$(2') \qquad -s + \frac{50st}{(1+x+y)^2} = 0,$$

$$(3') \qquad (1/2)t^{-1/2} - y - \frac{50s}{1+x+y} = 0,$$

$$(4') \qquad -t + \frac{50st}{(1+x+y)^2} = 0.$$

After some manipulation this gives:

$$(1'') \qquad s = (1/4)(1+2x+y)^{-2},$$

$$(2'') \qquad 50t = (1+x+y)^2,$$

$$(3'') \qquad t = (1/4)(1+x+2y)^{-2},$$

$$(4'') \qquad 50s = (1+x+y)^2.$$

From (2″) and (4″), we get $s = t$. Then from (1″) and (3″) we get $x = y$. The reduced system of equations has only one non-negative real solution, given by $x = 0.355473$ and $s = 0.0585468$. So, solving the first-order conditions, we get the following solution:

$$(s^*, x^*, t^*, y^*) = (0.0585468, 0.355473, 0.0585468, 0.355473).$$

All numerical computations in the paper are done using Mathematica. *The detailed calculations are appended at the end of this file.*

For Specification 2, the first-order conditions for maximizing NSB, corresponding to equations (2) and (4) above, are the exactly the same as (2′) and (4′). However, the first-order conditions corresponding to equations (1) and (3) above will be:

$$(1''') \qquad (1/2)s^{-1/2} - x - 0.01 - \frac{50t}{1+x+y} = 0,$$

$$(3''') \qquad (1/2)t^{-1/2} - y - 0.01 - \frac{50s}{1+x+y} = 0,$$

For Specification 2, the system of the first-order conditions has only one non-negative solution. Now we have the following:

$$(s^*, x^*, t^*, y^*) = (0.0582945, 0.353629, 0.0582945, 0.353629).$$

2.4 The Standard Liability Rules

Under the standard approach, The court sets due care standards by using the solution to the first-order condition equations for (s^*, x^*, t^*, y^*). The court cannot observe s and t, but it can observe x and y. Accordingly, under standard negligence based liability rules, the court sets care standards for negligence-based rules at x^* and y^* for X and Y, respectively. That is, for the standard negligence based rules, the following axiom holds:

Axiom (A1): Party X will be found *negligent* if and only if $x < x^*$, and similarly, party Y will be found *negligent* if and only if or $y < y^*$.

A *standard negligence-based liability rule* determines shares of damages to fall on each of the two parties, contingent on x, x^*, y and y^*. The leading negligence-based liability rules can be described as:

1. Simple negligence. This rule says $w_X = 1$ and $w_Y = 0$ (all the loss is placed on the injurer) if and only if $x < x^*$ (the injurer is negligent). Otherwise, $w_X = 0$ and $w_Y = 1$ (all the loss stays with the victim).

2. Negligence with a defense of contributory negligence. This rule says $w_X = 1$ and $w_Y = 0$ if and only if $x < x^*$ and $y \geq y^*$ (the injurer is negligent and the victim is non-negligent). Otherwise, $w_X = 0$ and $w_Y = 1$.

3. Strict liability with a defense of contributory negligence. This rule says $w_X = 1$ and $w_Y = 0$ if and only if $y \geq y^*$. Otherwise, $w_X = 0$ and $w_Y = 1$.

4. Comparative Negligence. This rule says $w_X = 1$ and $w_Y = 0$ if and only if $x < x^*$ and $y \geq y^*$; $w_X = 0$ and $w_Y = 1$ if and only if $x \geq x^*$ (the injurer is non-negligent); and when $x < x^*$ and $y < y^*$ (both are negligent) the loss is split in proportion to their degrees of negligence.

Besides, the above rules we consider the following negligence liability based rules.

5. The 50/50 split liability when both are negligent. This rule says $w_X = 1$ and $w_Y = 0$ if and only if $x < x^*$ and $y \geq y^*$; $w_X = 0$ and $w_Y = 1$ if and only if $x \geq x^*$; and $w_X = 1/2$ and $w_Y = 1/2$ (the loss is split 50/50) when $x < x^*$ and $y < y^*$ (both are negligent).

6. The 50/50 split liability when both are non-negligent. This rule says $w_X = 1$ and $w_Y = 0$ if and only if $x < x^*$; $w_X = 0$ and $w_Y = 1$ if and only if $y < y^*$ and $x \geq x^*$; and $w_X = 1/2$ and $w_Y = 1/2$ (the loss is split 50/50) when $x \geq x^*$ and $y \geq y^*$ (both are non-negligent).

Another important property of the standard negligence-based liability rules (rules 1-4 above) is this: When an accident occurs, if a court finds that one of the parties is negligent while the other is not, it places all the damages on the negligent party. We state this property as:

Axiom (A2): For all $x \in X,\ y \in Y$,

$[x \geq x^{*}\& \;\; y < y^{*} \Rightarrow \;\; w_X = 0]$ and $[x < x^{*}\& \;\; y \geq y^{*} \Rightarrow \;\; w_X = 1]$.

Both of our 50/50 split liability rules (5 and 6) also satisfy this property. In contrast two real-world negligence rules do not satisfy these properties. These are:

7. *Strict liability for injurer.* This rule says $w_X = 1$ and $w_Y = 0$ for any x, x^*, y, or y^* (always).

8. *No liability for injurer.* This rule says $w_X = 0$ and $w_Y = 1$ for any x, x^*, y, or y^* (always).

3. Existing Claims Revisited

In this section we show that the claims in the existing literature about equilibrium under negligence based liability rules and their efficiency properties do not hold for many accident contexts under the standard models.

To put the existing claims in perspective, Version 1 of the standard model, as described above, is used in Miceli (1997), Cooter and Ulen (2004), and Singh (2006), and Shavell (2007 a and b), among others. Version 2 as described above is used in Shavell (1980, 1987 pages 27-29 and 45) and Miceli (2017). It arises as a special case of the models in Endres (1989), Parisi and Fon (2004), Parisi and Singh (2010) and in Dari-Mattiacci at el (2014, see their page number 575).

These works have argued that under each of the standard negligence based rules (rules 1-4 mentioned in Section 2.4) an equilibrium exists. Specifically, the existing literature uses the solution of first order conditions to identify the socially optimum care and activity levels. Using the social optimum care levels set due care standards, the literature makes the following unproved claims: Equilibrium exists under the rule of negligence as well as under the rule of negligence with the defense of contributory negligence. In equilibrium, injurers will opt for the due care levels but their activity levels will be excessive. Given these choice of the injurer, the choices of the victims will be efficient. The equilibrium is also claimed to exist under the rule of strict liability with defense of contributory negligence, such that the victims will take optimal care but their activity levels will be excessive; given this, the injurers' behavior is claimed to be efficient.

We work with NSB functions that meets all the requirements of standard models. Moreover, we strictly follow the standard procedure towards the identification of the first best and the setting of due care standards for the parties. Yet, we show that an equilibrium does not exist under any of the standard negligence based rules; neither it exists under the rule of strict liability with defense. Since our specifications of the NSB function follow from the standard models, this means that claims in the literature about existence of equilibrium do not hold for the standard model, as such.

As to the benefits of splitting accident loss between the parties, for a special types of accident contexts where $l(s, x, t, y)$ can be separated as $f(s, x) + g(t, y)$, Dari-Mattiacci, Lovat, and Parisi (2014) argue for the sharing of liability between the non-negligent parties as it can enhance efficiency of liability rules by incentivizing the parties to moderate their activity levels.[12] Carbonara, Guerra and Parisi (2016) has extended the scope of this argument to more general settings.

We show that under the standard models in many contexts where expected loss function is not separable, all-or-nothing rules of strict liability and no liability are significantly more efficient than the loss sharing rules.

Another strand of literature suggests that efficiency of rule of negligence can be improved by raising the due care standard for injurers. See Goerke (2002) and Shavell (2007).[13] Contrary to these claims, in the next section we show that for a large class of accident contexts, economic efficiency is improved by reducing the injurer's due care standard to absolute zero. Similarly, efficiency of the rule of strict liability with a defense of contributory negligence is increased by reducing the due care standard for the victim to zero.

Below, we work with Specifications 1 and 2 to prove that none of the standard negligence liability rules has a Nash equilibrium. Moreover, we show that the strict liability and no-liability rules are more efficient than the standard negligence based rules as well the liability sharing rules. First we prove the claims about non-existence of equilibria under the negligence liability based rules and relative inefficiency of these rules. Later, we show how these claims extend to a much wider class of accident contexts under the standard models.

Claim 1. There is no Nash equilibrium under the standard rule negligence.

Here is why the claim holds. Let's start with Specification 1. Consider a choice of care level, say x, by party X. Recall, under this specification, $x^* = 0.355473$. The following cases arise.

Case 1: $x > x^*$. Obviously, under simple negligence there cannot be a Nash equilibrium in which party X opts for $x > x^*$.

Case 2: $x = x^*$. Under simple negligence rule by choosing x^*, X ensures that all damages falls on Y. So he will solve for the payoff maximizing s to go along with x^*, i.e. he will solve: $\max_s\{s^{1/2} - sx^* = s^{1/2} - 0.355473s\}$; it can easily be checked that his payoff maximizing activity choice is $s = 1.978455$. That is, in this case he will choose the pair $(x^* = 0.355473, s = 1.978455)$. Given these choices by X, party will choose t and y that solves

$$\max_{t,y}\{t^{1/2} - yt - \frac{50 \times 1.978455 \times t}{1 + 0.355473 + y}\}.$$

Party Y's best response, identified by the FOCs, is to choose $t = 0.000727586$ and $y = 8.59052$. In other words, for the choice of x^* by party X to be part of a Nash equilibrium, the following should hold: the choice of $(x^* = 0.355473, s = 1.978455)$ by party X and choice of $(y = 8.590, t = 0.000727586)$ by party y should be mutually best responses for party X and Y, respectively. But given that $(y = 8.590, t = 0.000727586)$ is chosen by

[12] Splitting of liability between the vigilant parties can potentially dilute the incentives of parties to adhere to the due care levels. See Parisi and Fon (2004) and Parisi and Singh (2010). However, when activity levels are constant, it is possible to design rules that allow sharing of liability between both negligent and vigilant parties, in proportion their degrees of negligence/vigilance, and yet maintain incentives for efficient care. See Feldman and Singh (2009) and (2011); Singh (2007a), (2007b).

[13] The latter work makes this argument for unilateral accidents. For discussion see Singh (2006) and Dari-Mattiacci, Lovat, and Parisi (2014).

Y, ($x^* = 0.355473, s = 1.978455$) is not a best response for X. At x^* and $s = 1.978455$ payoff of X is 0.70329. However, if he opts for $x = 0$ with $s = 17374.66$, his payoff increases to 65.9065. (See lines 1 and 2 of Table 1) This means that a Nash equilibrium cannot have party X choosing x^*.

Table 1 (Spcification 1)

	s	x	t	y	Liability	X's benefit Net of Damages	Y's benefit Net of Damages
1	1.978455	0.355473	0.000727586	8.5905226	All Y	0.70329	0.0134869
2	17374.66	0.000000	0.000727586	8.5905226	All X	65.9065	0.0207
3	any s	$x < x^*$	1000000	0.000000	All X	< 0.0000176789	1000.000
4	1.978455	0.355473	1000000	0.000000	All Y	0.70329	-72979213

Case 3: $x < x^*$. Finally, consider the case, $x < x^*$. This would make X negligent under simple negligence liability. So, all damages will fall on X, no matter what Y does. Therefore Y will set his care level $y = 0$, and will choose the largest t possible, which is $t = 1,000,000$. Now, given $y = 0$ and $t = 1,000,000$, in the region $x < x^*$ the optimization problem for X is

$$\max_{s,x}\{s^{1/2} - xs - \frac{50s \times 1,000,000}{1 + x + 0}\}.$$

This is a concave optimization problem with the global maximum identified by the first-order conditions at $x = 7070.07$, $s = 1.25018E^{-9}$. At this point, value of the payoff function is 0.0000176789. Elsewhere it takes a lower value. Specifically, for any given s and x such that $x < x^*$, an arbitrarily small movement in the direction of $x = 7070.07$, $s = 1.25018E^{-9}$ will give higher payoffs to X. Hence, in the domain $x < x^*$ there is no best response for Party X. Moreover, given $y = 0$ and $t = 1,000,000$ opted by Y, party X is better off choosing $x^* = 0.355473$ and $s = 1.978455$ as it gives hims payoff of 0.70329; a choice involving $x < x^*$ gives him less than 0.0000176789. See lines 3 and 4 of Table 1. Therefore, a Nash equilibrium is not possible at $x < x^*$.

Similarly, a Nash equilibrium is not possible under Specification 2. Recall, in this case $x^* = .353629$. Repeating the steps in Cases 1 and 2 above but using the Table 2 below, it can easily be seen that there cannot be that a Nash equilibrium with $x > x^*$ and $x = x^*$ opted by X.

Table 2 (Specification 2)

	s	x	t	y	Liability	X's benefit Net of Damages	Y's benefit Net of Damages
1	1.890707	0.353629	0.000762925	8.3693700	All Y	0.68751	0.0138105
2	1262.607	0.000000	0.000762925	8.3693700	All X	17.76659	0.02123
3	any s	$x < x^*$	2500.000	0.000000	All X	< 0.000354049	25.0000
4	1.890707	0.353629	2500.000	0.000000	All Y	0.68751	-174571

So, consider the case of $x < x^*$. Now Y will set his care level $y = 0$, and will choose the $t = 2,500$ to maximize his payoff. In the region $x < x^*$ the optimization problem for X is

$$\max_{s,x}\{s^{1/2} - xs - 0.01s - \frac{50s \times 2,500}{1 + x + 0}\}.$$

This is also a concave problem. The FOCs lead to solution: $x = 352.553$, $s = 5.01E^{-7}$. The function takes a maximum value of 0.000354049 at the solution of the FOCs, i.e., $x = 352.553$, $s = 5.01E^{-7}$. Again there is no best response for X in the $x < x^*$ region. Moreover, given $y = 0$ and $t = 2,500$, opted by Y, choice of $x^* = 0.353629$ and $s = 1.890707$ gives X payoff of 0.68751; any other choice gives him strictly lower payoffs. Again, a Nash equilibrium is not possible at $x < x^*$. □

Next, consider the rule strict liability with a defense of contributorily negligence rule. This rule is the mirror image of the simple negligence. Swapping party X for party Y and Y for party X, in view the symmetry of the functional forms, arguing along the lines in the above claim, it can be seen that the following claim holds.

Claim 2. There is no Nash equilibrium under the standard rule of strict liability with a defense of contributorily negligence.

Next, we have the following claim.

Claim 3. There is no Nash equilibrium under the standard rule of negligence with a defense of contributory negligence.

As under simple negligence, under this rule also party X has no liability as long as $x \geq x^*$. Moreover, as long $x \geq x^*$, the payoffs and the incentive structures are the same for both parties as under the simple rule of negligence, regardless of the choice of s by X and of t and y by party Y. So, it is easy to see that there cannot be a Nash equilibrium involving choice of $x \geq x^*$ under the rule of negligence with a defense of contributory negligence. Therefore, only possibility is a choice of $x < x^*$.

Suppose there is a Nash equilibrium in which party X chooses a $x < x^*$. As to the choice of y by party Y, when $x < x^*$ a choice of $y > y^*$ is never a best response. So, there are two possibilities for a Nash equilibrium: $y < y^*$ or $y = y^*$. In the former case, i.e., when $y < y^*$, party Y liable is under the rule of Negligence with a defense of contributory negligence, regardless of the choices made by party X. In such a scenario, under Specification 2, in equilibrium X must set his care level $x = 0$, and $s = 2,500$. So, under in the region $y < y^*$ the optimization problem for Y is

$$\max_{t,y}\{t^{1/2} - yt - 0.01t - \frac{50t \times 2,500}{1 + y + 0}\}.$$

This is a concave optimization problem. (Note. it is easy to see that it has the same solution as the last problem in proof of Claim 1.) The system of FOCs identifies solution at $y = 352.553$, $t = 5.01E^{-7}$. Value of the function at the solution is 0.000354049. Elsewhere it takes a lower value. Specifically, given $x = 0$ and $s = 2,500$ opted by X, a choice involving $y < y^*$ by party Y gives him a payoff less than 0.000354049. However, given $x = 0$ and $s = 2,500$ opted by X, party Y is better off choosing y^* and $t = 1.890707$ as it gives him payoff of 0.68751. Therefore, when $x < x^*$ and $y < y^*$ a Nash equilibrium is not possible.

Finally consider the case where party X chooses some $x < x^*$ but party Y opts for $y = y^*$. But, this would mean that all damages will fall on X, as long as Y keeps his $y = y^*$. With $y = y^*$, the unique best choice for party Y is to choose $t = 1.890707$.

However, given these choices by Y, as long as $x < x^*$ party X remains fully liable. So, in the region $x < x^*$ party X would want to maximize

$$\max_{s,x}\{s^{1/2} - xs - 0.01s - \frac{50s \times 1.890707}{1 + x + 0.353629}\}.$$

The first-order conditions with respect to s and x have a unique solution: $s = 0.000762925$ and $x = 8.36937$. In particular, in the region $x < x^* = 0.353629$, there is no best response choice for party X. It can be seen that party X is better of choosing $x^* = 0.353629$ and $s = 1.890707$ and getting a payoff of 0.22025, rather than any other choice involving $x < x^*$ that will give him less than 0.0138105. Again, a Nash equilibrium with $x < x^*$ is not possible.

Similarly, it can be seen that there is no Nash equilibrium with Specification 1 either. □

In fact, arguing along the lines of the above claim, for this rule we can make the following claim about the rule of 50/50 split liability when both are negligent.

Claim 4. There is no Nash equilibrium under the standard rule of 50/50 split liability when both are negligent.

This rule differs from the rule of simple negligence and the rule of negligence with a defense of contributory negligence only in the sub-domain of $x < x^*$ and $y < y^*$. Specifically, in the view of the above proofs, it is straightforward to see that under the rule of 50/50 split liability there cannot be an equilibrium in which X opts for $x \geq x^*$; or when X opts for $x < x^*$ and Y opts for $y \geq y^*$. Therefore, we have check existence of a Nash equilibrium only in the region $x < x^*$ and $y < y^*$. In this region, for Specification 1, the optimization problem for X is: Given t, y, solve

$$\max_{s,x}\{s^{1/2} - xs - \frac{50st}{2(1 + x + y)}\}.$$

The optimization problem for Y is: Given s, x, solve

$$\max_{t,y}\{t^{1/2} - yt - \frac{50st}{2(1 + x + y)}\}.$$

These optimization problems give us the following set of first-order conditions:

$$\begin{aligned} (1/2)s^{-1/2} - x - \frac{25t}{1 + x + y} &= 0, \\ -s + \frac{25st}{(1 + x + y)^2} &= 0 \end{aligned}$$

and

$$\begin{aligned} (1/2)t^{-1/2} - y - \frac{25s}{1 + x + y} &= 0, \\ -t + \frac{25st}{(1 + x + y)^2} &= 0. \end{aligned}$$

This system of FOCs has a unique solution: $s = 0.086245$ and $x = 0.234187$, $t = 0.086245$ and $y = 0.234187$. Moreover, in the region $x < x^*$ and $y < y^*$, the choice of

$s = 0.086245$ and $x = 0.234187$ by party X is a best response to the choice of $t = 0.086245$ and $y = 0.234187$ by party Y, and vice-versa. At these symmetric choices, each party gets a payoff of 0.14948185. However, if X unilaterally deviates to $x^* = 0.355473$ and $s = 1.978455$, it gives him a higher payoff, 0.70329. Hence, in the sub-domain of $x < x^*$ and $y < y^*$ there cannot exist a Nash equilibrium under the rule.

Similarly, it can be seen that for Specification 2 also, there is no Nash equilibrium under the rule.

□

Next, turn to the rule of comparative negligence. In view of the above arguments, it is obvious that under the comparative negligence rule there cannot be an equilibrium in which X opts for $x \geq x^*$. Moreover, under the rule of comparative negligence there cannot exist a symmetric Nash equilibrium. To see, consider Specification 2. In view of the arguments presented for Claim 4, it is easy to see that the only candidate for a symmetric Nash equilibrium is: $s = 0.086245$ and $x = 0.234187$ opted by party X, and $t = 0.086245$ and $y = 0.234187$ chosen by party Y. However, if X unilaterally deviates to $x^* = 0.353629$ and $s = 1.890707$, he gets higher payoff. Similarly, under Specifications 2 there cannot exist a *symmetric* Nash equilibrium under the standard rule of comparative negligence.

Due to the complexity of the calculations involved, we have not been able to rule out possibility of an asymmetric Nash equilibrium with both parties negligent. However, an equilibrium with both parties negligent is unlikely. In particular, it is obvious that there cannot be an equilibrium in which X opts for $x \geq x^*$.

Next we turn to the relative efficiency of the rules discussed above. We have the following claim.

Claim 5. The rule of strict liability and the rule of no liability are more efficient than any of the negligence-based rules 1-6 described above, including the rule of 50/50 split liability when both are non-negligent.

For complete proof see Appendix I. As the proof shows, under the strict liability there is a Nash equilibrium and the value of the NSB at the equilibrium is as high as it can be under any liability rule. In contrast, in view of the above results, there is no Nash equilibrium under the negligence-based rules 1-5. Whatever choices parties end up making, they are not as efficient as under strict liability. A similar logic applies vis-a-vis the rule of no liability.

Next, consider the rule of 50/50 split when both parties are non-negligent. This rule allows sharing of liability between non-negligent parties. Moreover, for the NSBs under the two specifications, a Nash equilibria does exists under the rule and is seemingly along expected lines; each party adheres to the due care standard and opts for moderate activity levels. But there are surprises here also. As is shown in the Appendix, the rule of strict liability for the injurer, and the rule of no liability for the injurer, are both more efficient than the negligence-based 50/50 split rules. That is, loss sharing rule is significantly less efficient than the all-or-nothing rules of strict liability and the rule of no-liability for the injurer.

Beyond Specifications 1 and 2: The problem of non-existence of Nash equilibrium is not confined to the above specifications. Even if we modify the functional forms for instance by substituting $0.01s$ and $0.01t$ with $0.001s$ and $0.001t$, respectively, or if we use non-symmetric benefit functions, say $s^{1/2} - xs - .01s$ for party X and $t^{1/2} - yt - .001t$ for party Y, a Nash equilibrium continues to elude us. We have also worked with $st = s + t$. This leads to the following form of

$$NSB = s^{1/2} - xs + t^{1/2} - yt - \frac{50(s+t)}{1+x+y}.$$

Under this specification as well there are no equilibria under any of the negligence rules.[14] We will return this issue again in Section 5.

Moreover, as is shown in Appendix II, for a wide class of accident contexts rule of strict liability and the rule of no liability for the injurer are more efficient than negligence based rule. In fact, these all or nothing rules will be more efficient than any of the negligence-based rules 1-6, even if the due care standards were set at levels different from x^* and y^*, respectively.

Examples

Here we present some illustrative examples. These examples provide several counter-intuitive insights on functioning of the standard liability rules. As in the standard negligence liability rules, (x^*, y^*) is used to set due care standards. Our examples show what happens when the parties X and Y make th reacting to the legal rule and to each other. For illustration we use a dynamic process, called a *Cournot dynamic process*, in which X and Y take turns moving. However, it should be noted that the process is used only for illustrative purpose. As is shown above, the results do not depend on use of this process.

Our examples deals with the accident settings as described below.

	Specification	$\phi(s,t)$	Liability Rule	What Happens?
Example 1	Spec. 1	st	Simple negligence	Cycle but no Nash Equilibrium
Example 2	Spec. 2	st	Simple negligence	Cycle but no Nash Equilibrium
Example 3	Spec. 2	st	negligence with defense of contributory negligence	Cycle but no Nash Equilibrium
Example 4	Spec. 2	st	50/50 split, both negligence	Cycle but no Nash Equilibrium
Example 5	Spec. 2 $\delta 1 \neq \delta 2$	st	Simple negligence	Cycle but no Nash Equilibrium
Example 6	Spec. 1	s+t	Simple negligence	Cycle but no Nash Equilibrium
Example 7	Spec. 2	s+t	Simple negligence	Cycle but no Nash Equilibrium

It is plausible to assume that under a negligence liability rule, (s^*, x^*, t^*, y^*), specifically the due care levels, (x^*, y^*), provide a 'focal' point for the parties.[15] Therefore, let the dynamic process starts at (s^*, x^*, t^*, y^*), and then allow X and Y to make a sequence of moves, starting with X, followed by Y, followed by X, and so on. When it's X's turn to move, he responds to the legal rule, (how will his choice of x affect w_X?), and he

[14] See Feldman and Singh (2019).

[15] On role of focal points in games see Basu (2018)

responds to what Y did at the previous stage. When it's Y's turn to move, he responds to the legal rule (how will his choice of y affect w_Y?) and to what X did at the previous stage.

As is shown on line 2 of each table, the injurer X reacts to solution FOC: He takes as given Y's choice of (t^*, y^*) as in the solution FOC, and chooses a benefit maximizing pair (s, x) for himself. In making his choice, X uses his benefit function, the expected damages function, and the liability rule which governs how expected damages are distributed. On line 2 of each table we will show the (s, x) numbers that X chooses in **bold**, and on the same line we also show the (t^*, y^*) numbers that Y is assumed to start with in non-bold.

Line 2 shows how the injurer X reacts to the variables in line 1, Solution FOC. He takes as given victim Ys variables (t*,y*), and chooses a benefit maximizing pair (s,x) for himself. In making his choice, Y uses his benefit function, the expected damages function, and the liability rule which governs how expected damages are distributed. On the line 3 of each table we show the (t, y) numbers that Y chooses in **bold**, and the (s, x) numbers to which he responds, in non-bold.

This dynamic process goes on, with X's move, followed by Y's move, followed by X's move, until it reaches a Nash equilibrium, or a repeating cycle, or some other strange result.

In each table for each example, and on each line, we also show the liability assignment given the variables of that line. ("All X" means that given the (s, x, t, y) variables of the line - and only x and y matter for liability - and given the liability rule, all the damages will fall on X; "All Y" means all the damages will fall on Y; "50/50" means half the damages will fall on X and half on Y.) Also in each table on each line we show the net benefit amounts for X and for Y; that is, each party's benefits net of any damages for which that party is liable at that stage.

In the notes beneath each table in each example we show the liability rule (e.g., for simple negligence, "X is liable if and only if $x < x^*$".) We also show any special constraints on the variables. A general constraint, used in all the examples, is that s, x, t, and y are all always non-negative. We will not repeat this in each of the examples.

Example 1 Assumptions: Simple negligence, Specification 1.
Result: No equilibrium but a Cycle!

Consider the following table based on the simple negligence rule and Specification 1. For this case we discuss all the steps in details. For the other cases discussed through remaining examples, the process is similar and hence we will discuss only the main points.

Line 1 of the table shows the first-order conditions solution, i.e., (s^*, x^*, t^*, y^*). This profile serves a the focal point for the remaining steps. At line 2 X chooses s and x. With respect to his choice of x, because of the liability rule he will not increase it to any level above x^* because x is costly and there would be no gain: under simple liability all damages fall on Y once X is choosing any x greater than or equal to x^*. So, he will either keep his care level at $x^* = 0.355473$ or reduce it. We have calculated what (s, x) would be the best choice for him if he (1) opts to have all damages fall on Y by choosing $x = x^* = 0.355473$, or if he (2) opts to have all damages fall on himself by choosing a smaller x. It turns out option 1 is better. So he sets $x = x^*$ and then solves for the best s to go along with x^*; this leads to $s = 1.978455$. At line 3, Y chooses t and y, in response to x^* and $s = 1.978455$ chosen by X. All damages will fall on Y because X has chosen $x = x^*$, and in response Y chooses the t and y shown in bold.

At line 4, it is again X's turn. He again makes the choice between (1) opting to have all the damages fall on Y by choosing $x = x^* = 0.355473$, or (2) opting to have all damages fall on himself by choosing a smaller x. This time he finds option 2 is better. His first-order maximization conditions point to a negative x, which is not allowed. He choses $x = 0$ instead, and the large $s = 17374$ shown - in any case, he is better off choosing $x = 0$ and $s = 17374$, given the t and y from line 3.

At line 5 Y chooses. Since X has just chosen $x = 0$, by simple liability all damages will fall on X, no matter what Y does. Therefore Y sets his care level $y = 0$, and chooses the largest t allowed. If we use the constraint $t \leq 1,000,000$, Y will choose $t = 1,000,000$; otherwise he will raise activity level beyond limits.

Example 1 Table

	s	x	t	y	Liability	X's benefit Net of Damages	Y's benefit Net of Damages
1	0.058547	0.355473	0.058547	0.355473	All Y	0.22115	0.12098
2	**1.978455**	**0.355473**	0.058547	0.355473	All Y	0.70329	-3.16388
3	1.978455	0.355473	**0.000728**	**8.590526**	All Y	0.70329	0.01349
4	**17374.66**	**0.000000**	0.000728	8.590526	All X	65.9065	0.0207
5	17374.66	0.000000	**1000000**	**0.000000**	All X	-8.69E+11	1000.000
6	**1.978455**	**0.355473**	1000000	0.000000	All Y	0.70329	-72979213
7	1.978455	0.355473	**0.000728**	**8.590526**	All Y	0.70329	0.013486

Notes:
X is liable if and only if $x < x^*$.
Constraint: $s, t \leq 1,000,000$.
Line 1 solves first order conditions to get (s^*, x^*, t^*, y^*).
Line 2: X chooses s and x. He chooses to keep x at x^*, and increase s to 1.978.
Line 3: Y chooses t and y. FOC's give interior solution: $t = .000728$ and $y = 8.5905$.
Line 4: X chooses s and x. FOC implies negative x, so $x = 0$ is used. FOC implies $s = 17374.6$.
Line 5: Y chooses t and y. Since $x = 0$, X pays damages; therefore Y sets $y = 0$, and wants t large.
Line 6: X chooses s and x. He must choose $x = 0.355473$ to escape huge damages. FOC gives s= 1.978.
Line 7: Y chooses t and y. Back to line 3! Cycle!

At line 6, X chooses. He sets $x = x^*$ to escape the huge damages, and solves a first-order condition to get $s = 1.978455$. At line 7 it is Y's turn to choose. But Y's choice at line 7 is exactly the same as his choice had been at line 3. We are in a cycle![16]

[16]It is easy to check that if we fix $s = t = 1$ a cycle will not be possible; instead, a unique Nash equilibrium will exits under negligence rule.

We conclude that a Cournot dynamic process, starting from a reasonably chosen initial point, may lead to a bizarre cycle rather than to a Nash equilibrium. Also, note that the logic of the cycle in lines 3 to 7 will remain even if we relax the constraint $t \leq 1,000,000$.

In fact, as is proved in the next subsection for above Specifications 1 and 2 there cannot exist a Nash equilibrium under negligence liability, regardless of whether parties make their choices sequentially as described above, or otherwise.

Example 2 Assumptions: Simple negligence, Specification 2. Result: No equilibrium but a Cycle!

We now proceed with our 2nd example. In this and subsequent examples we will try to be more brief than we were in example 1.

This example is much like example 1 above, except that it is based on Specification 2, taking $\delta = 0.01$. That is, the benefit functions are modified so that they have maxima in s and t. The benefit functions are now: $u(s) - xs = s^{1/2} - xs - .01s$ and $v(y) - yt = t^{1/2} - yt - .01t$. With this assumption it should no longer be necessary to constrain s and t in order to find a maximum for NSB.

Example 2 Table

	s	x	t	y	Liability	X's benefit Net of Damages	Y's benefit Net of Damages
1	0.058295	0.353629	0.058295	0.353629	All Y	0.22025	0.12072
2	**1.890707**	**0.353629**	0.058295	0.353629	All Y	0.68751	-3.00768
3	1.890707	0.353629	**0.000763**	**8.369300**	All Y	0.68751	0.01381
4	**1262.607**	**0.000000**	0.000763	8.369300	All X	17.76659	0.02123
5	1262.607	0.000000	**2500.000**	**0.000000**	All X	-1.58E+08	25.0000
6	**1.890707**	**0.353629**	2500.000	0.000000	All Y	0.68751	-174571
7	1.890707	0.353629	**0.000763**	**8.369300**	All Y	0.68751	0.01381

Notes:
X is liable if and only if $x < x^*$.
Line 1 solves first order conditions to get (s^*, x^*, t^*, y^*).
Line 2: X chooses s and x. X stays with $x = .3536$, but increases s to $s = 1.8907$.
Line 3: Y chooses t and y. FOC's give interior solution: $t = .000763$ and $y = 8.3693$.
Line 4: X chooses s and x. FOC implies negative x, so $x = 0$ is used. FOC implies $s = 1262.607$.
Line 5: Y chooses t and y. Since $x = 0$, X pays damages; Y sets $y = 0$, and uses FOC to get $t = 2500$.
Line 6: X chooses s and x. He sets $x = 0.3536$ to avoid huge accident losses, then chooses $s = 1.89071$.
Line 7: Y chooses t and y. Back to line 3! Cycle!

Next, consider the rule strict liability with a defense of contributorily negligence. This rule is the mirror image of the simple negligence. If we swap party X with party Y and vice-versa, then whatever claim is valid for party X under simple negligence, a similar claim will be true for party Y under strict liability with a defense of contributorily negligence. In view of the symmetry of the functional forms, it is easy to produce an example similar to Example 1 to show existence of cycle under the rule strict liability with a defense of contributorily negligence. As is proved in the next subsection, there cannot exist a Nash equilibrium under this rule as well.

Next we show the existence of a cycle under the rule of negligence with a defense of contributory negligence.

Example 3 Assumptions: Negligence with a defense of contributory negligence, Specification 2. Result: No equilibrium but a Cycle!

We now proceed with our 3rd example. This example is much like example 2 above, i.e., is based on Specification 2, except the liability rule used here is negligence with a defense of contributory negligence. What happens under this rule? Once again, another bizarre cycle, shown on lines 3 through 7 of the following Example 3 Table.

Example 3 Table

	s	x	t	y	Liability	X's benefit Net of Damages	Y's benefit Net of Damages
1	0.058295	0.353629	0.058295	0.353629	All Y	0.22025	0.12072
2	**1.890707**	**0.353629**	0.058295	0.353629	All Y	0.68751	-3.00768
3	1.890707	0.353629	**0.000763**	**8.369300**	All Y	0.68751	0.01381
4	**1262.607**	**0.000000**	0.000763	8.369300	All X	17.76659	0.02123
5	1262.607	0.000000	**1.890707**	**0.353629**	All X	-88155.64	0.687515
6	**1.890707**	**0.353629**	1.890707	0.353629	All Y	0.687515	-104.006
7	1.890707	0.353629	**0.000763**	**8.369300**	All Y	0.687515	0.01381

Notes:
X is liable if and only if $x < x^*$ and $y \geq y^*$.
Line 1 solves first order conditions to get (s^*, x^*, t^*, y^*).
Line 2: X chooses s and x. X stays with $x = .3536$, but increases s to $s = 1.8907$.
Line 3: Y chooses t and y. FOC's give interior solution: $t = .000763$ and $y = 8.3693$.
Line 4: X chooses s and x. FOC implies negative x, so $x = 0$ is used. FOC implies $s = 1262.607$.
Line 5: Y chooses t and y. Since $x = 0$, X pays damages as long as $y \geq y^*$; otherwise Y pays. Y is better off setting $y = 0.353629$ than using the FOC to set $t = 1.890707$.
Line 6: X chooses s and x. He sets $x = 0.3536$ to avoid huge accident losses, then chooses $s = 1.89071$.
Line 7: Y chooses t and y. Back to line 3! Cycle!

Working along the lines described in Example 3 above, but substituting Specification 1 for Specification 2, it is easy to produce another cycle for Specification 1. As is explained below, between the two specifications, Specification 2 is more plausible. Therefore, in the following, we will use it more frequently. However, corresponding to a cycle shown for Specification 2, a similar or worse cycle can be shown to exist for Specification 1.

Example 4 Assumptions: 50/50 split liability when both negligent, Specification 2. Result: No equilibrium but a Cycle!

Now we turn to an example in which the liability rule involves the sharing of liability by the 2 parties. In particular, the benefit functions and the expected damages function is as in Specification 2, with (delta) = 0.01 for both parties. And the liability rule is of 50/50 split liability when both parties are negligent. More formally: X bears all the loss when he is negligent and Y is not; that is, when $x < x^*$ and $y^* \leq y$. Y bears all the loss when X is non-negligent; that is, when $x^* \leq x$. When both X and Y are negligent, that is, when $x < x^*$ and $y < y^*$, the loss is split 50/50. Turning to the table:

Example 4 Table

	s	x	t	y	Liability	X's benefit Net of Damages	Y's benefit Net of Damages
1	0.058295	0.353629	0.058295	0.353629	All Y	0.220245	0.12072
2	**1.890707**	**0.353629**	0.058295	0.353629	All Y	0.687515	-3.00768
3	1.890707	0.353629	**0.000763**	**8.369300**	All Y	0.687515	0.01381
4	**1262.607**	**0.000000**	0.000763	8.369300	All X	17.76659	0.02123
5	1262.607	0.000000	**1.890707**	**0.353629**	All X	-88155.64	0.68751
6	**1.890707**	**0.353629**	1.890707	0.353629	All Y	0.687515	-104.006
7	1.890707	0.353629	**0.000763**	**8.369300**	All Y	0.687515	0.01381

Notes:
X is liable when X is negligent ($x < x^*$) and Y is non-neg; Y is liable when X is non-neg. However when both X and Y are negligent, liability is split 50/50.
Line 1 solves first order conditions to get (s^*, x^*, t^*, y^*).
Line 2: X chooses s and x. He chooses to keep x at x^*, and increase s to 1.890707.
Line 3: Y chooses t and y. FOC's give interior solution: $t = .000763$ and $y = 8.369300$.
Line 4: X chooses s and x. FOC implies negative x, so $x = 0$ is used. FOC implies $s = 1262.607$
Line 5: Y chooses t and y. Since $x = 0$, Y sets $y = 0.353629$ to avoid damages, and uses FOC to get $t = 1.890707$.
Line 6: X chooses s and x. Back to s and x of line 2.
Line 7: Y chooses t and y. Back to line 3.
Cycle! And they never enter the 50/50 split region.

In the table line 1 again shows the (s^*, x^*, t^*, y^*) that solves the first-order conditions. In line 2 X chooses (s, x) to maximize his utility, taking Y's (t, y) from line 1 as given; in line 3 Y reacts to what X did in line 2, and so on. The result? We once again have a cycle, in lines 3 to 6.

Example 5. Assumptions: Simple negligence, Specification 2, but with different δ's. Result: No equilibrium but a Cycle!

Example 5 is a variation of Example 2. All the assumptions are the same as in example 2, except that now the $.01t$ factor in Y's benefit function, which guarantees he wants a finite amount of t, is smaller than it was by a factor of 10. The benefit functions are now:

$u(s) - xs = s^{1/2} - xs - .01s$ and $v(t) - yt = t^{1/2} - yt - .001t$. This gives the following specification of the net social benefit function:

$$NSB = s^{1/2} - xs - 0.01s + t^{1/2} - yt - 0.001t - \frac{50st}{1 + x + y}.$$

The liability rule is simple negligence. The Example 5 conclusion is like the Example 2 conclusion: A bizarre cycle and no equilibrium.

Example 5 Table

	s	x	t	y	Liability	X's benefit Net of Damages	Y's benefit Net of Damages
1	0.058408	0.349958	0.058408	0.358958	All on Y	0.22065	0.12084
2	**1.929460**	**0.349958**	0.058408	0.358958	All on Y	0.69453	-3.07663
3	1.929460	0.349958	**0.000747**	**8.472134**	All on Y	0.69453	0.01366
4	**1286.024**	**0.000000**	0.000747	8.472134	All on X	17.93059	0.02100
5	1286.024	0.000000	**250000.0**	**0.000000**	All on X	-1.61E+10	250.000
6	**1.929460**	**0.349958**	250000.0	0.000000	All on Y	0.69453	-17865675
7	1.929460	0.349958	**0.000747**	**8.472134**	All on Y	0.69453	0.01366

Notes:
X is liable if and only if $x < x^*$.
Line 1 solves first order conditions to get (s^*, x^*, t^*, y^*).
Line 2: X chooses s and x. X stays with $x = .3500$ but increases s to s= 1.9295.
Line 3: Y chooses t and y. FOC's give interior solution: $t = .000747$ and $y = 8.4721$.
Line 4: X chooses s and x. FOC implies negative x, so $x = 0$ is used. FOC implies s= 1286.024.
Line 5: Y chooses t and y. X pays damages; Y sets $y = 0$; FOC gives $t = 250{,}000$.
Line 6: X chooses s and x. Sets $x = 0.3500$ to escape huge accident losses; uses FOC to get s= 1.9295.
Line 7: Y chooses t and y. Back to line 3! Cycle!

Before proceeding further let us consider a case where $\phi(s, t) = s + t$. Therefore in this example expected damages are given by $E(D) = 50(s + t)/(1 + x + y)$. Now, the NSB function takes the following form:

$$NSB = s^{1/2} - xs + t^{1/2} - yt - \frac{50(s + t)}{1 + x + y}.$$

This specification also leads to strange outcomes.To see why, consider simple negligence for the liability rule. We can continue to use the benefit functions we have used in Example 1, and we continue to constrain s and t to be less than or equal to 1,000,000. We have also added a constraint on the size of x and y. The crucial difference in this example is that we have changed the $\phi(s, t)$ function from st to $s + t$.

Example 6. Assumptions: Simple negligence, Specification 1, but with $\phi(s,t) = s+t$. Result: No equilibrium but a Cycle!

Example 6 Table

	s	x	t	y	Liability	X's benefit Net of Damages	Y's benefit Net of Damages
1	0.002770	4.500000	0.0027701	4.500000	All Y	0.040166	0.012465
2	**0.012346**	**4.500000**	0.0027701	4.500000	All Y	0.055556	-0.035413
3	0.012346	4.500000	**2.50E-07**	**1000.00**	All Y	0.055556	-0.000364
4	**100.2004**	**0.000000**	2.50E-07	1000.00	All X	5.005000	0.000250
5	100.2004	0.000000	**1000000**	**0.000000**	All X	-5.00E+07	1000.000
6	**0.012346**	**4.500000**	1000000	0.000000	All Y	0.055556	-9089909
7	0.012346	4.500000	**2.50E-07**	**1000.00**	All Y	0.055556	-0.000364

Notes:
X is liable if and only if $x < x^*$. Constraints: $s, t \leq 1,000,000$. $x, y \leq 1,000$
Line 1 solves first order conditions to get (s^*, x^*, t^*, y^*).
Line 2: X chooses s and x. X chooses $x = 4.5$ to avoid liability, and FOC gives $s =$ 0.1235.
Line 3: Y chooses t and y. Y chooses $t =$ 2.5E-7 and y very large, e.g. $y =$ 1,000.
Line 4: X chooses s and x. He sets $x = 0$ and becomes liable. FOC quickly gives s = 100.2004.
Line 5: Y chooses t and y. X is liable since $x = 0$. Y wants to max $t^{\frac{1}{2}} - yt$. Solution: set $y = 0$ and t large.
Line 6: X chooses s and x. He sets $x = 4.5$ to avoid liability and solves FOC for $s =$ 0.0123457
Line 7: Y chooses t and y. Back to line 3! Cycle!

What happens in this example? No Nash equilibrium, but another bizarre cycle in lines 3 to 7. Note the huge variations in the net benefits for X and Y. These are caused by huge variations in the payoffs.

Example 7. Assumptions: Simple negligence, Specification 2, but with $\phi(s,t) = s+t$. Result: No equilibrium but a Cycle!

Example 7 is just like Example 6, except it substitutes Specification 2 for Specification 1. Once again, the result is a big cycle.

Example 7 (Specification 2)

	s	x	t	y	Liability	X's benefit Net of Damages	Y's benefit Net of Damages
1	0.002764	4.500000	0.002764	4.500000	All Y	0.040109	0.012467
2	**0.012291**	**4.500000**	0.002764	4.500000	All Y	0.055432	-0.035167
3	0.012291	4.500000	**2.50E-07**	**1000.00**	All Y	0.055432	-0.000361
4	**69.560214**	**0.000000**	2.50E-07	1000.00	All X	4.170138	0.000250
5	69.560214	0.000000	**2500**	**0.000000**	All X	-128470	25
6	**0.012291**	**4.500000**	2500	0.000000	All Y	0.055432	-22702.4
7	0.012291	4.500000	**2.50E-07**	**1000.00**	All Y	0.055432	-0.000361

Notes:
X is liable if and only if $x < x^*$. Constraints: $x, y \leq 1,000$
Line 1 solves first order conditions to get (s^*, x^*, t^*, y^*).
Line 2: X chooses s and x. X chooses $x = 4.500000$ to avoid liability, and FOC gives $s = 0.012291$.
Line 3: Y chooses t and y. Y chooses $t =$ 2.50E-07 and y very large, e.g. $y =$ 1,000.000.
Line 4: X chooses s and x. He sets $x = 0$ and becomes liable. FOC quickly gives $s =$ 69.56021.
Line 5: Y chooses t and y. X is liable since $x = 0.000000$. Y wants to max $t^{\frac{1}{2}} - yt - .01t$. Solution: set $y = 0$ and $t = 2500$.
Line 6: X chooses s and x. He sets $x = 4.500000$ to avoid liability and solves FOC for $s = 0.012291$
Line 7: Y chooses t and y. Back to line 3! Cycle!

4. Problems with the Standard Models

The NSB functions examined by us, through Specification 1 and 2 and other related versions, meet all the requirements of the standard models. Moreover, we have strictly followed the standard procedure for identification of the first best, and for setting the due care standards for the parties. Yet, as we have shown above, the prevalent claims about the equilibrium outcomes under negligence based liability rules and their efficiency properties, do not hold.

What is even more problematic is this: The equilibrium values of NSB under strict liability for the injurer are much higher than the NSB values at the solution to the first order conditions for the social optimization problems. Similar is the case with value of NSB under no liability for the injurer vis-a-vis the first order conditions solution. For Specification 2, the value of NSB at (s^*, x^*, t^*, y^*) is just 0.340966, while at the equilibrium under strict liability it is 25.00035! (See Appendix). For Specification 1 the difference is even more bizarre. The value of NSB at (s^*, x^*, t^*, y^*) is just 0.34213, while at the Nash equilibrium under no-liability, it is 1,000.000017! (see Appendix) It is clear that the first-order conditions for the maximization of NSB have been misleading us. Therefore, we have to ask: What exactly has gone wrong with the standard models?

Consider again the problem of maximizing $NSB = s^{1/2} - xs + t^{1/2} - yt - 50st\pi(x, y)$. There are serious problems with this function used in the literature. *There are no* *maxima.* Here's why: set $x = y = t = 0$. Then the NSB function reduces to $NSB(s, 0, 0, 0) = s^{1/2}$, which is *unbounded* without a maximum! Similarly, $NSB(0, 0, 0, t) = t^{1/2}$, is *unbounded.* The maxima in our examples resulted from the constraints that we had imposed, i.e., $s, t \leq 1,000,000$. Take away these constraints, and there are no maxima.

Since the first order conditions to do not identify a global maximum, if a mamixmum does exist it must lie on one or the other boundary. With $s, t \leq 1,000,000$ constraints, there are two corner global maxima; one requires $x = 0$ along with $s = 1,000,000$, and the other requires $y = 0$ along with $t = 1,000,000$. Now, it is easy to see why the value of NSB is highest under the no-liability or the strict liability rule. Under strict liability rule, Y will set $y = 0$ and maximum possible level of $t = 1,000,000$. As is shown in the Appendix, the outcome under strict liability approximates the second global maximimum arbitrarily closely. A similar argument applies to the rule of no liability; it approximates the first of the maxima. It is straightforward to see that these problems persist even if we replace $s^{1/2}$ and $t^{1/2}$ with functions like $s^{1/k}$ and $t^{1/k}$, for $k > 1$.

Specification 2 suffers from similar problems. Note that under Specification 2, we get $NSB(s, 0, 0, 0) = s^{1/2} - \delta s$. It can be made arbitrarily large by choosing sufficiently small δ. In other words, it is easy to find a range of δ for which global maxima cannot be interior. In fact, as is shown in the Appendix, even when δ is bounded away from 0 as in Specification 2, the maxima are not interior.

These problems are not restricted to our specifications. We have tried variations of specifications 1 and 2, e.g., by using different values of D and δ. For both specifications, we have examined the outcomes by replacing $\phi(s, t) = st$ with $\phi(s, t) = \sqrt{st}$ and $\phi(s, t) = (st)^2$, but the above problem persists. The first order conditions, do not identify a global maxima.

In fact, the "bad behavior" extends to the standard model itself. For version 1 of the model, $NSB = u_0(s) - xs + v_0(t) - yt - sth(x, y)$. If we set $x = y = t = 0$, then $NSB = u_0(s) + v_0(0)$. The function $u_0(s)$ is monotonically increasing. Clearly, there can be no maxima if $u_0(s)$ or $v_0(t)$ is unbounded.

Next, consider the case when $u_0(s)$ and $v_0(t)$ are bounded. Clearly, a global maximum will not be interior as long as the upper bound for either of the functions is sufficiently large. In fact, even when the upper bound on $u_0(s)$ and $v_0(t)$ is reasonably small, an interior maximum is not guaranteed. For instance, $u_0(s)\frac{\sqrt{s}}{\sqrt{1+s}}$ and $v_0(t) = \frac{\sqrt{t}}{\sqrt{1+t}}$. Clearly, $u_0(s) \leq 1$ and $v_0(t) \leq 1$ for all s and t. Still, there is no interior maximum. Specifically, with

$$NSB(s, x, t, y) = \frac{\sqrt{s}}{\sqrt{1+s}} + \frac{\sqrt{t}}{\sqrt{1+t}} - sx - ty - \frac{50st}{1+x+y}$$

there two global maxima; these are corner points $(s, x, t, y) = (3180.99, 0, 3.93946 \times 10^{-7}, 397.81)$ and its mirror image $(s, x, t, y) = (3.93946 \times 10^{-7}, 397.81, 3180.99, 0)$. At these points $NSB = 1.00016$. In addition, we have worked with the following specifications of the NSB:

$$NSB(s,x,t,y) = \frac{\sqrt{s}}{\sqrt{1+s}} + \frac{\sqrt{t}}{\sqrt{1+t}} - sx - ty - \frac{50st}{1+\sqrt{x}+\sqrt{y}}$$
$$NSB(s,x,t,y) = \log s + \log t - sx - ty - \frac{50st}{1+x+y}$$
$$NSB(s,x,t,y) = \log s + \log t - sx - ty - \frac{50st}{1+\sqrt{x}+\sqrt{y}}$$

These functions also fail to generate an interior maximum. The solution to the relevant FOCs does not satisfy the second order conditions for a maximum.

From the above arguments (detailed in Appendix I) it follows that whenever NSB has corner maxima, the rule of strict liability or the rule of no-liability will be more efficient than the rules of negligence including the rule of comparative negligence. A negligence rule will not be able to approximate the first best, unless it mimics the rule of strict liability or the rule of no-liability rule,[17].

In fact, to see the inherent problem with the standard model fix $t = 0$. This means the care costs for the victim is zero, even when y is very high. So, it makes sense to set $x = 0$ to bring down injurer's costs of care. Still, expected accident costs remains zero since $t = 0$. That is, under both versions of the standard model, fixing $t = 0$ has two direct and significant social benefits. First, the cost of care for can be reduced to zero, for at least one party. Second, the expected accident loss also becomes zero, even if the injurer opts for very high level of activity and very little care. Our analysis shows that for a large set of functional forms, consistent with the above specifications, these two gains dominate the opportunity cost, i.e, keeping the net gains to the victim at $v_0(0) - 0y = 0$. This logic applies to the positive but arbitrarily small levels of t.

It seems, under the standard models the problem will persist as long as $\lim_{s,t\to 0} \phi(s,t) = 0$ holds. This problem does not arise when $\phi(s,t) = s+t$. So, let us consider $\phi(s,t) = s+t$. Now also we have the usual problems of non-existence of Nash equilibrium under negligence liability. See Feldman an Singh (2019). In fact, it gets worse. Consider a specification corresponding to Specification 1 above, except now we will let $st = s + t$ instead of $st = st$. So, the net social benefit function is now

$$NSB = s^{1/2} - xs + t^{1/2} - yt - \frac{(s+t)50}{1+x+y}.$$

The system of first order conditions for this specifications has one non-negative solution: $(0.00277083, 4.5, 0.00277083, 4.5)$. That is, the apparent first-best optimum is now $(s^*, x^*, t^*, y^*) = (0.002770083, 4.5, 0.002770083, 4.5)$. At this point, $NSB = 0.0526316$. However, holding $t = x = 0$, $NSB = s^{1/2} - s \times 0 + 0 - y \times 0 - \frac{(s+0)50}{1+0+y} = s^{1/2} - \frac{s50}{1+y}$. So, by increasing s and y such that $y = 50s - 1$, the NSB can be increased without any upper limit! Again, a global maximum does not exist. Next, we consider

$$NSB = s^{1/2} - xs - 0.01s + t^{1/2} - yt - 0.01t - \frac{(s+t)50}{1+x+y}.$$

The system of first order conditions for this specifications has one non-negative solution: $(s, x, t, y) = (.002764, 4.5, .002764, 4.5)$. At this point, $NSB = .0525762$. However,

[17]It is well known that when the due care is set very high, it will resemble the rule of strict liability as the injurer will prefer to bear liability rather than comply with the high due care level. When the due care is set at 0, it is the same as no-liability.

holding $t = x = 0$, the maximum value of NSB is $\bar{s}^{1/2}(\delta) - \bar{s}(\delta) \times 0 - \delta\bar{s}(\delta) + 0 - y \times 0 - \delta \times 0 - \frac{(\bar{s}(\delta)+0)50}{1+0+y} = \bar{s}^{1/2}(\delta) - \delta\bar{s}(\delta) - \frac{\bar{s}(\delta)50}{1+y}$. So, by choosing arbitrarily small δ and large y, the NSB can be increased without any upper limit, leading to a corner global maximum.

5. Looking Beyond the Standard Models

The above analysis shows that the models used in the existing literature are inherently flawed. Moreover, the commonly used approach of using the first order condition to identify the first best care levels is misleading in many accident contexts. Consequently, the first order conditions are not a good guide when it comes to choosing the due care levels. Even if we completely disregard these issues, there is yet another serious problem with the existing literature.

To explain, suppose the NSB function is 'well-behaved' with a unique solution identifiable by the first order conditions. That is, assume that the solution to the first order conditions, (s^*, x^*, t^*, y^*), is indeed a unique global maximum, and the due care standards for the injurer and victim are set at x^* and y^*, respectively. The literature argues/assumes that under the rule of negligence, the injurer would choose x^* as care level. Specifically, it is implicitly assumed that a downward deviation from x^* to some $x < x^*$ cannot be in the interest of injurer. Similar assumptions are made for the other negligence based rules.[18] But, to our knowledge, not a single work has provided proof behind such a critical beliefs.

Therefore, the question arises: Assuming the social optimization problem has a unique interior solution, does a Nash equilibrium necessarily exist under the standard negligence based rules?

To address this question, in this section we assume that the individual payoff functions and the expected accident loss functions are such that an interior first best exists. In particular, the social optimization problem

$$\max_{u,x,t,y} \{u(s,x) + v(t,y) - l(s,x,t,y)\}$$

is concave with a unique interior solution given the following set of first order conditions:

$$\frac{\partial u(s,x)}{\partial s} - \frac{\partial l(s,x,t,y)}{\partial s} = 0 \tag{5.1}$$

$$\frac{\partial u(s,x)}{\partial x} - \frac{\partial l(s,x,t,y)}{\partial x} = 0 \tag{5.2}$$

$$\frac{\partial v(t,y)}{\partial t} - \frac{\partial l(s,x,t,y)}{\partial t} = 0 \tag{5.3}$$

$$\frac{\partial v(t,y)}{\partial y} - \frac{\partial l(s,x,t,y)}{\partial y} = 0 \tag{5.4}$$

As above, the unique solution be denoted by (s^*, x^*, t^*, y^*), where $(s^*, x^*, t^*, y^*) > (0, 0, 0, 0)$. Further, assume that the following optimization problems are well defined.

[18] See Shavell (1980), Shavell (1987), Endres (1989), Miceli (1997 p. 29), Cooter and Ulen (2004, pp. 332-33), a Dari-Mattiacci (2002), Delhaye (2002), Goerke (2002), Parisi and Fon (2004), Singh (2006), Parisi and Singh (2010). For discussion see Dari-Mattiacci, Lovat and Parisi (2014, page 572), and Miceli (2017, Ch 2).

Specifically, for any given (t, y), the optimization problem

$$\max_{u,x}\{u(s,x) - l(s,x,t,y)\} \tag{5.5}$$

is concave with a unique and interior solution. Similarly, for any given (s, x), the optimization problem

$$\max_{t,y}\{v(t,y) - l(s,x,t,y)\} \tag{5.6}$$

has a unique and interior solution.

Recall, the standard liability rules satisfy Axioms (A1) and (A2). Another essential feature of the negligence-criterion based liability rules is captured by the following Axiom.

Axiom (A3): Let $w_X(x^*, y^*) = w_X^*$. For all $x \geq x^* \& y \geq y^*$

$$w_X(x,y) = w_X^*,\ w_Y(x,y) = 1 - w_X^*,\ \text{where } w_X^* \in \{0,1\}.$$

Axiom (A3) says that in the region where parties are both non-negligent, the entire burden of liability falls on one and only one party. Depending on the rule, this party can be the injurer or the victim. $w_X^* = 0$, under the rule of negligence, the rule of negligence with the defense of contributory negligence, the rule of 50/50 split when both parties are negligent, and the rule of the comparative negligence. However, $w_X^* = 1$ under the rule of strict liability with the defense of contributory negligence.

Let us investigate the behavior of parties under rules that satisfy Axioms (A1)-(A3). When $x \geq x^*$ and $y < y^*$, the victim is solely negligent. In such an event, due to Axiom (A2), the injurer has no liability. So, for given s his payoff is $u(s, x)$. Note that $u(s, x)$ deceases with x. Therefore, regardless of the s opted by him whenever $x > x^*$, the injurer can increase his payoff simply by reducing x until he reaches at x^*. This means that if the victim opts for some y such that $y < y^*$, the injurer is better off opting x^* rather than any $x > x^*$. As a result, any tuple $((\bar{s}, \bar{x}), (\bar{t}, \bar{y})))$ such that $\bar{x} > x^*$ & $\bar{y} < y^*$ cannot be a N.E. Similarly, a tuple $((\bar{s}, \bar{x}), (\bar{t}, \bar{y})))$ with $\bar{x} < x^*$ & $\bar{y} > y^*$ cannot be a N.E.

Next, consider $((\bar{s}, \bar{x}), (\bar{t}, \bar{y})))$ such that $\bar{x} > x^*$ & $\bar{y} > y^*$. Suppose $w_X(x^*, y^*) = 0$. In that case, as long as the injurer is non-negligent, his liability is nil. This means that regardless of the levels of s chosen by him and (t, y) opted by the victim, the injurer can increase his payoff by choosing x^* over $\bar{x}$. Therefore, $\bar{x} > x^*$ cannot be the best choice for the injurer. That is, $((\bar{s}, \bar{x}), (\bar{t}, \bar{y})))$ cannot be a N.E. Similarly, when $w_X(x^*, y^*) = 1$, $\bar{y} > y^*$ cannot be a best choice for the victim. In any case, when $\bar{x} > x^*$ & $\bar{y} > y^*$, $((\bar{s}, \bar{x}), (\bar{t}, \bar{y})))$ cannot be a N.E.

In fact, under a standard liability rule satisfying Axioms (A1)-(A2), if an equilibrium exists it will lie either on the line defined by $x = x^*$ or on the line defined by $y = y^*$.

Proposition 1. *Suppose Axioms (A1)-(A3) hold. A profile* $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ *can be a Nash equilibrium only if:* $[\bar{x} = x^* \quad OR \quad \bar{y} = y^*]$.

For proof see Appendix III. Let, s_p^* solves

$$\max_{s}\{u(s,x^*)\} \tag{5.7}$$

That is, s_p^* is the optimum activity level for the injurer when he simply opts for x^* as care level and does not bear any amount of the accident costs. Likewise, let t_p^* solves

$$\max_t \{v(t, y^*)\} \tag{5.8}$$

It is easy to show that in most contexts $s_p^* > s^*$ and $t_p^* > t^*$ will hold. Our next result is as follows. Proof is given in the Appendix III.

Proposition 2. *Suppose Axioms (A1)-(A3) hold. Let $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ be a N.E. If $w_X^* = 0$ then $\bar{x} = x^*$; if $w_X^* = 1$ then $\bar{y} = y^*$.*

Let, the pair $(\hat{t}, \hat{y})$, uniquely solve (5.9):

$$\max_{t,y} \{v(t, y) - l(s_p^*, x^*, t, y))\} \tag{5.9}$$

To interpret, suppose the injurer has opted for (s_p^*, x^*) and the victim has to bear the entire loss. Under such a scenario, $(\hat{t}, \hat{y})$ uniquely maximizes payoffs of party Y. Likewise for the injurer and the pair $(\hat{s}, \hat{x})$, where $(\hat{s}, \hat{x})$ uniquely solves (5.10):

$$\max_{s,x} \{u(s, x) - l(s, x, t_p^*, y^*))\} \tag{5.10}$$

That is, if the victim has opted for (t_p^*, y^*) and party X has to bear the entire loss, choice of $(\hat{s}, \hat{x})$ is uniquely best best choice for party X. By definition, for any $(\bar{s}, \bar{x})$ such that $\bar{x} < x^*$, we have

$$u(\hat{s}, \hat{x}) - l(\hat{s}, \hat{x}, t_p^*, y^*)) \geq u(\bar{s}, \bar{x}) - l(\bar{s}, \bar{x}, t_p^*, y^*))$$

Moving on, suppose $(\hat{\hat{s}}, \hat{\hat{x}})$ and $(\hat{\hat{t}}, \hat{\hat{y}})$, uniquely solve (5.12) and (5.11), respectively:

$$\max_{s,x} \{u(s, x) - l(s, x, \hat{t}, \hat{y}))\}, \tag{5.11}$$

$$\max_{t,y} \{v(t, y) - l(\hat{s}, \hat{x}, t, y))\}, \tag{5.12}$$

where $(\hat{t}, \hat{y})$ and $(\hat{s}, \hat{x})$ are as defined above.

Condition: No Downward Deviations (NDD). Functions $u(.)$, $v(.)$ and $l(.)$ are such that the following holds:

$$NDDX: \quad u(\hat{\hat{s}}, \hat{\hat{x}}) - l(\hat{\hat{s}}, \hat{\hat{x}}, \hat{t}, \hat{y}) \quad \leq \quad u(s_p^*, x^*) \tag{5.13}$$

$$NDDY: \quad v(\hat{\hat{t}}, \hat{\hat{y}}) - l(\hat{s}, \hat{x}, \hat{\hat{t}}, \hat{\hat{y}}) \quad \leq \quad v(t_p^*, y^*) \tag{5.14}$$

For an intuitive interpretation, suppose Axioms (A1)-(A3) hold with $w_X^* = 0$. That, a non-negligent injurer has no liability at all. The condition (5.13) ensures that a unilateral downward deviation by the injurer X from $(s_p^*, x^*, \hat{t}, \hat{y})$ to a care level where X will become fully liable is not strictly profitable to him. In particular, the condition implies that a unilateral downward deviation by the injurer X cannot be incentive compatible. Similarly, condition (5.14) ensures that when Axioms (A1)-(A3) hold with $w_X^* = 1$, unilateral downward deviation from $(\hat{s}, \hat{x}, t_p^*, y^*)$ cannot be profitable for the victim.

Below we show that the NDDX is a necessary condition for existence of Nash equilibrium under the rule of negligence and the other negligence based rules that have $w_X^* = 0$. That is, if in an accident context, if the condition is not satisfied then there will be no Nash equilibrium in that context. If $w_X^* = 1$ holds under a liability rule, then a Nash equilibrium is possible only if condition NDDY holds. Formally we have the following result. For proof see Appendix.

Proposition 3. *Suppose a liability satisfies Axioms (A1)-(A3). When* $w_X^* = 0$ *[resp.* $w_X^* = 1$*], condition NDDX [resp. condition NDDY] is necessary for existence of a Nash equilibrium.*

It is easy to see that the condition NDDX is sufficient for existence of a Nash equilibrium under the rule of negligence. Suppose the injurer has opted for (s_p^*, x^*), and the victim has opted for $(\hat{t}, \hat{y})$ as defined above. Since the injurer has opted for (s_p^*, x^*), under the negligence rule the entire burden of accident loss falls on the victim. Moreover, given that (s_p^*, x^*) is opted by X, by definition $(\hat{t}, \hat{y})$ is a best response choice by Y. Now, consider the unilateral deviations by the injurer from $((s_p^*, x^*), (\hat{t}, \hat{y}))$. Clearly, the injurer is worse off choosing a $x > x^*$. Moreover, if the injurer deviates downward below x^* to say $\bar{x} < x^*$ along with some activity levels say $\bar{s}$, he will become negligent and therefore fully liable for the accident loss. Given that $(\hat{t}, \hat{y})$ is opted by Y, when X has to bear the entire loss, his uniquely best choice is given by $(\hat{\hat{s}}, \hat{\hat{x}})$, as defined above. But, due to condition NDDX, he is worse off opting for $(\hat{\hat{s}}, \hat{\hat{x}})$ over (s_p^*, x^*). At $(\bar{s}, \bar{x})$, his payoff will be even lesser than at $(\hat{\hat{s}}, \hat{\hat{x}})$. That is, given $(\hat{t}, \hat{y})$ opted by the victim, (s_p^*, x^*) is a best response for the injurer. Formally, $((s_p^*, x^*), (\hat{t}, \hat{y}))$ is a N.E.

An argument similar to the one used above shows that $((\hat{s}, \hat{x})(t_p^*, y^*))$ is a N.E. under the Rule of Strict Liability with Defense of Contributory Negligence.

Before proceeding further we make the following remarks.

Remark 1: It should be noted that while establishing the above results in this section, we have considered a very general form of functions $u(s, x)$, $v(t, y)$ and $l(s, x, t, y)$ function. The claims hold for any functional forms that satisfy the assumptions stated above. In particular, the above claims are equally valid for continuous as well as discrete care and activity levels. In the interest of simplicity, now on we will assume that the variables are continuous and functions are differentiable.

Remark 2: So far we have not made any statement about how the pair $(\hat{t}, \hat{y})$ in the Nash equilibrium profile under negligence rule, $((s_p^*, x^*), (\hat{t}, \hat{y}))$, compares with the pair (t^*, y^*). Similarly, we do not if the N.E. under the Rule of Strict Liability with Defense of Contributory Negligence, $((\hat{s}, \hat{x})(t_p^*, y^*))$, would mean more than first best care on the part of the injurer.

By definition, $(\hat{t}, \hat{y})$ maximizes the SBF (and also the payoffs for the victim), given that the injurer has opted for x^* as care level and s_p^* as his activity level. Since the activity level of the injurer is excessive, it seems plausible that the victim will choose care level more than y^* and activity level less than t^*. However, we cannot be sure that it will actually be the case.

To get $\hat{t} < t^*$ and $\hat{y} \geq y^*$ we need to impose additional structure on the payoff and the expected loss functions. For given (s, x) opted by the injurer, consider the following optimization problem

$$\max_{t,y}\{v(t,y) - l(s,x,t,y))\} \tag{5.15}$$

By assumption, the solution is unique and interior. Suppose, we fix x at x^*. Now, the solution, say $(\tilde{t}(s), \tilde{y}(s))$, is identified by the following set of equations:

$$\begin{aligned} V_1(.) \equiv \frac{\partial v(t,y)}{\partial t} - \frac{\partial l(s,x^*,t,y))}{\partial t} &= 0 \\ V_2(.) \equiv \frac{\partial v(t,y)}{\partial y} - \frac{\partial l(s,x^*,t,y))}{\partial y} &= 0 \end{aligned}$$

Differentiating these first order conditions with respect to s and rearranging we get:

$$\frac{\partial \tilde{t}}{\partial s} = \frac{\frac{\partial V_1}{\partial s}\frac{\partial V_2}{\partial y} - \frac{\partial V_2}{\partial s}\frac{\partial V_1}{\partial y}}{\frac{\partial V_1}{\partial y}\frac{\partial V_2}{\partial t} - \frac{\partial V_2}{\partial y}\frac{\partial V_1}{\partial t}} \tag{5.16}$$

$$\frac{\partial \tilde{y}}{\partial s} = \frac{\frac{\partial V_2}{\partial s}\frac{\partial V_1}{\partial t} - \frac{\partial V_1}{\partial s}\frac{\partial V_2}{\partial t}}{\frac{\partial V_1}{\partial y}\frac{\partial V_2}{\partial t} - \frac{\partial V_2}{\partial y}\frac{\partial V_1}{\partial t}} \tag{5.17}$$

To get

$$\frac{\partial \tilde{t}(s)}{\partial s} < 0, \ \& \ \frac{\partial \tilde{y}(s)}{\partial s} > 0 \tag{5.18}$$

we need the following conditions to hold:

$$\frac{\partial V_1}{\partial s} \leq 0 \quad \frac{\partial V_2}{\partial s} \geq 0, \text{ and } \frac{\partial V_1}{\partial \tilde{y}} = \frac{\partial V_2}{\partial \tilde{t}} \leq 0.$$

In that case, $s_p^* > s^*$ implies $\hat{t} < t^*$ and $\hat{y} > y^*$. Similar assumption on the payoff function for the injurer can ensure that $\hat{s} < s^*$ and $\hat{x} > x^*$.

Equipped with these assumptions, we show that the condition NDD guarantees existence of a Nash equilibrium.

Proposition 4. *Suppose Axioms (A1)-(A3) hold for a liability rule. When* $w_X^* = 0$ *and the condition NDDX holds, the profile* $((s_p^*, x^*), (\hat{t}, \hat{y}))$ *is a unique Nash equilibrium. When* $w_X^* = 1$ *and the condition NDDY holds, the profile* $((\hat{s}, \hat{x}), (t_p^*, y^*))$ *is a unique Nash equilibrium.*

From Proposition 4, it follows that when the condition NDDX holds the profile $((s_p^*, x^*), (\hat{t}, \hat{y}))$, $\hat{y} > y^*$ is a unique Nash equlibrium under the following rules: the rule of negligence, the rule of negligence with the defense of contributory negligence, the rule of 50/50 split when both parties are negligent, and the rule of the comparative negligence. Moreover, the profile $((\hat{s}, \hat{x}), (t_p^*, y^*))$, $\hat{x} > 0$, is the sole equilibrium under the rule of strict liability with the defense of contributory negligence.

6. Concluding Remarks

Contrary to claims in the existing literature, for a large set of accident contexts admissible under the standard model we have shown the following: (1) There are no Nash equilibria under any of the standard negligence based liability rules; (2) the rules of strict liability for the injurer and no liability for the injurer are more efficient than the standard negligence based rules, such as simple negligence or negligence with a defense of contributory negligence; (3) all-or-nothing rules are more efficient than the rules that require sharing of liability between non-negligent parties (presumably to incentivise them to moderate their activities); efficiency of negligence based rules can significantly be increased if the due care standard for one party is reduced to zero.[19] In other words, our finding show that for many accident contexts, the claims in the relevant literature do not follow from the standard accident models.

Moreover, the standard models do not gel with the prevalence of negligence liability rules in the real world. For many accident contexts, the standard models imply that the rules of strict liability and no liability are more efficient than the negligence liability based rules. But, in the real world we observe that negligence based rules are much more prevalent than the all-or-nothing rules of strict liability and no-liability of injurer. Besides, the due care standards used by the negligence based rules are significantly greater than zero.[20]

In fact, there is a serious problem with the standard models. We have shown that: Even when the individual optimization problems are well defined, for the existing models the social optimization problem does not possess properties assumed in the literature. For large class of functions consistent with the standard model, the standard approach towards identification of the first best does not work. In many contexts, the standard social benefit functions do not induce any interior maximum. Either a global maximum does not exist or it is corner - in requires very high activity with zero care from one of the parties, along with almost zero activity with very high care from the other party. Due to this problem with the standard models, rules of strict liability and no liability emerge as more efficient outcome than the negligence liability based liability rules.

Going beyond the standard models, we have addressed the following question: If we assume the social benefit functions to be 'well behaved' with an interior solution, would a Nash equilibrium necessarily exist under negligence based rules? We have shown that the answer is 'No.' Ensuring existence of an equilibrium requires conditions more stringent than discussed in the literature. We have provided a necessary and sufficient condition for existence of a Nash equilibrium under the negligence based rules.

However, some issues still remain unaddressed. For the economic analysis to be a guide to legal decision making, it will be useful to know functional forms of individual payoff functions and the accident loss functions that lead to a social benefit function whose maximum can be found by using the usual first order condition approach. The knowledge about the possible functional forms of payoff and expected loss function will equip us in satisfactorily addressing issues such as desirability of splitting liability or raising the negligence standards to enhance efficiency of liability rules. We have tried several different combination of functional forms without success. Further research is required to answer the questions and issues arising from our study. We hope our findings will be a useful guide for the future research on these subjects.

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[19]This finding is in complete contrast to a strand of literature that argues for raising due care standards to improve efficiency.

[20]When due care is set at zero level or at very high level, the negligence rule reduces to the rule of no-liability and the rule of strict liability, respectively. For details see Singh (2004).

APPENDIX

Appendix I: Proofs of Claim 5:

First, we prove the claim for standard negligence based liability rules 1-6 as described above. These rules set due care standards at x^* and y^* as described above. Towards the end of this section, we discuss why the claim would hold even if the due care standards were set at other care levels.

For Specification 1 with constraints $s, t \leq 1,000,000$, the social optimization problem has two global maxima: $(s, x, t, y) = (1.25E^{-9}, 7070.18455, 1,000,000, 1.385E^{-17})$; and $(1,000,000, 1.385E^{-17}, 1.25E^{-9}, 7070.18455)$. That is, NSB maximization problem does not have an interior solution.[21] At each of these solution points the value of $NSB =$ 1000.00002. Note that the first NSB maxima requires very high activity and zero care from the from the victim, along with almost zero activity and very high care from the injurer, and vice-versa for the second solution.

The first of the two global maximum is achieved as a Nash equilibrium under the rule of strict liability for the injurer. Since X pays damages; therefore Y sets $y =$ 0 and t as large as possible, i.e., 1,000,000. Now, given $y = 0$ and $t = 1,000,000$, the best response for X is to choose $x = 7070.07$, $s = 1.25018E^{-9}$. With these choices, Y gets payoff of 0.0000176789. Specifically, under this rule the equilibrium is: $(s, x, t, y) = (1.25018E^{-9}, 7070.07, 1000000, 0)$, and at the equilibrium point value of $NSB = 1000.0000176789$. That is, the rule of strict liability almost attains the first best. By symmetry, under the rule of no liability for the injurer the equilibrium will be at $(s, x, t, y) = (1000000, 0, 1.25018E^{-9}, 7070.07)$; and the value of NSB at this equilibrium will again be $NSB = 1000.0000176789$.

In contrast, under the standard negligence liability based rules number 1-3 and 5, there is no Nash equilibrium. The actual choices by the parties will vary from contexts to context. In any case, the actual choice by the parties and hence the value of NSB will almost always be different from their first best levels.

As far as the standard rule of comparative negligence is concerned, even if an equilibrium exists, it must have each party taking more than 0 but less than x^* care. The global maxima, on the other hand, requires one party to take zero care and the other one to take very high care. This mean that even under the unlikely scenario of existence of an equilibrium under the comparative negligence, the global maximum will not be achieved.

Next, consider the rule of 50/50 split liability when both parties are non-negligent. Specifically, this liability rule works like this: X bears all the loss when he is negligent; that is, when $x < x^*$. Y bears all the loss when $y < y^*$ and $x^* \leq x$. When $x^* \leq x$ and $y^* \leq y$, the loss is split 50/50.

From the arguments in Claim 1 above, it can be seen that as under the rule of negligence, under this rule as well there cannot be a Nash equilibrium involving choice of $x < x^*$ by party X. Consider Specification 1. Suppose $x^* = 0.355473$ and $y^* = 0.355473$ are chosen by X and Y, respectively. Keeping their liability shares fixed at 1/2, the choices of s and t by the parties are characterized by the following FOCs.

[21] Use of Mathematica throws up several solutions to the global maximization problem with $NSB =$ 1000.00002 approximately. All of the global maxima seem to converge to these two points.

$$
\begin{aligned}
(1/2)s^{-1/2} - 0.355473 - \frac{25t}{1 + 0.355473 + 0.355473} &= 0, \\
(1/2)t^{-1/2} - 0.355473 - \frac{25s}{1 + 0.355473 + 0.355473} &= 0.
\end{aligned}
$$

This system has a unique solution with $s = t = 0.089838$ Now, it can be seen that $(0.089838, 0.355473, 0.089838, 0.355473)$ is a Nash equilibrium. However, at the equilibrium point the value of $NSB = 0.2989637$ is much less than $NSB = 1000.0000176789$.

Under Specification 2 with $\delta = 0.01$ also there are two global maxima: $(s, x, t, y) = (5.01409E^{-7}, 352.551, 2499.97, 0)$, and $(s, x, t, y) = (2499.97, 0, 5.01409E^{-7}, 352.551)$. At these points $NSB = 25.0004$. Now, under strict liability for the injurer, the Nash equilibrium is: $(s, x, t, y) = (5.01E^{-7}, 352.553, 2500, 0)$, resulting in NSB=25.00035. Again, the strict liability approximates the second of the global maxima. The second global maximum is approximated as the Nash equilibrium outcome under the rule of no liability for the injurer.

In contrast, under the negligence liability rules 1-5 either there is no equilibrium and parties' choices are almost always diverge from the maxima points. Under the the rule of 50/50 split liability when both parties are non-negligent, following arguments similar to the case of Specification 1, it can be seen that $(0.089379, 0.353629, 0.089379, 0.353629)$ is a Nash equilibrium but at this point the value of $NSB = 0.29973$ is much less than $NSB = 1000.0000176789$ under strict liability.

The outcome under the rule of no-liability is will be a mirror image of the outcome under the strict liability. For instance, under Specification 1 and the rule of no liability for the injurer the equilibrium will be at $(s, x, t, y) = (1000000, 0, 1.25018E^{-9}, 7070.07)$; and the value of NSB at this equilibrium will the same as under strict liability. For the same reasons as in case of strict liability, the no-liability rule is more efficient than any of the other negligence rules. In contrast, under the negligence liability rules either there is no equilibrium or it is far away from either of the maxima.

□

Appendix II: Problems with the standard models

Here we show that the standard models are not appropriate for many accident contexts. To start with we use our specifications to show the exact nature of the problems. Thereafter we show how these problems extend to the standard models.

To see the problem with Version 1, let us start with Specification 1. As shown in Section 3.2, the system of first order conditions has only one non-negative solution given by $(s^*, x^*, t^*, y^*) = (0.0585468, 0.355473, 0.0585468, 0.355473)$. However, the point $(0.0585468, 0.355473, 0.0585468, 0.355473)$ is in fact neither a global maximum for NSB, nor a local maximum. It is not even a global or local minimum point for NSB. Given $(x^*, y^*) = (0.355473, 0.355473)$, $(s^*, t^*) = (0.0585468, 0.0585468)$ is actually a *saddle point* in s/t space for the NSB function. See Plot 1. In other words, the first-order conditions led us astray.

But the problem is worse than that. As is shown above, at $(s, 0, 0, 0)$, $NSB = s^{1/2}$, is *unbounded*. The maxima under Specification 1 resulted from the constraints that

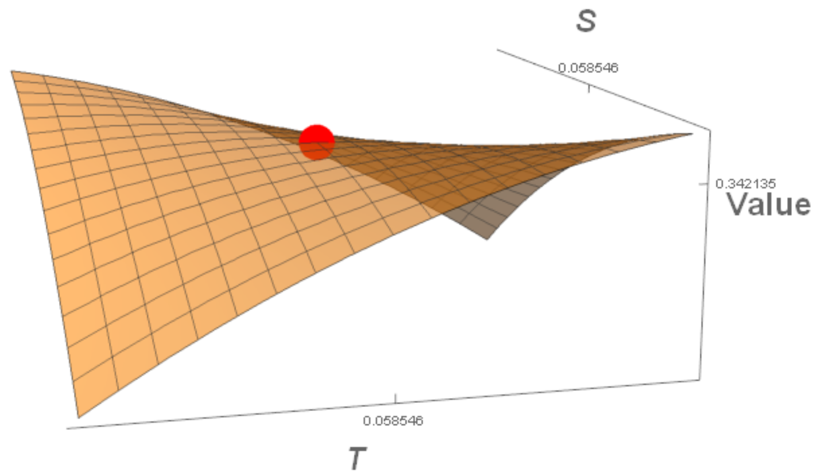

FIGURE 1
Value of NSB at (x^*, y^*) about (s^*, t^*) in (s, t) space.

we had imposed, i.e., $s, t \leq 1,000,000$. Take away these constraints, and there are no maxima. With these constraints also, since the FOCs fail to identify a maximum. There is no interior solution. A maximum has to be on the boundary. But, it is easy to see that $s = 0$ or $t = 0$ cannot be part of maximum. There are two global maxima: $(s, x, t, y) = (1.25E^{-9}, 7070.18455, 1,000,000, 1.385E^{-17})$; and $(s, x, t, y) = (1,000,000, 1.385E^{-17}, 1.25E^{-9}, 7070.18455)$.[22]

This problem with the NSB in Specification 1 is regardless of the choice of constrains used. Since the NSB function is continuous, therefore on a compact constraint set, a global maximum exits. But, note that the first order conditions completely fail to identify even a local maximum, therefore a maximum has to lie on a boundary, regardless of the size the constrains set as long as it is nonempty and convex. It is straightforward to see that the problems of Specification 1 continue to hold even if we replace $s^{1/2}$ and $t^{1/2}$ with functions like $s^{1/k}$ and $t^{1/k}$, for $k > 1$. In fact, this "bad behavior" extends to version 1 of the standard model as such. Under Version 1 if we set $x = y = t = 0$, then $NSB(s, 0, 0, 0) = u_0(s)$. Similarly, $NSB(0, 0, t, 0) = u(t)$. The functions $u_0(s)$ and $v_0(t)$ are monotonically increasing. Clearly, as long as $u_0(s)$ or $v_0(t)$ is unbounded, no maximum can exists and use of the first order conditions will be misleading.

Next, we consider non-monotonic and bounded above functions, such as $u_0(s) = s^{1/2} - \delta s$ and $v_0(t) = t^{1/2} - \delta t$, where $x, y > 0$. But this is exactly what we have for specification 2, which gives us $NSB = s^{1/2} - xs - \delta s + t^{1/2} - yt - \delta t - \frac{50st}{1+x+y}$. Let $\delta = 0.01$. Clearly, the NSB function is bounded and a global maximum does exist. However, as seen above, the first order conditions give $(s^*, x^*, t^*, y^*) = (0.0582945, 0.353629, 0.0582945, 0.353629)$. This point satisfies the following property: given (s^*, x^*), the point (t^*, y^*) maximizes NSB, and vice-versa. See Plots 2 and 3. Still, the point $(0.0582945, 0.353629, 0.0582945, 0.353629)$

[22]Use of Mathematica throws up several solutions to the global maximization problem with $NSB =$ 1000.00002 approximately. All of the global maxima seem to converge to these two points.

is neither a maximum nor a minimum for NSB. As in the case of Specification 1, here too (s^*, x^*, t^*, y^*) is a saddle point in s/t space. See Plot 4.

With this NSB, there are two global maxima, the points $(s, x, t, y) = (5.01409E^{-7}, 352.551, 2499.97, 0)$ and $(s, x, t, y) = (2499.97, 0, 5.01409E^{-7}, 352.551)$. Neither is an interior point. The value of NSB at each of global maxima is approximately 25.0004.

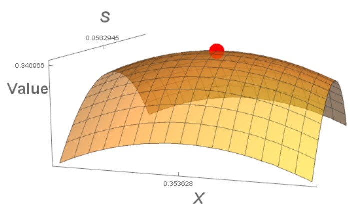

FIGURE 2
Value of NSB at (t^*, y^*) about (s^*, x^*) in (s, x) space.

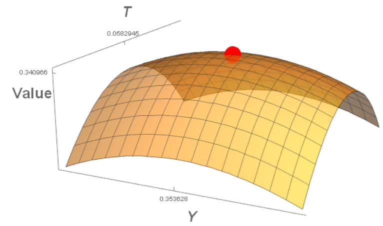

FIGURE 3
Value of NSB at (s^*, x^*) about (t^*, y^*) in (t, y) space.

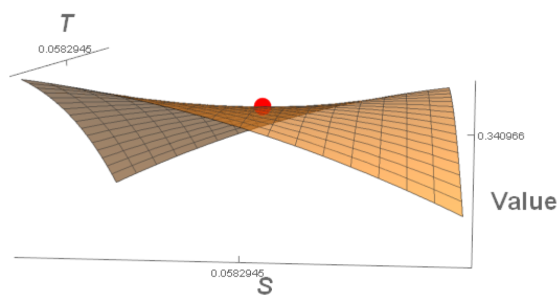

FIGURE 4
Value of NSB at (x^*, y^*) about (s^*, t^*) in (s, t) space.

In fact, specification 2 suffers from problems similar to specification 1. Under Specification 2, $NSB(s, 0, 0, 0) = s^{1/2} - \delta s$. Let $\bar{s}(\delta)$ solve $\max_s\{s^{1/2} - \delta s\}$. That is, $\bar{s}(\delta) = \frac{1}{4\delta^2}$. Clearly, $NSB(\bar{s}, 0, 0, 0) = \bar{s}^{1/2}(\delta) - \delta s$ can be made arbitrarily large by choosing sufficiently small δ. The analysis above shows that even when $\bar{s}$ is moderate ($\bar{s} = 2500$), the first best is not interior. The problem only becomes worse when $\bar{s}$ is large.

It should be noted that for each specification, as marginal gain from s, t approaches ∞ as $s, t \to 0$, we can rule out the maxima at boundary where $s, t = 0$. Therefore, a corner maximum will have either $x = 0$ or $y = 0$.

Summing up, the outcomes under Specifications 1 and 2 show similar features. Under each specification, a global maximum for NSB requires either: very high activity with zero care for the injurer, along with almost zero activity with very high care for the victim, or vice-versa. Moreover, none of the maxima is interior. Also, as is shown in Appendix I, as long as there is a corner global maximum, the rules of strict liability for the injurer and no liability for the injurer are more efficient than any of the negligence based rules 1-6.

Problems beyond Specifications 1 and 2:

Things do not change if $s^{1/2}$ and $t^{1/2}$ are replaced with $s^{1/k}$ and $t^{1/k}$, for any $k > 1$. For Specification 2, we have tried several variations; e.g., by replacing terms $-0.01s$ and $-0.01t$ with terms like $-0.001s$ and $-0.001t$. The above problems continue to persist - there is no interior global maxima. For both specifications, we have also examined the outcomes by replacing $\phi(s,t) = st$ with $\phi(s,t) = \sqrt{st}$ and $\phi(s,t) = (st)^2$, but qualitative nature of results does not change.

Appendix III: Proofs of Propositions

Proof of Proposition 1: We want to prove that a Nash equilibrium must lie either on the line defined by $x = x^*$ or on the line defined by $y = y^*$. In view of the discussion preceding Proposition 1, we just need to prove that: A profile $((\bar{s},\bar{x}),(\bar{t},\bar{y})))$, such that $\bar{x} < x^*$ & $\bar{y} < y^*$, cannot be an equilibrium. Take any $((\bar{s},\bar{x}),(\bar{t},\bar{y}))$ such that $\bar{x} < x^*$ and $\bar{y} < y^*$. Suppose, the injurer opts for $(\bar{s},\bar{x})$ and the victim for $(\bar{t},\bar{y})$. At $((\bar{s},\bar{x}),(\bar{t},\bar{y}))$, let $w_X(\bar{x},\bar{y})$ be the injurer's share of loss, and $w_Y(\bar{x},\bar{y}) = 1 - w_X(\bar{x},\bar{y})$ be the share of the victim. At $((\bar{s},\bar{x}),(\bar{t},\bar{y}))$, the expected payoff of the victim is

$$v(\bar{t},\bar{y}) - (1 - w_X(\bar{x},\bar{y}))l(\bar{s},\bar{x},\bar{t},\bar{y})).$$

On the other hand, given that $(\bar{s},\bar{x})$ is opted by the injurer, if the victim instead opts for (t^*,y^*), then the injurer will be solely negligent. In that case, in view of (A1), the injurer's liability is full and that of the victim is none. Therefore, given that $(\bar{s},\bar{x})$ is opted by the injurer, if the victim opts for (t^*,y^*), his payoff will be $v(t^*,y^*)$. Similarly, at $((\bar{s},\bar{x}),(\bar{t},\bar{y}))$ the expected payoff of the injurer is $u(\bar{s},\bar{x}) - w_X(\bar{x},\bar{y})l(\bar{s},\bar{x},\bar{t},\bar{y}))$. But, given that $(\bar{t},\bar{y})$ is opted by the victim, should the injurer instead opt for (s^*,x^*), his payoff will be $u(s^*,x^*)$. At $((\bar{s},\bar{x}),(\bar{t},\bar{y}))$ if

$$u(s^*,x^*) > u(\bar{s},\bar{x}) - w_X(\bar{x},\bar{y})l(\bar{s},\bar{x},\bar{t},\bar{y}),$$

a unilateral deviation by the injurer to (s^*,x^*) is strictly profitable. In that case, $((\bar{s},\bar{x}),(\bar{t},\bar{y}))$ cannot be a N.E. Thus, let us assume

$$u(\bar{s},\bar{x}) - w_X(\bar{x},\bar{y})l(\bar{s},\bar{x},\bar{t},\bar{y})) \geq u(s^*,x^*). \tag{6.1}$$

Since $((\bar{s},\bar{x}),(\bar{t},\bar{y})) \neq ((s^*,x^*),(t^*,y^*))$, by assumption, we know that

$$u(s^*,x^*) + v(t^*,y^*) - l(s^*,x^*,t^*,y^*) > u(\bar{s},\bar{x}) + v(\bar{t},\bar{y}) - l(\bar{s},\bar{x},\bar{t},\bar{y})). \tag{6.2}$$

Subtracting $u(s^*,x^*)$ from the LHS and $u(\bar{s},\bar{x}) - w_X(\bar{x},\bar{y})l(s,x,t,y)$ from the RHS of (6.2), in view of (6.1), we get

$$v(t^*,y^*) - l(s^*,x^*,t^*,y^*) > v(\bar{t},\bar{y}) - (1 - w_X(\bar{x},\bar{y}))l(\bar{s},\bar{x},\bar{t},\bar{y})). \tag{6.3}$$

Now, since $l(s^*,x^*,t^*,y^*) \geq 0$, from (6.3) we have

$$v(t^*,y^*) > v(\bar{t},\bar{y}) - (1 - w_X(\bar{x},\bar{y}))l(\bar{s},\bar{x},\bar{t},\bar{y})).$$

That is, given $(\bar{s},\bar{x})$ opted by the injurer, , the victim is better off opting (t^*,y^*) rather than $(\bar{t},\bar{y})$. Again, $((\bar{s},\bar{x}),(\bar{t},\bar{y}))$ cannot be a N.E.

That is, when $\bar{x} < x^*$ & $\bar{y} < y^*$, $((\bar{s}, \bar{x}), (\bar{t}, \bar{y})))$ cannot be an equilibrium. Hence, if an an equilibrium exists it must lie either on the line defined by $x = x^*$ or on the line defined by $y = y^*$. □

Proof of Proposition 2: Given $w_X^* = 0$, when $\bar{x} > x^*$ profile $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ cannot be a N.E. So, consider the case of $\bar{x} < x^*$. As to $\bar{y}$, there are three mutually exclusive and jointly exhaustive possibilities: $\bar{y} > y^*$, $\bar{y} = y^*$ or $\bar{y} < y^*$. We have already ruled out N.E. with $\bar{x} < x^*$ and $\bar{y} > y^*$, we well as with $\bar{x} < x^*$ and $\bar{y} < y^*$. Therefore, consider a N.E. involving a choice of $\bar{x} < x^*$ by X with $y = y^*$ opted by Y. With these choices, in view of Axiom (A2), the entire burden falls X, so Y will choose $t = t_p^*$. This means, $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ is N.E. implies $((\bar{s}, \bar{x}), (\bar{t}, \bar{y})) = ((\bar{s}, \bar{x}), (t_p^*, y^*))$. But, $((\bar{s}, \bar{x}), (t_p^*, y^*))$ is a N.E. and $\bar{x} < x^*$ imply:

$$u(\bar{s}, \bar{x}) - l(\bar{s}, \bar{x}, t_p^*, y^*) \quad \geq \quad u(s_p^*, x^*).$$

The right hand side of the inequality follows from the fact that when $w_X^* = 0$ under Axioms (A1)-(A3), if X opts for (s_p^*, x^*), the entire burden of accident loss falls on Y. Adding $v(t_p^*, y^*)$ to both sides, we get: $u(\bar{s}, \bar{x}) + v(t_p^*, y^*) - l(\bar{s}, \bar{x}, t_p^*, y^*) \geq u(s_p^*, x^*) + v(t_p^*, y^*)$. But, $u(s_p^*, x^*) + v(t_p^*, y^*) > u(s^*, x^*) + v(t^*, y^*)$, since $s_p^* > s^*$ and $t_p^* > t^*$. This gives us,

$$u(\bar{s}, \bar{x}) + v(t_p^*, y^*) - l(\bar{s}, \bar{x}, t_p^*, y^*) > u(s^*, x^*) + v(t^*, y^*) - l(s^*, x^*, t^*, y^*)),$$

which cannot be true. Therefore, $((\bar{s}, \bar{x}), (t_p^*, y^*)$ cannot be a N.E. Hence, under Axioms (A1)-(A3) with $w_X^* = 0$, $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ can be a N.E only if $\bar{x} = x^*$.

Similarly, it can be shown that when $w_X^* = 1$, the profile $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ can be a N.E. only if: $\bar{y} = y^*$. □

Proof of Proposition 3: Suppose a liability satisfies Axioms (A1)-(A3) with $w_X^* = 1$. To see that the condition NDDX is necessary, suppose there exists a N.E., say $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ when Axioms (A1)-(A3) hold with $w_X^* = 0$. In view of Proposition 2, we just have to consider the case $\bar{x} = x^*$, i.e., in equilibrium X chooses x^*. Given $w_X^* = 0$ and Axiom 3, for X the best choice of activity level is s_p^*, as defined above. That is, $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ is N.E. implies $(\bar{s}, \bar{x}) = (s_p^*, x^*)$. Furthermore, given that X opts for (s_p^*, x^*) and $w_X^* = 0$, the entire burden of accident loss falls on Y, regardless of the choice of activity and care level by Y. So, Y's best response is to choose $(\hat{t}, \hat{y})$, as defined above. That is, $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ is N.E. and $\bar{x} = x^*$ implies $((\bar{s}, \bar{x}), (\bar{t}, \bar{y})) = ((s_p^*, x^*), (\hat{t}, \hat{y}))$. To sum up, if X chooses x^*, the N.E. will have (s_p^*, x^*) opted by X and $(\hat{t}, \hat{y})$ opted by Y.

However, when condition NDDX does not hold, we have $u(\hat{\hat{s}}, \hat{\hat{x}}) - l(\hat{\hat{s}}, \hat{\hat{x}}, \hat{t}, \hat{y}) > u(s_p^*, x^*)$. That is, given that $(\hat{t}, \hat{y})$ is opted by Y, even if deviation from (s_p^*, x^*) makes the injurer fully liable, he is better off unilaterally deviating to $(\hat{\hat{s}}, \hat{\hat{x}})$. Hence, (s_p^*, x^*) cannot a best response for X, given $(\hat{t}, \hat{y})$ opted by Y. By implication, $((\bar{s}, \bar{x}), (\bar{t}, \bar{y}))$ with $\bar{x} = x^*$ cannot be a N.E.

Similarly, we can prove that when $w_X^* = 1$, if condition NDDY does not hold the liability rule cannot have a Nash equilibrium. □

Proof of Proposition 4: First consider the case when $w_X^* = 0$. It implies that if the injurer's opts for x^* his best choice of activity level is s_p^*. Under Axioms (A1)-(A3), if the injurer opts for (s_p^*, x^*), the entire burden of accident loss falls on the victim, regardless of the choice of activity and care level by the victim.

Now, consider the choice of (s_p^*, x^*) by X and of $(\hat{t}, \hat{y})$ by Y. At these choices the entire burden of accident loss falls on the victim. Moreover, in view of the above, $\hat{y} > y^*$.

By definition $(\hat{t}, \hat{y})$ is a best response choice by Y, if (s_p^*, x^*) is chosen by X. So, consider the unilateral deviations by the injurer from $((s_p^*, x^*), (\hat{t}, \hat{y}))$ to some $(\bar{s}, \bar{x})$. Clearly, the injurer is worse off choosing a $\bar{x} > x^*$. If the injurer unilaterally deviates downward to $\bar{x} < x^*$, he will become fully liable for the accident loss as $\hat{y} > y^*$. So, if X deviates to some $(\bar{s}, \bar{x})$, $\bar{x} < x^*$, his payoff will be $u(\bar{s}, \bar{x}) - l(\bar{s}, \bar{x}, \hat{t}, \hat{y}))$.

Note that when $(\hat{t}, \hat{y})$ is opted by Y, if X has to bear the entire loss, the injurer's best choice is given by $(\hat{\hat{s}}, \hat{\hat{x}})$ as defined above; any other choice will be (weakly) worse. In particular, for any $(\bar{s}, \bar{x})$ such that $\bar{x} < x^*$ the condition NDDX, by definition, gives us:

$$u(\bar{s}, \bar{x}) - l(\bar{s}, \bar{x}, \hat{t}, \hat{y})) \leq u(\hat{\hat{s}}, \hat{\hat{x}}) - l(\hat{\hat{s}}, \hat{\hat{x}}, \hat{t}, \hat{y})$$

But, due to the condition NDDX, even $(\hat{\hat{s}}, \hat{\hat{x}})$ is worse than (s_p^*, x^*). That is, given $(\hat{t}, \hat{y})$ opted by the victim, (s_p^*, x^*) is a best response for the injurer. Formally, $((s_p^*, x^*), (\hat{t}, \hat{y}))$ is a N.E. Moreover, in view of Proposition 1, a Nash equilibrium will necessarily have x^* opted by the injurer. As is seen above, this implies that $((s_p^*, x^*), (\hat{t}, \hat{y}))$ is the only N.E. possible.

When $w_X^* = 1$, using a reasoning similar to the one above with the roles of X and Y swapped, it can be seen that the profile $((\hat{s}, \hat{x}), (t_p^*, y^*))$ is a unique Nash equilibrium under the rule. $\square$

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In[11]:=
```
s == (1/4) / (1 + 2 * x + y) ^2
50 * t == (1 + x + y) ^2
t == (1/4) / (1 + x + 2 * y) ^2
50 * s == (1 + x + y) ^2
NSolve[{s == (1/4) / (1 + 2 * x + y) ^2, 50 * t == (1 + x + y) ^2, t == (1/4) / (1 + x + 2 * y) ^2,
  50 * s == (1 + x + y) ^2, s == t, x == y, s ≥ 0, t ≥ 0, x ≥ 0, y ≥ 0}, {s, t, x, y}]
```

Out[11]= $s == \frac{1}{4\left(1+2x+y\right)^2}$

Out[12]= $50t == \left(1+x+y\right)^2$

Out[13]= $t == \frac{1}{4\left(1+x+2y\right)^2}$

Out[14]= $50s == \left(1+x+y\right)^2$

Out[15]= $\{\{s \to 0.0585468, t \to 0.0585468, x \to 0.355473, y \to 0.355473\}\}$

In[6]:=
```
(1/2) * s^(-1/2) == x + 0.01 + 50 * t / (1 + x + y)
50 * t == (1 + x + y) ^2
(1/2) * t^(-1/2) == y + 0.01 + 50 * s / (1 + x + y)
50 * s == (1 + x + y) ^2
NSolve[{(1/2) * s^(-1/2) == x + 0.01 + 50 * t / (1 + x + y),
  50 * t == (1 + x + y) ^2, (1/2) * t^(-1/2) == y + 0.01 + 50 * s / (1 + x + y),
  50 * s == (1 + x + y) ^2, s == t, x == y, s ≥ 0, t ≥ 0, x ≥ 0, y ≥ 0}, {s, t, x, y}]
```

Out[6]= $\frac{1}{2\sqrt{s}} == 0.01 + x + \frac{50t}{1+x+y}$

Out[7]= $50t == \left(1+x+y\right)^2$

Out[8]= $\frac{1}{2\sqrt{t}} == 0.01 + y + \frac{50s}{1+x+y}$

Out[9]= $50s == \left(1+x+y\right)^2$

Out[10]= $\{\{s \to 0.0582945, t \to 0.0582945, x \to 0.353629, y \to 0.353629\}\}$

- Specification 1

```
In[51]:= N[FindMaximum [{Sqrt[s] - x * s + Sqrt[t] - y * t - 50 * s * t / (1 + x + y),
            1000000 ≥ s ≥ 0, 1000000 ≥ t ≥ 0, 1000000 ≥ x ≥ 0, 1000000 ≥ y ≥ 0},
          {s, x, t, y}, WorkingPrecision → 50], 9]
```

Out[51]= $\{1000.00002,\ \{s \to 1.25029110 \times 10^{-9},\ x \to 7070.18455,\ t \to 1.00000000 \times 10^{6},\ y \to 1.38532753 \times 10^{-17}\}\}$

- Specification 2

```
In[52]:= N[FindMaximum [{Sqrt[s] - x * s - 0.01 * s + Sqrt[t] - y * t - 0.01 * t - 50 * s * t / (1 + x + y),
            s ≥ 0, t ≥ 0, x ≥ 0, y ≥ 0}, {s, x, t, y}, WorkingPrecision → 50], 6]
```

Out[52]= $\{25.0004,\ \{s \to 2499.97,\ x \to 0,\ t \to 5.01409 \times 10^{-7},\ y \to 352.551\}\}$

- Claim 1, Case 2

```
In[53]:= NMaximize [{Sqrt[t] - y * t - 50 * 1.978455 * t / (1 + 0.355473 + y), t ≥ 0, y ≥ 0}, {t, y}]
```

Out[53]= {0.0134869 , {t → 0.000727586 , y → 8.59052 }}

- Claim 1, Case 2

```
In[54]:= N[FindMaximum [{Sqrt[t] - y * t - 50 * 1.978455 * t / (1 + 0.355473 + y), t ≥ 0, y ≥ 0},
          {t, y}, WorkingPrecision → 50], 10]
```

Out[54]= {0.01348689671 , {t → 0.0007275855320 , y → 8.590518655 }}

- Claim 1, Case 3

```
In[58]:= NMaximize [
          {Sqrt[s] - x * s - 50 * s * 1000000 / (1 + x), 1000000 ≥ s ≥ 0, 1000000 ≥ x ≥ 0}, {s, x}]
```

Out[58]= $\{0.0000176789,\ \{s \to 1.25018 \times 10^{-9},\ x \to 7070.07\}\}$

- Claim 1, Case 3

```
In[45]:= N[FindMaximum [{Sqrt[s] - x * s - 50 * s * 1000000 / (1 + x),
            1000000 ≥ s ≥ 0, 1000000 ≥ x ≥ 0}, {s, x}, WorkingPrecision → 50], 10]
```

Out[45]= $\{0.00001767891962,\ \{s \to 1.250176795 \times 10^{-9},\ x \to 7070.067812\}\}$

- Claim 1, Case 3, Spf 2

```
In[59]:= NMaximize [{Sqrt[s] - x * s - 0.01 * s - 50 * s * 2500 / (1 + x), s ≥ 0, x ≥ 0}, {s, x}]
```

Out[59]= $\{0.000354049,\ \{s \to 5.01403 \times 10^{-7},\ x \to 352.553\}\}$

- Claim 1, Case 3, Spf 2

```
In[60]:= N[FindMaximum [{Sqrt[s] - x * s - 0.01 * s - 50 * s * 2500 / (1 + x), s ≥ 0, x ≥ 0},
          {s, x}, WorkingPrecision → 50], 10]
```

Out[60]= $\{0.0003540490846,\ \{s \to 5.014030172 \times 10^{-7},\ x \to 352.5533906\}\}$

- Claim 3

In[61]:=
```
NMaximize [
 {Sqrt[s] - x*s - 0.01 *s - 50*s*1.890707 /(1 + x + 0.353629 ), s ≥ 0, x ≥ 0}, {s, x}]
```

Out[61]=
```
{0.0138105 , {s → 0.000762915 , x → 8.3693 }}
```

- Claim 3

In[62]:=
```
N[FindMaximum [{Sqrt[s] - x*s - 0.01 *s - 50*s*1.890707 /(1 + x + 0.353629 ),
    s ≥ 0, x ≥ 0}, {s, x}, WorkingPrecision → 50], 10]
```

Out[62]=
```
{0.01381045382 , {s → 0.0007629145388 , x → 8.369300085 }}
```