ISSN No. 2454 - 1427

CDE
June 2020

Unraveling of Value-Rankings in Auctions with Resale

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Working Paper No. 308

http://econdse.org/wp-content/uploads/work308.pdf

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Unraveling of value-rankings in auctions with resale

Sanyyam Khurana[*]

Abstract

Consider a single-unit auction with resale and two risk neutral bidders. The ranking of the valuations is known to both the bidders—that is, the bidders know the identity of the highest and lowest valuation bidders. We show that, with the revelation of value-rankings, the classic result of "bid symmetrization" does not hold. Surprisingly, the bidder with the lowest valuation produces a stronger bid distribution than the bidder with the highest valuation. We also show that the revelation of value-rankings in auctions with resale *asymmetrizes* the bidding strategies. Finally, under restrictive assumptions, we compare seller's and bidders' ranking of a first-price and second-price auction.

JEL classification: D44, D82

Keywords: resale, asymmetry, private values, value-rankings

1 Introduction

Consider an indivisible object for sale through an auction mechanism. Two risk neutral bidders participating in the auction have private information about their valuations. Their valuations are drawn from independent and asymmetric probability distributions. Furthermore, the ranking of the valuations is revealed to both the bidders—that is, the bidders know the identity of the highest and lowest valuation bidders. In this sense, we distinguish the bidders as *high-type* and *low-type* where the realized valuation of the high-type bidder is greater than that of the low-type bidder. The game is played in two stages. Stage 1 is the *bidding stage* where the seller conducts a sealed-bid auction. The bidder with the highest bid wins the auction and pays according to the underlying auction mechanism. Stage 2 is the *resale stage* where the winner of the auction may make a single offer to the loser.

Common knowledge of the value-rankings were first considered in auctions without resale by Landsberger et al. [7]. It increases the information set of every bidder which in turn changes their bidding behavior and other properties of equilibrium. However, it leads to inefficiency of a first-price auction—the

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bidder with the lowest valuation also wins the auction with positive probability.[1] Thus, bidders can be made better off if there are resale opportunities. Some of the instances where value-rankings and resale opportunities arise naturally include antique auctions, art auctions, spectrum license auctions, real estate auctions, old car auctions, procurement auctions and takeover auctions. In all of these auctions, bidders may imperfectly predict their position in the game in terms of the true valuation of other bidders in any of the following way: (a) past data, (b) competitive intelligence, and (c) economic/corporate espionage. (a) means interactions in the past with particular bidder(s); (b) can be gathered through newspapers, media, business magazines, publicly available data, etc; and (c) is a common practice conducted by an organization on other organizations which may be misused at different levels. A well known example of corporate espionage is acquisition of Clairol by Procter and Gamble through an auction where Unilever also participated. In 2001, Procter and Gamble was accused of spying Unilever and paid $10 million as compensation. Later, in 2016, Clairol was resold to Coty, Inc.

Consider the following two features in single-unit auctions: (a) value-rankings are revealed to the bidders, and (b) there are resale markets. In this paper, we study the bidding behavior and other important properties in the presence of both the features. The main findings of the paper are as follows:
(A) The low-type bidder produces a stronger bid distribution than the high-type bidder, i.e., the *ex-ante* probability of winning the object is more for the low-type bidder.
(B) Revelation of value-rankings in auctions with resale *asymmetrizes* the bidding strategies.
(C) Under a parametric restriction on the distribution functions, a first-price auction is revenue superior to a second-price auction.
(D) The low-type bidder always prefers a first-price auction over a second-price auction, and under a parametric restriction on the valuation of the high-type bidder, the high-type bidder also prefers a first-price auction over a second-price auction.

In contrast to the literature, when only feature (a) is considered, Landsberger et al. [7] shows that, with uniform distributions, (i) the seller's *ex-ante* expected revenue is more in a first-price auction than in a second-price auction, and (ii) the low-type bidder prefers a first-price auction over a second-price auction and the high-type bidder prefers a second-price auction over a first-price auction. It is important to note that when probability distributions of the valuations are *ex-ante* asymmetric, it is not possible to compare the bid distributions. One implication of our result (A) is that the inclusion of resale markets allow us to compare the bid distributions.

When only feature (b) is considered, Hafalir and Krishna [3] shows that the *ex-ante* winning probability of both the bidders is equal. This fact is known as *bid symmetrization.* It is important to note that bid symmetriza-

[1] Corollary 1 of Landsberger et al. [7].

tion result holds without the assumption of stochastic orders on the probability distributions. They also show that the seller's *ex-ante* expected revenue generated from a first-price auction is more than that from a second-price auction.

In the absence of both the features (a) and (b), Maskin and Riley [11] shows that the bidder who is more likely to get a higher valuation ends up winning the auction more often than the bidder who is less likely to get a higher valuation. In other words, the bidder with a stochastically higher value distribution produces a stronger bid distribution than the bidder with a stochastically lower value distribution. As it is evident from the above result, stochastic orders on the probability distribution of the valuations are necessary to compare the bid distributions. They also show that the general revenue rankings of the seller for the first-price and second-price auction cannot be established. Furthermore, the bidder who is more likely to get a higher valuation prefers a second-price auction over a first-price auction and the bidder who is less likely to get a higher valuation prefers a first-price auction over a second-price auction.

We discuss the reason for the low-type bidder to produce a stronger bid distribution. There is a trade-off between the chances of winning the auction and utility generated from it. In equilibrium, the conditional belief of a bidder that the other bidder loses the auction by a small margin is equal to the inverse of his marginal utility. Due to resale markets, the marginal utility of both the bidders is equal. But, the belief of low-type bidder that high-type bidder loses the auction by a small margin conditional on the fact that his valuation is lowest is greater than his unconditional belief, whereas the belief of high-type bidder that low-type bidder loses the auction by a small margin conditional on the fact that his valuation is highest is equal to his unconditional belief. In other words, the formation of conditional beliefs are binding only for the lowest valuation bidder.

We are also interested in capturing the impact of revealed value-rankings in auctions with resale. The best possible way to do this is to compare the bidding strategies in the presence and absence of value-rankings. However, the nature of the differential equations characterizing the equilibria restrict us to compare the bidding strategies. Nevertheless, we can compare the *connecting functions* in the presence and absence of value-rankings. A connecting function is a map describing the valuation required by a bidder to match the bid made by the other bidder. For instance, a higher connecting function of a particular bidder with respect to the other bidder represents that the bidding strategies of the two bidders are more asymmetric. Theorem 2 of this paper shows that the valuation required by the high-type bidder to match the bid made by the low-type bidder is more in the presence of value-rankings. As a special case, whenever the probability distribution of the high-type bidder dominates that of the low-type bidder in terms of reverse hazard rate, the low-type bidder increases his level of aggression against the high-type bidder in the presence of value-rankings. In other words, revelation of value-rankings in auctions with resale *asymmetrizes* the bidding strategies.

It is a well-known fact that, in the absence both the features (a) and (b), bid-your-own-value is a weakly dominant strategy for every bidder in a second-price auction. Hafalir and Krishna [3] notices that, in the presence of feature (b)—the inclusion of resale markets, bid-your-own-value is no longer a weakly dominant strategy. Nevertheless, it is still an equilibrium strategy. The above proposition is still valid if we consider both the features (a) and (b). In fact, the opportunity of resale markets is the only driving force which is leading to the violation of a dominant strategy. Since bid-your-own-value is an equilibrium strategy, it always lead to an efficient outcome—the object is won by the bidder with the highest valuation. Thus, the game never reaches the resale stage despite having the resale markets. We compare the ranking of a first-price and second-price auction for the bidders and seller. We show that, under parametric conditions on the distribution functions of the valuations, the seller generates more expected revenue from a first-price auction than from a second-price auction. We also show that the low-type bidder always prefer a first-price auction and, for not so high realized valuation, the high-type bidder also prefers a first-price auction.

1.1 The literature

Landsberger et al. [7] characterizes and proves the existence of a unique Bayesian equilibrium in a first-price auction when only feature (a) is present. Feature (b) has been considered by Hafalir and Krishna [3, 4]; Virág [12, 13]; Lebrun [10]; Cheng and Tan [2]; and Cheng [1]. Hafalir and Krishna [3] characterizes and proves the existence of a unique perfect Bayesian equilibrium in a first-price auction with two bidders. Virág [12] extends the analysis of Hafalir and Krishna [3] for more than two bidders. They show that bid symmetrization does not hold when there are more than two bidders. Cheng and Tan [2] shows that a common value auction without resale is bid equivalent to a first-price auction with resale. Lebrun [10] studies the effect of revelation of bids after the bidding stage on equilibrium behavior. He constructs a behavioral equilibrium when the bids are not revealed and shows that this equilibrium is equivalent to a separating equilibrium where the bids are revealed. Virág [13] studies the impact of reserve price on bidding behavior and expected revenue. Maskin and Riley [11]; Lebrun [9]; Lebrun [8]; and Kirkegaard [5] considers auctions without features (a) and (b).

1.2 Outline

The outline of the paper is as follows. In section 2, we formalize the model and describe the equilibria. In section 3, we study the comparative results. In section 4, we compare every bidders' and the seller's ranking of the first-price and second-price auction. Section 5 concludes the paper. The proofs are collected in Appendix A.

2 The environment

Consider a first-price sealed bid auction for an indivisible object. There are two bidders with risk neutral preferences. The set of bidders is denoted by $N = \{1, 2\}$. The valuation (or type) space is same for both the bidders and is given by $T = [0, \bar{a}] \subset \Re_+$ where $\bar{a} > 0$. The random variables of the valuation for bidder 1 and 2 are given by $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. The distribution functions of $\mathcal{T}_1$ and $\mathcal{T}_2$ are independently distributed and are given by $F_1 : T \to \Re_+$ and $F_2 : T \to \Re_+$ respectively. We assume that the distribution functions are twice continuously differentiable and the corresponding density functions, denoted by f_1 and f_2, are positive and always bounded away from zero. For simplicity, we also assume that the seller is risk neutral and there is no reserve price.

The timing of the game is as follows:
(1) At $t = 0$, every bidder knows the distribution functions of the valuations.
(2) At $t = 1/2$, nature draws the valuation of each bidder and informs them privately. Moreover, it also reveals the ranking of the valuations.
(3) At $t = 1$, the seller of the object conducts a first-price auction. We call this as the *bidding stage*.
(4) At $t = 2$, the winner of the auction may make a single offer to the loser. We call this as the *resale stage*.

The game ends at $t = 2$ and there is no further resale of the object. Notice that $t = 0$ is the *ex-ante* stage and $t = 1/2$ is the *interim* stage. So, at $t = 1/2$, every bidder knows (a) his own valuation, (b) the distribution function of the other bidder and (c) the ranking of the valuations, i.e., the fact that his realized valuation is greater or less than that of the other bidder. However, he does not know the difference between the two valuations. In this sense, we distinguish the bidders as *high-type* and *low-type* where the realized valuation of the high-type bidder is more than that of the low-type bidder. We assume that the losing bid is not revealed after the bidding stage.[2] Furthermore, we do not assume any kind of stochastic orders on the distributions of the valuation. So, without loss of generality, we assume that bidder 1 is of high-type and bidder 2 is of low-type.

We make the following assumption on the distribution functions.

Assumption 1. *The hazard function of the valuation, given by*

$$\frac{f_i(t)}{1 - F_i(t)}$$

[2]If the losing bid was revealed after the bidding stage, then this becomes a game of complete information. In that case, the winner makes a single offer equal to the valuation of the loser, which is always accepted by the loser. Therefore, efficiency is always attained under this scenario. However, assuming that losing bid is revealed is too strong an assumption.

is strictly increasing in t for every $i \in N$.[3]

We restrict our attention to the class of bidding strategies that are measurable, strictly increasing and continuous functions. For every $i \in N$, the bidding and inverse bidding strategies are denoted by β_i and ϕ_i respectively. The following Lemma discusses the direction of trade. It tells that the low-type bidder makes a resale offer and the high-type bidder does not make a resale offer only if the low-type bidder bids more aggressively than the high-type bidder.

Lemma 1. *Whenever $\beta_2(t) > \beta_1(t)$ for every $t \in T - \{0, \bar{a}\}$, the low-type bidder (bidder 2) makes a resale offer whereas the high-type bidder (bidder 1) does not make a resale offer.*

To see why the above result is true, first consider the low-type bidder. Since the low-type bidder bids more aggressively than the high-type bidder, the valuation of the low-type bidder is less than the valuation required by the high-type bidder to match the bid made by the low-type bidder. Therefore, there are potential gains from trade if the low-type bidder makes a resale offer. Similarly, we can argue that the high-type bidder never makes a resale offer.

We begin the analysis by assuming that the low-type bidder bids more aggressively than the high-type bidder and later show that this is indeed the case. We require this assumption before setting up the optimization problems of both the bidders because we need to know the direction of the trade.

Since a bidder knows his own valuation and the ranking of the valuations, he updates his belief about the other bidder. So, the high-type bidder now knows that the valuation of low-type bidder is drawn from $[0, t_1]$. Therefore, the *truncated density function* (left-truncation) of the low-type bidder is

$$g_2(t_2|\mathcal{T}_2 < t_1) = \begin{cases} \frac{f_2(t_2)}{F_2(t_1)} & \text{if } 0 \leq t_2 \leq t_1 \\ 0 & \text{otherwise} \end{cases}$$

The corresponding truncated distribution is

$$G_2(t_2|\mathcal{T}_2 < t_1) = \frac{F_2(t_2)}{F_2(t_1)}$$

for $t_2 \in [0, t_1]$. On the other hand, the low-type bidder now knows that the valuation of high-type bidder is drawn from $[t_2, \bar{a}]$. Therefore, the *truncated*

[3]This assumption is equivalent to assuming that the virtual valuation of the bidders are regular—that is,

$$t - \frac{1 - F_i(t)}{f_i(t)}$$

is strictly increasing in t for every $i \in N$. This assumption is sufficient to establish the uniqueness of the resale price in equilibrium.

density function (right-truncation) of the high-type bidder is

$$g_1(t_1|\mathcal{T}_1 > t_2) = \begin{cases} \frac{f_1(t_1)}{1-F_1(t_2)} & \text{if } t_2 \leq t_1 \leq \bar{a} \\ 0 & \text{otherwise} \end{cases}$$

The corresponding truncated distribution is

$$G_1(t_1|\mathcal{T}_1 > t_2) = \frac{F_1(t_1) - F_1(t_2)}{1 - F_1(t_2)}$$

for $t_1 \in [t_2, \bar{a}]$.

We solve the game by backward induction. First, consider the resale stage at $t = 2$. From Lemma 1, bidder 1 does not make any resale offer. Therefore, there is no optimization problem for bidder 1. Consider bidder 2 with valuation t_2. Suppose he bids b, chooses a resale price p and bidder 1 follows his equilibrium inverse bidding strategy ϕ_1. We claim that $t_2 \leq \phi_1(b)$. To see this, suppose $t_2 > \phi_1(b)$. Since $\phi_1(b) > \phi_2(b)$, bidder 2 will win the auction with probability 0. Hence, it is profitable to raise his bid. Therefore, the optimization problem of bidder 2 is

$$\max_p \left[G_1(\phi_1(b)|\mathcal{T}_1 > t_2) - G_1(p|\mathcal{T}_1 > t_2)\right](p - b) + G_1(p|\mathcal{T}_1 > t_2)(t_2 - b)$$

The first-term in the optimization problem is the expected utility of bidder 2 when his offer is accepted, and the second term is the expected utility when his offer is rejected. Thus, the optimization problem of bidder 2 can be rewritten as

$$\max_p \frac{F_1 \circ \phi_1(b) - F_1(p)}{1 - F_1(t_2)}(p - b) + \frac{F_1(p) - F_1(t_2)}{1 - F_1(t_2)}(t_2 - b)$$

The first-order condition leads to the following equation

$$t_2 = p - \frac{F_1 \circ \phi_1(b) - F_1(p)}{f_1(p)} \tag{1}$$

From Assumption 1, the right hand side of the above equation is strictly increasing in the resale price. Hence, a unique p exists. Thus,

$$p(t_2, b) = \arg\max_p \frac{F_1 \circ \phi_1(b) - F_1(p)}{1 - F_1(t_2)}(p - b) + \frac{F_1(p) - F_1(t_2)}{1 - F_1(t_2)}(t_2 - b)$$

Notice that

$$t_2 < p(t_2, b) < \phi_1(b)$$

Moreover, from Assumption 1, $p(t_2, b)$ is increasing in t_2 and b. To save on notations, we write $p(b) := p(t_2, b)$. We now turn to the bidding stage problem.

First, consider bidder 2 with valuation t_2. Suppose he bids b and bidder 1 follows his equilibrium inverse bidding strategy. The optimization problem of bidder 2 is

$$\max_b \frac{F_1 \circ \phi_1(b) - F_1(p)}{1 - F_1(t_2)}(p - b) + \frac{F_1(p) - F_1(t_2)}{1 - F_1(t_2)}(t_2 - b)$$

Using Envelope theorem, we have the following first-order condition

$$F_1 \circ \phi_1(b) = \mathrm{D}F_1 \circ \phi_1(b)(p(b) - b) + F_1 \circ \phi_2(b) \tag{2}$$

Notice that

$$\frac{F_1 \circ \phi_1(b)}{\mathrm{D}F_1 \circ \phi_1(b)} > p(b) - b \tag{3}$$

Now, consider bidder 1 with valuation t_1. Suppose he bids b and bidder 2 follows his equilibrium inverse bidding strategy. We claim that $\phi_2(b) \leq t_1$. To see this, suppose $\phi_2(b) > t_1$. Since $\phi_1(b) > \phi_2(b)$, bidder 1 can reduce his bid slightly and still win with probability 1. Thus, the optimization problem of bidder 1 is

$$\max_b G_2(\phi_2(b)|\mathcal{T}_2 < t_1)(t_1 - b) + \int_{\phi_2(b)}^{t_1} \mathrm{d}t_2 \max\{t_1 - p(\beta_2(t_2), t_2), 0\} g_2(t_2|\mathcal{T}_2 < t_1)$$

The first-term in the optimization problem is the expected utility of bidder 1 when he wins the auction, and the second-term is the expected utility when he loses the auction and buys the object in the resale stage. Thus, we can re-write the optimization problem as

$$\max_b \frac{F_2 \circ \phi_2(b)}{F_2(t_1)}(t_1 - b) + \frac{1}{F_2(t_1)} \int_{\phi_2(b)}^{t_1} \mathrm{d}t_2 \max\{t_1 - p(\beta_2(t_2), t_2), 0\} f_2(t_2)$$

Using Leibniz integral rule, we get the following first-order condition

$$\frac{F_2 \circ \phi_2(b)}{\mathrm{D}F_2 \circ \phi_2(b)} = p(b) - b \tag{4}$$

In the following Proposition, we describe the equilibria.

Proposition 1. (ϕ_1, ϕ_2, p) *is an equilibrium profile if and only if it solves the following system*

$$\begin{aligned}
&F_1 \circ \phi_1(b) = \mathrm{D}F_1 \circ \phi_1(b)(p(b) - b) + F_1 \circ \phi_2(b) \\
&F_2 \circ \phi_2(b) = \mathrm{D}F_2 \circ \phi_2(b)(p(b) - b) \\
&\phi_2(b) = p(b) - \frac{F_1 \circ \phi_1(b) - F_1 \circ p(b)}{f_1 \circ p(b)} \\
&\phi_1(0) = \phi_2(0) = 0 \quad \& \quad \phi_1(\bar{b}) = \phi_2(\bar{b}) = \bar{a} \quad \exists \quad \bar{b} \in \Re_{++}.
\end{aligned} \tag{5}$$

The first two equations are the first-order differential equations derived during the bidding stage, the third equation determines the optimal resale price during the resale stage, and the fourth equation gives the relevant boundary conditions. The above result conveys that the first-order differential equations are both necessary and sufficient for an equilibrium. The sufficiency part states that local deviations are not profitable for bidder 1 and 2. Remark that, for bidder 2, local deviations are strictly worse off and, for bidder 1, local deviations are weakly worse off.

Until now, we have assumed that bidder 2 bids more aggressively than bidder 1. In what follows, we show that this is indeed the case.

Proposition 2. *Suppose Assumption 1 is satisfied. Then, the low-type bidder bids more aggressively than the high-type bidder, i.e.,*

$$\beta_2(t) > \beta_1(t)$$

for every $t \in (0, \bar{a})$.

Let us gain some intuition of the above result. To see why bidder 2 bids more aggressively than bidder 1, suppose this is not true. Instead, suppose that there exists some interval such that bidder 1 bids more aggressively than bidder 2. Then, for any particular valuation, bidder 2 bids less than bidder 1, and as we know that the realized valuation of bidder 1 is more than that of bidder 2, bidder 2 always loses the auction. Moreover, bidder 2 cannot purchase the object in the resale market because the resale price will be more than the valuation of bidder 1 which is again more than the valuation of bidder 2. Therefore, bidder 2 gets an assured utility of 0. In order to get a strictly positive utility, bidder 2 has to bid more aggressively than bidder 1.

The following result shows that bid symmetrization does not hold.

Theorem 1. *Suppose Assumption 1 is satisfied. Then, the low-type bidder produces a stronger bid-distribution than the high-type bidder, i.e.,*

$$F_2 \circ \phi_2(b) < F_1 \circ \phi_1(b)$$

for every $b \in (0, \bar{b})$.

The above result conveys that, for any $b \in (0, \bar{b})$, the probability of bidding at least b is more for bidder 2 as compared to bidder 1, i.e., bidder 2 has a higher *surviving rate* than bidder 1. In other words, the *ex-ante* probability of winning is more for the low-type bidder than the high-type bidder. Let us explore the reason why the above result holds. Let $\mathcal{Y}_i := \beta_i(\mathcal{T}_i)$ for every $i \in N$. Let the bid distributions be $\Sigma_i(b) := \Pr(\mathcal{Y}_i < b)$ and the corresponding bid density functions be denoted by σ_i for every $i \in N$. Consider a bid b and a real number $\epsilon > 0$ such that a bidder wins the auction if he bids b and loses the auction if he bids $b - \epsilon$. We argue that the *marginal utility* is same for both the bidders. Consider bidder 2 with valuation t_2. Whenever he bids

$b-\epsilon$, he loses the object and gets a utility of 0. Whenever he bids b, he wins the object and makes a resale offer to bidder 1. Bidder 1 accepts the offer since bidder 2 is at the margin. Thus, the utility of bidder 2 by bidding b is $p-b$. Therefore, the marginal utility by increasing his bid from $b-\epsilon$ to b is $(p-b)/\epsilon$. We can re-write (2) as

$$\lim_{\epsilon\downarrow 0}\frac{\int_{b-\epsilon}^{b}\sigma_1(x)\mathrm{d}x}{\epsilon[\Sigma_1(b)-\Sigma_1\circ\beta_1\circ\phi_2(b)]}=\frac{1}{p(b)-b} \tag{6}$$

For a given ϵ, $\int_{b-\epsilon}^{b}\sigma_1(x)\mathrm{d}x/[\Sigma_1(b)-\Sigma_1\circ\beta_1\circ\phi_2(b)]$ is the probability that bidder 1 bids between $b-\epsilon$ and b conditional on the event that his bid is between $\beta_1(t_2)$ and b. The left-hand side is the *conditional* reverse hazard function of the bid made by bidder 1 and it equals the inverse of change in utility of bidder 2. Notice that the unconditional reverse hazard function for bidder 1 is $\lim_{\epsilon\downarrow 0}\int_{b-\epsilon}^{b}\sigma_1(x)\mathrm{d}x/\epsilon\Sigma_1(b)$ which is less than the conditional reverse hazard function. For a given ϵ, (6) can be re-written as

$$\frac{\epsilon\sigma_1(b)}{\Sigma_1(b)-\Sigma_1\circ\beta_1\circ\phi_2(b)}=\left[\frac{u_2(b;t_2)-u_2(b-\epsilon;t_2)}{\epsilon}\right]^{-1}$$

where $u_2(.;t_2)$ is the utility generated by bidder 2. This is equivalent to

$$\Pr(\mathcal{Y}_1\in(b-\epsilon,b)|\mathcal{Y}_1<b\wedge\mathcal{Y}_1>\beta_1(t_2))=[\text{Marginal utility of bidder 2}]^{-1}$$

The left-hand side is interpreted as the probability that bidder 1 loses the auction by a small margin conditional on the fact that the valuation of bidder 1 is more than that of bidder 2. Notice that

$$\Pr(\mathcal{Y}_1\in(b-\epsilon,b)|\mathcal{Y}_1<b\wedge\mathcal{Y}_1>\beta_1(t_2))>\Pr(\mathcal{Y}_1\in(b-\epsilon,b)|\mathcal{Y}_1<b)$$

Thus,

$$\Pr(\mathcal{Y}_1\in(b-\epsilon,b)|\mathcal{Y}_1<b)<[\text{Marginal utility of bidder 2}]^{-1} \tag{7}$$

Now consider bidder 1 with valuation t_1. Whenever he bids $b-\epsilon$, he loses the auction but he is successfully able to buy the object in the resale market since he loses the auction by a small margin. Thus, his utility from bidding $b-\epsilon$ is t_1-p. Whenever he bids b, he wins the auction and retains the object by himself thereby getting a utility of t_1-b. Therefore, the marginal utility by increasing his bid from $b-\epsilon$ to b is $[t_1-b-(t_1-p)]/\epsilon=(p-b)/\epsilon$. We can re-write (4) as

$$\lim_{\epsilon\downarrow 0}\frac{\int_{b-\epsilon}^{b}\sigma_2(x)\mathrm{d}x}{\epsilon\Sigma_2(b)}=\frac{1}{p(b)-b} \tag{8}$$

For a given ϵ, $\int_{b-\epsilon}^{b}\sigma_2(x)\mathrm{d}x/\Sigma_2(b)$ is the probability that bidder 2 bids between $b-\epsilon$ and b conditional on the event that his bid is less than b as well as less than $\beta_2(t_1)$. The left-hand side is the *conditional* reverse hazard function of the bid made by bidder 2 and it equals the inverse of change in utility of bidder 1. Notice that the conditional reverse hazard function for bidder 2 is equivalent to his unconditional reverse hazard function. For a given ϵ, (6) can be re-written as

$$\frac{\epsilon\sigma_2(b)}{\Sigma_2(b)} = \left[\frac{u_1(b;t_1) - u_1(b-\epsilon;t_1)}{\epsilon}\right]^{-1}$$

where $u_1(.;t_1)$ is the utility generated by bidder 1. This is equivalent to

$$\Pr(\mathcal{Y}_2 \in (b-\epsilon, b)|\mathcal{Y}_2 < b \wedge \mathcal{Y}_2 < \beta_2(t_1)) = [\text{Marginal utility of bidder 1}]^{-1}$$

The left-hand side is interpreted as the probability that bidder 2 loses the auction by a small margin conditional on the fact that the valuation of bidder 2 is less than that of bidder 1. Notice that

$$\Pr(\mathcal{Y}_2 \in (b-\epsilon, b)|\mathcal{Y}_2 < b \wedge \mathcal{Y}_2 < \beta_2(t_1)) = \Pr(\mathcal{Y}_2 \in (b-\epsilon, b)|\mathcal{Y}_2 < b)$$

Thus,

$$\Pr(\mathcal{Y}_2 \in (b-\epsilon, b)|\mathcal{Y}_2 < b) = [\text{Marginal utility of bidder 1}]^{-1} \tag{9}$$

From (7), (9), and the fact that marginal utilities are equal, we have

$$\Pr(\mathcal{Y}_1 \in (b-\epsilon, b)|\mathcal{Y}_1 < b) < \Pr(\mathcal{Y}_2 \in (b-\epsilon, b)|\mathcal{Y}_2 < b)$$

Therefore, the belief of bidder 2 that bidder 1 loses the auction by a small margin is less than the belief of bidder 1 that bidder 2 loses the auction by a small margin.

3 Comparative results

In this section, we study some comparative results. First, we analyze the impact of revelation of value-rankings in auctions with resale. Second, we analyze the impact of revelation of value-rankings and inclusion of resale markets in auctions. Third, we analyze the impact of asymmetry on the bidding behavior of bidder 2. Fourth, we analyze the bidding behavior when the distribution function of bidder 2 changes stochastically.

We begin by defining a *connecting function* $\Omega_i : T \to T$ as

$$\Omega_i(t) := \phi_i \circ \beta_j(t)$$

for every $i \in N$. We interpret Ω_i as the valuation required by bidder i in order to match the bid made by bidder j. Similarly, by $\Theta_i : T \to T$, we define the connecting function of bidder i in the absence of value-rankings. In order to find the impact of revelation of value rankings in auctions with resale, we compare the connecting function of bidder 1 in the presence and absence of value-rankings. The reason for comparing the connecting functions rather than the bid functions is that the system of differential equations expressed in terms of connecting function is traceable, unlike the case with the bid functions. The differential equation of the connecting function of a bidder is independent of the connecting function of the other bidder and this is true both in the presence and absence of value-rankings. This independence property allows us to make the relevant comparison.

In the following result, we compare the connecting function of bidder 1 in the presence and absence of value-rankings.

Theorem 2. *Suppose* (ϕ_1, ϕ_2, p) *is an equilibrium profile and* (Ω_1, Ω_2) *is the corresponding connecting function profile when the value-rankings are revealed. Suppose* (μ_1, μ_2, r) *is an equilibrium profile and* (Θ_1, Θ_2) *is the corresponding connecting function profile when the value-rankings are not revealed. Then,*

$$\Omega_1(t) > \Theta_1(t)$$

for every $t \in (0, \bar{a})$.

The valuation required by bidder 1 in order to match the bid made by bidder 2 is more in the presence of value-rankings. Notice that $\Omega_1(t) > t$ since bidder 2 bids more aggressively than bidder 1 (Proposition 2). This means that the (absolute) difference in bid functions is more when value-rankings are revealed. Thus, the revelation of value-rankings in auctions with resale *asymmetrizes* the bidding strategies. However, $\Theta_1(t)$ can be either $>$ or $< t$. Whenever F_1 dominates F_2 in terms of reverse hazard rate, $\Omega_1(t) > \Theta_1(t) > t$ holds. On the other hand, whenever F_2 dominates F_1 in terms of reverse hazard rate, $\Omega_1(t) > t > \Theta_1(t)$ holds. We interpret both the cases.

Whenever $\Omega_1(t) > \Theta_1(t) > t$ is true, then (a) bidder 2 bids more aggressively than bidder 1 in both the scenarios, i.e., with and without revelation of value-rankings, and (b) bidder 2 increases his level of aggression against bidder 1 when the value-rankings are revealed.

Whenever $\Omega_1(t) > t > \Theta_1(t)$ is true, then (a) bidder 2 bids more aggressively than bidder 1 when the value-rankings are revealed and bidder 1 bids more aggressively than bidder 2 when the value-rankings are not revealed, and (b) the level of aggression of bidder 2 against bidder 1—when the value-rankings are revealed—is more than the level of aggression of bidder 1 against bidder 2—when the value-rankings are not revealed.

We explore the reason for this result to hold. Whenever value-rankings are not revealed, the equilibrium condition for the connecting function of bidder 1 is

$$\frac{\mathrm{D}F_1 \circ \Theta_1(t)}{F_1 \circ \Theta_1(t)} = \frac{f_2(t)}{F_2(t)} \tag{10}$$

For a given valuation t of bidder 2, $\Theta_1(t)$ is the valuation of bidder 1 such that there is bid equivalence and $F_1 \circ \Theta_1$ is the *connect distribution* of bidder

1. The left-hand side is the reverse hazard function of connect for bidder 1 and the right-hand side is the reverse hazard function of valuation for bidder 2. We can re-write (10) as

$$\lim_{\epsilon \downarrow 0} \frac{\int_{t-\epsilon}^{t} \mathrm{D}F_1 \circ \Theta_1(t)\mathrm{d}t}{\epsilon F_1 \circ \Theta_1(t)} = \lim_{\epsilon \downarrow 0} \frac{\int_{t-\epsilon}^{t} f_2(t)\mathrm{d}t}{\epsilon F_2(t)}$$

For a given ϵ and a given valuation t of bidder 2, $\int_{t-\epsilon}^{t} \mathrm{D}F_1 \circ \Theta_1(t)\mathrm{d}t / F_1 \circ \Theta_1(t)$ is the probability that bidder 1 gets a valuation very close to $\Theta_1(t)$ but not more than $\Theta_1(t)$. On the other hand, $\int_{t-\epsilon}^{t} f_2(t)\mathrm{d}t/\epsilon F_2(t)$ is the probability that bidder 2 gets a valuation very close to t but not more than t.

We can re-write the above equation as

$$\Pr(\mathcal{T}_1 \in (\Theta_1(t-\epsilon), \Theta_1(t)) | \mathcal{T}_1 < \Theta_1(t)) = \Pr(\mathcal{T}_2 \in (t-\epsilon, t) | \mathcal{T}_2 < t) \tag{11}$$

Whenever value-rankings are revealed, the equilibrium condition for the connecting function of bidder 1 is

$$\frac{\mathrm{D}F_1 \circ \Omega_1(t)}{F_1 \circ \Omega_1(t) - F_1(t)} = \frac{f_2(t)}{F_2(t)} \tag{12}$$

The left-hand side is the *conditional* reverse hazard function of connect for bidder 1 and the right-hand side is the reverse hazard function of valuation for bidder 2. We can re-write (12) as

$$\lim_{\epsilon \downarrow 0} \frac{\int_{t-\epsilon}^{t} \mathrm{D}F_1 \circ \Omega_1(t)\mathrm{d}t}{\epsilon [F_1 \circ \Omega_1(t) - F_1(t)]} = \lim_{\epsilon \downarrow 0} \frac{\int_{t-\epsilon}^{t} f_2(t)\mathrm{d}t}{\epsilon F_2(t)}$$

For a given ϵ and a given valuation t of bidder 2, $\int_{t-\epsilon}^{t} \mathrm{D}F_1 \circ \Omega_1(t)\mathrm{d}t/[F_1 \circ \Omega_1(t) - F_1(t)]$ is the probability that bidder 1 gets a valuation very close to $\Theta_1(t)$ but not more than $\Theta_1(t)$ conditional on the fact that $\mathcal{T}_1 > t$.

We can re-write the above equation as

$$\Pr(\mathcal{T}_1 \in (\Omega_1(t-\epsilon), \Omega_1(t)) | \mathcal{T}_1 < \Omega_1(t) \wedge \mathcal{T}_1 > t) = \Pr(\mathcal{T}_2 \in (t-\epsilon, t) | \mathcal{T}_2 < t)$$

Notice that

$$\Pr(\mathcal{T}_1 \in (\Omega_1(t-\epsilon), \Omega_1(t)) | \mathcal{T}_1 < \Omega_1(t) \wedge \mathcal{T}_1 > t) > \\ \Pr(\mathcal{T}_1 \in (\Omega_1(t-\epsilon), \Omega_1(t)) | \mathcal{T}_1 < \Omega_1(t))$$

Thus,

$$\Pr(\mathcal{T}_1 \in (\Omega_1(t-\epsilon), \Omega_1(t)) | \mathcal{T}_1 < \Omega_1(t)) < \Pr(\mathcal{T}_2 \in (t-\epsilon, t) | \mathcal{T}_2 < t) \tag{13}$$

From (11) and (13), we have

$$\Pr(\mathcal{T}_1 \in (\Omega_1(t-\epsilon), \Omega_1(t)) | \mathcal{T}_1 < \Omega_1(t)) < \Pr(\mathcal{T}_1 \in (\Theta_1(t-\epsilon), \Theta_1(t)) | \mathcal{T}_1 < \Theta_1(t))$$

The intuition behind this result is quite straightforward. From Hafalir and Krishna [3], we know that, with only resale markets and no revelation of value-rankings, the bid distributions of the two bidders are equal. From Theorem 1, we have seen that, with resale markets and revelation of value-rankings, the bid distributions of the two bidders are not identical. This is possible only if the bidding functions are more asymmetric in the latter than in the former.

Now, by $\Lambda_i : T \to T$, we denote the connecting function of bidder i in the absence of resale and value-rankings.

In the following result, we compare the connecting function of bidder 1 in the presence and absence of resale and value-rankings.

Theorem 3. *Suppose (ϕ_1, ϕ_2, p) is an equilibrium profile and (Ω_1, Ω_2) is the corresponding connecting function profile when there are resale opportunities and value-rankings are revealed. Suppose (ρ_1, ρ_2) is an equilibrium profile and (Λ_1, Λ_2) is the corresponding connecting function profile when there are no resale opportunities and the value-rankings are not revealed. Then,*

$$\Omega_1(t) > \Lambda_1(t)$$

for every $t \in (0, \bar{a})$.

The above result conveys that the valuation required by bidder 1 to match the bid of bidder 2 is more in the presence of resale and value-rankings. Since $\Omega_1(t) > t$, it follows that the revelation of value-rankings in auctions with resale *asymmetrizes* the bidding strategies. Note that whenever F_1 dominates F_2 in terms of reverse hazard rate, $\Omega_1(t) > \Lambda_1(t) > t$ and whenever F_2 dominates F_1 in terms of reverse hazard rate, $\Omega_1(t) > t > \Lambda_1(t)$. Whenever $\Omega_1(t) > \Lambda_1(t) > t$, bidder 2 increases his level of aggression against bidder 1 with the introduction of resale opportunities and the revelation of value-rankings. On the other hand, whenever $\Omega_1(t) > t > \Lambda_1(t)$, the level of aggression of bidder 2 against bidder 1—when there are resale opportunities and the value-rankings are revealed—is more than the level of aggression of bidder 1 against bidder 2—when there are no resale opportunities and the value-rankings are not revealed.

For every $k \in N$, let (Γ_k, Γ_k) and (θ_k, θ_k) denote the bidding strategy and inverse bidding strategy when both the bidders' valuation is drawn from a symmetric probability distribution and the value-rankings are not revealed. Notice that it does not matter if there are resale opportunities or not because first-price auction is efficient when bidders are *ex-ante* symmetric.

Theorem 4. *Suppose (θ_k, θ_k) is an equilibrium profile, when bidders are symmetric, for every $k \in N$. Suppose (ϕ_1, ϕ_2) is an equilibrium profile when bidders are asymmetric. Then,*

$$\theta_2(b) > \phi_2(b)$$

for every $b \in (0, \bar{b})$.

Asymmetry in the form of revelation of value-rankings induces bidder 2 to bid more aggressively. The idea of the proof is to show that $\theta_2 > \phi_2$ around the neighborhood of 0. After establishing this fact, we are left to show that the two functions do not intersect.

Proposition 3. *For small enough valuation,* $\Gamma_1(t) > \beta_1(t)$.

The above result says that, for small enough valuation, asymmetry in the form of revelation of value-rankings induces bidder 1 to bid less aggressively.

Now, we study the bidding behavior when the distribution function of bidder 2 changes stochastically. Fix the distribution function of bidder 1 and change the distribution function of bidder 2 in a manner that the new distribution function is dominant to the old distribution in terms of reverse hazard rate.

Formally, suppose the distribution function of bidder 2 changes from F_2 to H_2 such that H_2 is conditional stochastic dominant to F_2.

Theorem 5. *Suppose* (ϕ_1, ϕ_2, p) *and* $(\lambda_1, \lambda_2, q)$ *are an equilibrium profile when the distribution functions are* (F_1, F_2) *and* (F_1, H_2) *respectively. Suppose Assumption 1 is satisfied and* G_2 *dominates* F_2 *in terms of reverse hazard rate. Let* $F_2(0) > 0$ *and* $H_2(0) > 0$*. Then,*

$$\phi_1(b) > \lambda_1(b)$$

for every $b \in (0, \bar{b}]$.

The above result conveys that bidder 1 bids more aggressively than before when the distribution function of bidder 2 improves stochastically.

The next result compares the bid distribution of bidder 2 before and after the distribution change.

Proposition 4. *Suppose Assumption 1 is satisfied and* G_2 *dominates* F_2 *in terms of reverse hazard rate. Let* $F_2(0) > 0$ *and* $H_2(0) > 0$*. Then, for high enough valuation,*

$$H_2 \circ \lambda_2(b) < F_2 \circ \phi_2(b)$$

The above result tells that, for a high enough valuation, bidder 2 produces a stronger bid distribution after the change of the distribution function.

4 Ranking of the two auctions

In this section, we compare the seller's and every bidders' ranking of the two auction formats, i.e., a first-price and second-price auction. In a second-price auction, bid-your-own-value is a weakly dominant strategy for every bidder whenever there are no resale markets. This is true irrespective of whether bidders are symmetric or not and whether the value-rankings are revealed or not. However, bid-your-own-value is *not* a dominant strategy in the presence of resale markets; this is true even if the value-rankings are revealed. Nonetheless, bid-your-own-value is still an equilibrium strategy. Furthermore, since bidders are bidding their own valuation, the winner of the auction will be the bidder with highest valuation (bidder 1 in this case). Hence, efficiency is always attained and there will be no resale of the object despite having resale opportunities.

Proposition 5. *Suppose the auction format is a second-price auction. Then, truth-telling strategy is an equilibrium strategy.*

To compare seller's *ex-ante* expected revenue in a first-price and second-price auction, we make the following assumption on the distribution functions.

Assumption 2. $F_1 = F_2 \equiv F$ *and the following holds:*

$$\int_0^{\bar{a}} \mathrm{d}t(1 - F(t))(2F(t) - 1) \geq 0.$$

We state the following result.

Theorem 6. *Suppose Assumptions 1 and 2 hold. Then, a first-price auction is revenue superior to a second-price auction.*

For the family of distribution functions which satisfy Assumption 2, we observe that the expected revenue generated from bidder 2 in a first-price auction exceeds the total expected revenue generated in a second-price auction. In other words, in a first-price auction, bidder 2 bids so aggressively that he alone generates more expected revenue for the seller than the total revenue generated in a second-price auction. Since $t_1 > t_2$, bidder 2 may have an incentive to bid more than his valuation and resell the object to bidder 1. One of the plausible reason for revenue superiority of the first-price auction may be that bidder 2 bids more than his value.

We now compare the bidders preferences for the two auctions. To do so, we impose the following restriction on the valuation of bidder 1.

Condition 1. *The valuation of bidder 1,* t_1*, is such that*

$$1 - \frac{f_2(t_1)}{F_2(t_1)} \int_0^{t_1} \mathrm{d}z f_2(z)(t_1 - z) > F_2(t_1) - F_2(0).$$

We state the following result.

Theorem 7. *(A) The low-type bidder always prefers a first-price auction over a second-price auction.*
(B) Suppose Assumption 1 is satisfied and $F_2(0) > 0$*. Then, the high-type bidder prefers a first-price auction over a second-price auction as long as Condition 1 is satisfied.*

Bidder 2 always prefers a first-price auction over a second-price auction. This is because, in a second-price auction, bidder 2 always loses the auction and cannot buy the object in the resale market. Therefore, bidder 2 always gets a utility of 0. On the other hand, in a first-price auction, bidder 2 wins the auction with positive probability and therefore gets a strictly positive expected utility.

Since bidder 1 is a buyer in the resale stage, it is easy to see that he shades his bid below his valuation. In a second-price auction, he pays the valuation of bidder 2 but, in a first-price auction, he pays his own bid if he wins and he pays an amount greater than the valuation of bidder 2 if he loses. The only possible way bidder 1 prefers a first-price auction is that he shades his bid below the valuation of bidder 2.

5 Conclusion

We have shown that the classic result of bid symmetrization, as shown by Hafalir and Krishna [3], does not hold even with two risk neutral bidders if the value-rankings are common knowledge among the bidders. Specifically, the low-type bidder produces a *stronger* bid distribution than the high-type bidder. We also noticed that the stochastic order on value-distributions is *not* necessary to unambiguously rank the bidding strategies and bid distributions. The introduction of resale possibilities allow us to unanimously rank the bid distributions which were otherwise not possible. We have also shown that the presence of value-rankings in auctions with resale *asymmetrizes* the bid functions. Next, we have shown that, under parametric restrictions on the distribution functions, a first-price auction is revenue superior to a second-price auction. Finally, we have shown that the low-type bidder always prefers a first-price auction over a second-price auction and under certain restrictions, the high-type bidder also prefers a first-price auction over a second-price auction.

Acknowledgments

I would like to acknowledge Anirban Kar, Debasis Mishra, Gopakumar Achuthankutty, Parikshit Ghosh, Ram Singh, Shubhro Sarkar, Sudhir Shah, and Vijay Krishna for their valuable feedback and comments. I would also like to thank all the participants at Delhi Economic Theory Workshop (Indian Statistical Institute, Delhi), CoRe 2019 (Indira Gandhi Institute of Development and Research, Mumbai), Winter School 2019 (Delhi School of Economics), 29th Annual Conference on Contemporary Issues in Development Economics (Jadavpur University, Kolkata), and 15th Annual Conference on Economic Growth and Development (Indian Statistical Institute, Delhi).

A Appendix: Proofs

Proof of Lemma 1. Consider bidder 2 with valuation t_2. Suppose bidder 2 wins the auction by bidding $\beta_2(t_2)$. Then, $\beta_2(t_2) > \beta_1(t_1)$, and thus, $t_1 < \beta_1^{-1} \circ \beta_2(t_2)$. This means that the valuation of bidder 1 is less than the valuation required to bid the same as bidder 2. Since $\beta_2(t_2) > \beta_1(t_2)$, we have $t_2 < \beta_1^{-1} \circ \beta_2(t_2)$. This means that the valuation of bidder 2 is less than the valuation required by bidder 1 to bid the same as bidder 2. Therefore, bidder 2 makes a resale offer with price between t_2 and $\beta_1^{-1} \circ \beta_2(t_2)$.

Now, consider bidder 1 with valuation t_1. Suppose bidder 1 wins the auction by bidding $\beta_1(t_1)$. Since $\beta_2(t_1) > \beta_1(t_1)$, we have $t_1 > \beta_2^{-1} \circ \beta_1(t_1)$. Since the valuation of bidder 1 is less than the valuation required by bidder 2 to bid the same as bidder 1. Since $t_1 > \beta_2^{-1} \circ \beta_1(t_1)$ and given that bidder 1 wins the auction, it must be true that $t_1 > \beta_2^{-1} \circ \beta_1(t_1) > t_2$. Hence, bidder 1 does not make a resale offer. ■

Proof of Proposition 1. Suppose (ϕ_1, ϕ_2) is an equilibrium profile. We show $\beta_2(0) = \beta_1(0) = 0$. Consider bidder 2. Suppose $\beta_2(0) > \beta_1(0) \geq 0$. Then, bidder 2 makes a resale offer $p \circ \beta_2(0)$ such that $p \circ \beta_2(0) > \beta_2(0) > \beta_1(0)$. Then, it is profitable for bidder 1 to deviate and bid in $(\beta_2(0), p \circ \beta_2(0))$. This implies $\beta_2(0) > \beta_1(0)$ cannot be the case. Thus, $\beta_2(0) = \beta_1(0) \geq 0$.

Suppose $\beta_2(0) = \beta_1(0) > 0$. Consider a sequence $(t^n)_{n=1}^{\infty}$ such that $t^n \downarrow 0$. Then, $\beta_2(t^n) \geq \beta_1(t^n)$ for every $n \in \mathcal{N}$. For large enough n, $\beta_2(t^n) > t^n$. If bidder 2 wins, then he makes a resale offer which is lower than $\beta_2(t^n)$, and thus, utility is negative. On the other hand, if bidder 2 loses, then bidder 1does not make a resale offer, and hence, utility is zero. Therefore, $\beta_2(0) > 0$ is not profitable. Hence, $\beta_2(0) = \beta_1(0) = 0$.

We show there exist a common upper bound on the bidding space. Suppose there exists $\bar{b}_2, \bar{b}_1 > 0$ such that $\bar{b}_2 \neq \bar{b}_1$, $\beta_2(\bar{a}) = \bar{b}_2$ and $\beta_1(\bar{a}) = \bar{b}_1$. Since bidder 2 bids more aggressively, we have $\bar{b}_2 \geq \bar{b}_1$. If $\bar{b}_2 = \bar{b}_1$, then the result holds trivially. Suppose $\bar{b}_2 > \bar{b}_1$. Then, bidder 2 makes a resale offer of $p \circ \beta_2(\bar{a})$ such that $p \circ \beta_2(\bar{a}) > \beta_2(\bar{a}) > \beta_1(\bar{a})$. Then, it is profitable for bidder 1 to deviate and bid in $(\beta_2(\bar{a}), p \circ \beta_2(\bar{a}))$. Hence, $\bar{b}_2 > \bar{b}_1$ cannot be true.

Conversely, suppose (ϕ_1, ϕ_2) solves the system given by (5). Consider bidder 2. The value function is

$$V_2(t_2, b) = \frac{F_1 \circ \phi_2(b) - F_1(p)}{1 - F_1(t_2)}(p - b) + \frac{F_1(p) - F_1(t_2)}{1 - F_1(t_2)}(t_2 - b)$$

The first-order derivative is

$$\mathrm{D}V_2(t_2, b) = \mathrm{D}F_1 \circ \phi_2(b)(p - b) - F_1 \circ \phi_2(b) + F_1(t_2)$$

Suppose bidder 2 over bids by choosing b' such that $\phi_2(b') > t_2$. Then,

$$\begin{aligned}\mathrm{D}V_2(t_2, b') &= \mathrm{D}F_1 \circ \phi_2(b')(p - b') - F_1 \circ \phi_2(b') + F_1(t_2) \\ &< \mathrm{D}F_1 \circ \phi_2(b')(p - b') - F_1 \circ \phi_2(b') + F_1 \circ \phi_2(b') \\ &= 0\end{aligned}$$

Hence, it is not profitable for bidder 2 to deviate.

On the other hand, suppose bidder 2 under bids by choosing b'' such that $\phi_2(b'') < t_2$. Then,

$$\begin{aligned} \mathrm{D}V_2(t_2, b'') &= \mathrm{D}F_1 \circ \phi_2(b'')(p - b'') - F_1 \circ \phi_2(b'') + F_1(t_2) \\ &> \mathrm{D}F_1 \circ \phi_2(b'')(p - b'') - F_1 \circ \phi_2(b'') + F_1 \circ \phi_2(b'') \\ &= 0 \end{aligned}$$

Hence, it is not profitable for the bidder 2 to deviate.

Now consider bidder 1. The value function is

$$V_1(t_1, b) = \frac{F_2 \circ \phi_2(b)}{F_2(t_1)}(t_1 - b) + \frac{F_1(t_1) - F_2 \circ \phi_2(b)}{F_2(t_1)}(t_1 - p)$$

The first-order derivative is

$$\begin{aligned} \mathrm{D}V_1(t_1, b) &= \mathrm{D}F_2 \circ \phi_2(b)(p - b) - F_2 \circ \phi_2(b) \\ &= 0 \end{aligned}$$

Hence, it is not profitable for bidder 1 to deviate. Therefore, (ϕ_1, ϕ_2) is an equilibrium. ■

Proof of Propositon 2. We show bidder 2 bids more aggressively than bidder 1. Since $\phi_1'(\bar{b}) = 0 < \phi_2'(\bar{b})$, it follows that there exists $\epsilon > 0$ such that $\phi_1(b) > \phi_2(b)$ for every $b \in (\bar{b} - \epsilon, \bar{b})$. Suppose there exists b^* such that $\phi_1(b^*) = \phi_2(b^*)$ and $\phi_1(b) > \phi_2(b)$ for every $b \in (b^*, \bar{b})$. Then, $p(b^*) = \phi_1(b^*) = \phi_2(b^*)$. From (18), we have

$$\phi_2'(b^*) > \phi_1'(b^*)$$

This implies that there exists $\delta > 0$ such that $\phi_2(b^* + \delta) > \phi_1(b^* + \delta)$, which is a contradiction. Hence, $\phi_1(b) > \phi_2(b)$ for every $b \in (0, \bar{b})$. ■

Proof of Theorem 1. We show bidder 2 produces a stronger bid distribution than bidder 1. From (3) and (4), we have

$$\frac{F_1 \circ \phi_1(b)}{\mathrm{D}F_1 \circ \phi_1(b)} > \frac{F_2 \circ \phi_2(b)}{\mathrm{D}F_2 \circ \phi_2(b)}$$

Thus,

$$\mathrm{D}\left(\frac{F_2 \circ \phi_2(b)}{F_1 \circ \phi_1(b)}\right) > 0$$

Since $F_1 \circ \phi_1(\bar{b}) = F_2 \circ \phi_2(\bar{b}) = 1$, we have $F_2 \circ \phi_2(b) < F_1 \circ \phi_1(b)$ for every $b \in (0, \bar{b})$. ■

Proof of Theorem 2. We describe the connecting function in the following manner:

$$\begin{aligned}
\mathrm{D}\Omega_2(t) &= \frac{F_2 \circ \Omega_2(t)}{f_2 \circ \Omega_2(t)} \frac{f_1(t)}{F_1(t) - F_1 \circ \Omega_2(t)} \\
\mathrm{D}\Omega_1(t) &= \frac{F_1 \circ \Omega_1(t) - F_1(t)}{f_1 \circ \Omega_1(t)} \frac{f_2(t)}{F_2(t)} \\
\Omega_2(0) &= \Omega_1(0) = 0 \quad \& \quad \Omega_2(\bar{a}) = \Omega_1(\bar{a}) = \bar{a}
\end{aligned} \tag{14}$$

In the absence of value-rankings, the characterization of inverse bidding strategy is given by

$$\begin{aligned}
\mathrm{D}\mu_2(b) &= \frac{F_2 \circ \mu_2(b)}{f_2 \circ \mu_2(b)} \frac{1}{r(b) - b} \\
\mathrm{D}\mu_1(b) &= \frac{F_1 \circ \mu_1(b)}{f_1 \circ \mu_1(b)} \frac{1}{r(b) - b} \\
\mu_2(0) &= \mu_1(0) = 0 \quad \& \quad \mu_2(\hat{b}) = \mu_1(\hat{b}) = \bar{a} \quad \exists \quad \hat{b} \in \Re_{++}
\end{aligned}$$

The derivation of the above characterization can be found in Hafalir and Krishna [3]. We can describe the equilibrium in the following manner:

$$\begin{aligned}
\mathrm{D}\Theta_2(t) &= \frac{F_2 \circ \Theta_2(t)}{f_2 \circ \Theta_2(t)} \frac{f_1(t)}{F_1(t)} \\
\mathrm{D}\Theta_1(t) &= \frac{F_1 \circ \Theta_1(t)}{f_1 \circ \Theta_1(t)} \frac{f_2(t)}{F_2(t)} \\
\Theta_2(0) &= \Theta_1(0) = 0 \quad \& \quad \Theta_2(\bar{a}) = \Theta_1(\bar{a}) = \bar{a}
\end{aligned} \tag{15}$$

Notice that $\Omega_1'(\bar{a}) = 0$ and $\Theta_1'(\bar{a}) > 0$. Since $\Omega_1'(\bar{a}) < \Theta_1'(\bar{a})$, it follows that there exists $\epsilon > 0$ such that $\Omega_1(t) > \Theta_1(t)$ for every $t \in (\bar{a} - \epsilon, \bar{a})$. Suppose there exists $t^* > 0$ such that $\Omega_1(t^*) = \Theta_1(t^*)$ and $\Omega_1(t) > \Theta_1(t)$ for every $t \in (t^*, \bar{a})$. Then, from (14) and (15), we have

$$\Omega_1'(t^*) < \Theta_1'(t^*)$$

Thus, there exists $\delta > 0$ such that $\Omega_1(t^* + \delta) < \Theta_1(t^* + \delta)$, a contradiction. Hence, no such t^* exists. Therefore, $\Omega_1(t) > \Theta_1(t)$ for every $t \in (0, \bar{a})$. ■

Proof of Theorem 3. In the absence of resale and value-rankings, the characterization of inverse bidding strategy is given by:

$$\begin{aligned}
\mathrm{D}\rho_2(b) &= \frac{F_2 \circ \rho_2(b)}{f_2 \circ \rho_2(b)} \frac{1}{\rho_1(b) - b} \\
\mathrm{D}\rho_1(b) &= \frac{F_1 \circ \rho_1(b)}{f_1 \circ \rho_1(b)} \frac{1}{\rho_2(b) - b} \\
\rho_2(0) &= \rho_1(0) = 0 \quad \& \quad \rho_2(\underline{b}) = \rho_1(\underline{b}) = \bar{a} \quad \exists \quad \underline{b} \in \Re_{++}
\end{aligned}$$

The derivation of the above characterization can be found in Maskin and Riley [11] and Lebrun [9]. Let (α_1, α_2) be the corresponding bidding strategies. We can describe the equilibrium in the following manner:

$$\begin{aligned} \mathrm{D}\Lambda_2(t) &= \frac{F_2 \circ \Lambda_2(t)}{f_2 \circ \Lambda_2(t)} \frac{f_1(t)}{F_1(t)} \frac{\Lambda_2(t) - \alpha_1(t)}{t - \alpha_1(t)} \\ \mathrm{D}\Lambda_1(t) &= \frac{F_1 \circ \Lambda_1(t)}{f_1 \circ \Lambda_1(t)} \frac{f_2(t)}{F_2(t)} \frac{\Lambda_1(t) - \alpha_2(t)}{t - \alpha_2(t)} \\ \Lambda_2(0) &= \Lambda_1(0) = 0 \quad \& \quad \Lambda_2(\bar{a}) = \Lambda_1(\bar{a}) = \bar{a} \end{aligned} \tag{16}$$

This can also be found in Lebrun [9].

Notice that $\mathrm{D}\Omega_1(\bar{a}) = 0$ and $\mathrm{D}\Lambda_1(\bar{a}) > 0$. Since $\mathrm{D}\Omega_1(\bar{a}) < \mathrm{D}\Lambda_1(\bar{a})$, it follows that there exists $\epsilon > 0$ such that $\Omega_1(t) > \Lambda_1(t)$ for every $t \in (\bar{a} - \epsilon, \bar{a})$. Suppose there exists $t^* > 0$ such that $\Omega_1(t^*) = \Lambda_1(t^*)$ and $\Omega_1(t) > \Lambda_1(t)$ for every $t \in (t^*, \bar{a})$. Then, from (14), (16) and the fact that $\Lambda_1(t) > t$ for every t, we have

$$\begin{aligned} \Lambda_1'(t^*) &= \frac{F_1 \circ \Lambda_1(t^*)}{f_2 \circ \Lambda_1(t^*)} \frac{f_2(t^*)}{F_2(t^*)} \frac{\Lambda_1(t^*) - \alpha_2(t^*)}{t^* - \alpha_2(t^*)} \\ &> \frac{F_1 \circ \Lambda_1(t^*)}{f_1 \circ \Lambda_1(t^*)} \frac{f_2(t^*)}{F_2(t^*)} \\ &= \frac{F_1 \circ \Omega_1(t^*)}{f_1 \circ \Omega_1(t^*)} \frac{f_2(t^*)}{F_2(t^*)} \\ &> \frac{F_1 \circ \Omega_1(t^*) - F_1(t^*)}{f_1 \circ \Omega_1(t^*)} \frac{f_2(t^*)}{F_2(t^*)} \\ &= \Omega_1'(t^*) \end{aligned}$$

Thus, there exists $\delta > 0$ such that $\Lambda_1(t^* + \delta) > \Omega_1(t^* + \delta)$, a contradiction. Hence, no such t^* exists. Therefore, $\Omega_1(t) > \Lambda_1(t)$ for every $t \in (0, \bar{a})$. ■

Proof of Theorem 4. When bidders are symmetric, the characterization of inverse bidding strategy is given by

$$\begin{aligned} \mathrm{D}\theta_k(b) &= \frac{F_k \circ \theta_k(b)}{f_k \circ \theta_k(b)} \frac{1}{\theta_k(b) - b} \\ \theta_k(0) &= 0 \quad \& \quad \theta_k(\bar{b}_k) = \bar{a} \quad \exists \quad \bar{b}_k \in \Re_{++} \end{aligned} \tag{17}$$

for every $k \in N$.

Let $y_i : T_i \to \Re$ be defined as

$$y_i(t) = t \frac{f_i(t)}{F_i(t)}$$

for every $i \in N$. Then,

$$y_i(0) = 1 \quad \& \quad y_i'(0) = \frac{f_i'(0)}{2 f_i(0)}$$

So,

$$y_i \circ \phi_i(b) = \phi_i(b)\frac{f_i \circ \phi_i(b)}{F_i \circ \phi_i(b)}$$

Using (18) for $i = l$ in the above equation, we have

$$\phi_2'(b) y \circ \phi_2(b) = \frac{\phi_2(b)}{p(b) - b}$$

At $b = 0$, using L'Hôpital's rule, we have

$$\phi_2'(0) = \frac{\phi_2'(0)}{p'(0) - 1}$$

This implies $p'(0) = 2$. Similarly, for $i = 1$, we have

$$y_1 \circ \phi_1(b) = \frac{\phi_1(b) f_1 \circ \phi_1(b)}{\phi_1'(b) F_1 \circ \phi_1(b)(p(b) - b) + F_1 \circ \phi_2(b)}$$

At $b = 0$, using L'Hôpital's rule, we have

$$\phi_2'(0) = 0$$

Differentiating third equation of (18) and calculating it at $b = 0$, we have

$$\phi_1'(0) = 4$$

Similarly, we have

$$\theta_2'(0) = \theta_1'(0) = 2$$

Since $\theta_2'(0) > \phi_2'(0)$, it follows that there exists $\epsilon > 0$ such that $\theta_2(b) > \phi_2(b)$ for every $b \in (0, \epsilon)$. Suppose there exists b^* such that $\theta_2(b^*) = \phi_2(b^*)$ and $\theta_2(b) > \phi_2(b)$ for every $b \in (0, b^*)$. Then, $p(b^*) > \phi_2(b^*) = \theta_2(b^*)$. From (18) and (17), we have

$$\theta_2'(b^*) > \phi_2'(b^*)$$

This implies that there exists $\delta > 0$ such that $\theta_2(b^* - \delta) < \phi_2(b^* - \delta)$, which is a contradiction. Hence,

$$\theta_2(b) > \phi_2(b)$$

for every $b \in (0, \bar{b})$. ■

Proof of Proposition 3. Since $\Gamma_1'(0) > \theta_1'(0)$, it follows that there exists $\delta > 0$ such that $\phi_1(b) > \theta_1(b)$ for every $b \in (0, \epsilon)$. ■

Proof of Theorem 5. When the distribution functions are F_1 and F_2, the characterization of inverse bidding strategy is

$$\begin{aligned}
\mathrm{D}\phi_2(b) &= \frac{F_2 \circ \phi_2(b)}{f_2 \circ \phi_2(b)} \frac{1}{p(b) - b} \\
\mathrm{D}\phi_1(b) &= \frac{F_1 \circ \phi_1(b) - F_1 \circ \phi_2(b)}{f_1 \circ \phi_1(b)} \frac{1}{p(b) - b} \\
\phi_2(b) &= p(b) - \frac{F_1 \circ \phi_1(b) - F_1 \circ p(b)}{f_1 \circ p(b)} \\
\phi_1(0) &= \phi_2(0) = p(0) = 0 \\
\phi_1(\bar{b}) &= \phi_2(\bar{b}) = \bar{a} \quad \exists \quad \bar{b} \in \Re_{++}
\end{aligned} \tag{18}$$

When the distribution functions are H_2 and F_1, we denote the bidding strategy, inverse bidding strategy and resale price by ψ_i, λ_i and q respectively for every $i \in N$. The characterization of inverse bidding strategy after the distribution change is given by

$$\begin{aligned}
\mathrm{D}\lambda_2(b) &= \frac{H_2 \circ \lambda_2(b)}{h_2 \circ \lambda_2(b)} \frac{1}{q(b) - b} \\
\mathrm{D}\lambda_1(b) &= \frac{F_1 \circ \lambda_1(b) - F_1 \circ \lambda_2(b)}{f_1 \circ \lambda_1(b)} \frac{1}{q(b) - b} \\
\lambda_2(b) &= q(b) - \frac{F_1 \circ \lambda_1(b) - F_1 \circ q(b)}{f_1 \circ q(b)} \\
\lambda_1(0) &= \lambda_2(0) = q(0) = 0 \\
\lambda_1(\tilde{b}) &= \lambda_2(\tilde{b}) = \bar{a} \quad \exists \quad \tilde{b} \in \Re_{++}
\end{aligned} \tag{19}$$

Suppose $\phi_2(c) \leq \lambda_2(c)$ and $\phi_1(c) \leq \lambda_1(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$. Then, $p(c) \leq q(c)$.

We show that there exists $\epsilon > 0$ such that

$$\phi_1(b) < \lambda_1(b) \quad \& \quad \frac{F_2 \circ \phi_2(b)}{H_2 \circ \lambda_2(b)} < \frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)}$$

for every $b \in (c - \epsilon, c)$.

Since $\phi_2 \leq \lambda_2$, we have

$$\frac{F_2 \circ \phi_2(b)}{H_2 \circ \lambda_2(b)} < \frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(b)}$$

for every $b < c$.

Case 1: $\phi_2(c) < \lambda_2(c)$ and $\phi_1(c) < \lambda_1(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$

Then, $p(c) < q(c)$. It is straightforward to see that there exists $\epsilon > 0$ such that

$$\phi_1(b) < \lambda_1(b) \quad \& \quad \frac{F_2 \circ \phi_2(b)}{H_2 \circ \lambda_2(b)} < \frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)}$$

for every $b \in (c-\epsilon, c)$.

Case 2: $\phi_2(c) = \lambda_2(c)$ and $\phi_1(c) < \lambda_1(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$

Then, $p(c) < q(c)$. From the system given by (18) and (19), we have

$$\phi_2'(c) > \lambda_2'(c)$$

Then, there exists $\epsilon > 0$ such that

$$\phi_1(b) < \lambda_1(b) \quad \& \quad \frac{F_2 \circ \phi_2(b)}{H_2 \circ \lambda_2(b)} < \frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)}$$

for every $b \in (c-\epsilon, c)$.

Case 3: $\phi_2(c) < \lambda_2(c)$ and $\phi_1(c) = \lambda_1(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$

Then, $p(c) < q(c)$. From the system given by (18) and (19), we have

$$\phi_1'(c) > \lambda_1'(c)$$

Then, there exists $\epsilon > 0$ such that

$$\phi_1(b) < \lambda_1(b) \quad \& \quad \frac{F_2 \circ \phi_2(b)}{H_2 \circ \lambda_2(b)} < \frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)}$$

for every $b \in (c-\epsilon, c)$.

Case 4: $\phi_2(c) = \lambda_2(c)$ and $\phi_1(c) = \lambda_1(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$

Then, $p(c) = q(c)$. From the system given by (18) and (19), we have

$$\phi_2'(c) > \lambda_2'(c)$$

The second differential equation of the system given by (18) and (19) can be rewritten as

$$\mathrm{D} \log F_1 \circ \phi_1(b) = \frac{F_1 \circ \phi_1(b) - F_1 \circ \phi_2(b)}{F_1 \circ \phi_1(b)} \frac{1}{p(b) - b}$$
$$\mathrm{D} \log F_1 \circ \lambda_1(b) = \frac{F_1 \circ \lambda_1(b) - F_1 \circ \lambda_2(b)}{F_1 \circ \lambda_1(b)} \frac{1}{q(b) - b}$$

Taking the derivative of the above equations, we have

$$\mathrm{D}^2 \log F_1 \circ \phi_1 = \frac{F_1 \circ \phi_1 - F_1 \circ \phi_2}{F_1 \circ \phi_1} \frac{1 - \mathrm{D}p}{(p-b)^2} + \frac{1}{p-b} \left\{ \frac{\mathrm{D}\phi_1 f_1 \circ \phi_1 F_1 \circ \phi_2 - \mathrm{D}\phi_2 F_1 \circ \phi_1 f_1 \circ \phi_2}{(F_1 \circ \phi_1)^2} \right\}$$
$$\mathrm{D}^2 \log F_1 \circ \lambda_1 = \frac{F_1 \circ \lambda_1 - F_1 \circ \lambda_1}{F_1 \circ \lambda_1} \frac{1 - \mathrm{D}q}{(q-b)^2} + \frac{1}{q-b} \left\{ \frac{\mathrm{D}\lambda_1 f_1 \circ \lambda_1 F_1 \circ \lambda_2 - \mathrm{D}\lambda_2 F_1 \circ \lambda_1 f_1 \circ \lambda_2}{(F_1 \circ \lambda_1)^2} \right\}$$

The above two expressions are strictly decreasing in $\mathrm{D}\phi_2$ and $\mathrm{D}\lambda_2$ respectively. Taking the derivatives of third equation for the system given by (18) and (19), we have

$$\mathrm{D}\phi_2 = \mathrm{D}p - \frac{f_1 \circ p(\mathrm{D}\phi_1 f_1 \circ \phi_1 - \mathrm{D}p f_1 \circ p) - (F_1 \circ \phi_1 - F_1 \circ p)\mathrm{D}p f_1' \circ p}{(f_1 \circ p)^2}$$
$$\mathrm{D}\lambda_2 = \mathrm{D}q - \frac{f_1 \circ q(\mathrm{D}\lambda_1 f_1 \circ \lambda_1 - \mathrm{D}q f_1 \circ q) - (F_1 \circ \lambda_1 - F_1 \circ q)\mathrm{D}q f_1' \circ q}{(f_1 \circ q)^2}$$

Since $\mathrm{D}\phi_2 > \mathrm{D}\lambda_2$, $\mathrm{D}\phi_1 = \mathrm{D}\lambda_1$, $\phi_1 = \lambda_1$, $\phi_2 = \lambda_2$ and $p = q$, comparing the above two expressions, we have $\mathrm{D}p > \mathrm{D}q$. Using this fact in the expressions of $\mathrm{D}^2 \log F_1 \circ \phi_1$ and $\mathrm{D}^2 \log F_1 \circ \lambda_1$, we have

$$\mathrm{D}^2 \log F_1 \circ \phi_1 < \mathrm{D}^2 \log F_1 \circ \lambda_1$$

Then, there exists $\epsilon > 0$ such that

$$\phi_1(b) < \lambda_1(b) \quad \& \quad \frac{F_2 \circ \phi_2(b)}{H_2 \circ \lambda_2(b)} < \frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)}$$

for every $b \in (c - \epsilon, c)$.

Let

$$M := \inf\Bigg\{x \in [0, c-\epsilon] : \quad \phi_1(b) < \lambda_1(b) \quad \wedge \\ \frac{F_2 \circ \phi_2(b)}{H_2 \circ \lambda_2(b)} < \frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)} \text{ for every } b \in (c-\epsilon, c)\Bigg\}$$

We show $M = 0$. We show by contradiction. Suppose $M > 0$. Then, either

$$\phi_1(M) = \lambda_1(M) \quad \text{or} \quad \frac{F_2 \circ \phi_2(M)}{H_2 \circ \lambda_2(M)} = \frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)}$$

Since $\phi_1(b) < \lambda_1(b)$ and $\phi_2(b) \leq \lambda_2(b)$ for every $b \in (M, c-\epsilon)$, we have $p(b) < q(b)$. From the system given by (18) and (19), we have

$$\frac{F_2 \circ \phi_2(b)}{\mathrm{D}F_2 \circ \phi_2(b)} < \frac{H_2 \circ \lambda_2(b)}{\mathrm{D}H_2 \circ \lambda_2(b)}$$

This implies

$$\mathrm{D}\log F_2 \circ \phi_2(b) > \mathrm{D}\log\left\{\frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)} H_2 \circ \lambda_2(b)\right\}$$

Since $M < c - \epsilon$, we have

$$\log F_2 \circ \phi_2(c-\epsilon) - \log F_2 \circ \phi_2(M) > \log\left\{\frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)} H_2 \circ \lambda_2(c-\epsilon)\right\} - \\ \log\left\{\frac{F_2 \circ \lambda_2(c)}{H_2 \circ \lambda_2(c)} H_2 \circ \lambda_2(M)\right\}$$

Rearranging the above expression, we have

$$\log\left\{\frac{F_2\circ\lambda_2(c)}{H_2\circ\lambda_2(c)}H_2\circ\lambda_2(M)\right\}-\log F_2\circ\phi_2(M)>$$
$$\log\left\{\frac{F_2\circ\lambda_2(c)}{H_2\circ\lambda_2(c)}H_2\circ\lambda_2(c-\epsilon)\right\}-\log F_2\circ\phi_2(c-\epsilon)$$

From the definition of M, we have

$$\log\left\{\frac{F_2\circ\lambda_2(c)}{H_2\circ\lambda_2(c)}H_2\circ\lambda_2(c-\epsilon)\right\}-\log F_2\circ\phi_2(c-\epsilon)>0$$

Then,

$$\frac{F_2\circ\phi_2(M)}{H_2\circ\lambda_2(M)}<\frac{F_2\circ\lambda_2(c)}{H_2\circ\lambda_2(c)}$$

Therefore, $\phi_2(M)=\lambda_2(M)$ must be true. From the system given by (18) and (19), the definition of M and the assumption of conditional stochastic dominance, we have

$$\frac{F_2\circ\phi_2(M)}{H_2\circ\lambda_2(M)}<\frac{F_2\circ\lambda_2(c)}{H_2\circ\lambda_2(c)}<\frac{F_2\circ\lambda_2(M)}{H_2\circ\lambda_2(M)}$$

This implies $\phi_2(M)<\lambda_2(M)$. Since $\phi_1(M)=\lambda_1(M)$ and $\phi_2(M)<\lambda_2(M)$, we have $p(M)<q(M)$. Then $\phi_1'(M)>\lambda_1'(M)$. Thus, there exists $\delta>0$ such that $\phi_1(M+\delta)>\lambda_1(M+\delta)$, which is a contradiction. Therefore, $M=0$.

Hence,

$$\phi_1(b)<\lambda_1(b)\quad\&\quad\frac{F_2\circ\phi_2(b)}{H_2\circ\lambda_2(b)}<\frac{F_2\circ\lambda_2(c)}{H_2\circ\lambda_2(c)}$$

for every $b\in(0,c)$.

We show that, for every $c\in(0,\min\{\bar{b},\tilde{b}\})$, $\phi_2(c)\leq\lambda_2(c)$ and $\phi_1(c)\leq\lambda_1(c)$ cannot hold simultaneously. We show by contradiction. Suppose there exists c^* such that $\phi_2(c^*)\leq\lambda_2(c^*)$ and $\phi_1(c^*)\leq\lambda_1(c^*)$. Then,

$$\frac{F_2\circ\phi_2(b)}{H_2\circ\lambda_2(b)}<\frac{F_2\circ\lambda_2(c^*)}{H_2\circ\lambda_2(c^*)}$$

Taking the limit at $b\downarrow 0$, we have

$$\frac{F_2(0)}{H_2(0)}<\frac{F_2\circ\lambda_2(c^*)}{H_2\circ\lambda_2(c^*)}$$

Since $\lambda_2(c^*)>0$, it follows from the above expression that F_2/H_2 is strictly increasing, which is a contradiction. Hence, $\phi_2(c)\leq\lambda_2(c)$ and $\phi_1(c)\leq\lambda_1(c)$ cannot hold simultaneously.

We show $\tilde{b}>\bar{b}$. We show by contradiction. Suppose $\tilde{b}\leq\bar{b}$. Then, $\phi_2(\tilde{b})\leq\lambda_2(\tilde{b})$ and $\phi_1(\tilde{b})\leq\lambda_1(\tilde{b})$, which is a contradiction as $\phi_2(c)\leq\lambda_2(c)$ and $\phi_1(c)\leq\lambda_1(c)$ cannot hold simultaneously. Hence, $\tilde{b}>\bar{b}$.

Finally, we show $\phi_1(b) > \lambda_1(b)$ for every $b \in (0, \bar{b}]$. Since $\phi_1(\bar{b}) > \lambda_1(\bar{b})$, it implies that there exists $\epsilon > 0$ such that $\phi_1(b) > \lambda_1(b)$ for every $b \in (\bar{b} - \epsilon, \bar{b}]$. Suppose there exists b^* such that $\phi_1(b^*) = \lambda_1(b^*)$ and $\phi_1(b) > \lambda_1(b)$ for every $b \in (b^*, \bar{b}]$. Then, $\phi_2(b^*) > \lambda_2(b^*)$. As $\phi_1(b^*) = \lambda_1(b^*)$ and $\phi_2(b^*) > \lambda_2(b^*)$, we have $p(b^*) > q(b^*)$. From the system given by (18) and (19), we have

$$\phi_1'(b^*) < \lambda_1'(b^*)$$

Then, there exists $\delta > 0$ such that $\phi_1(b^* + \delta) < \lambda_1(b^* + \delta)$, which is a contradiction. Therefore, $\phi_1(b) > \lambda_1(b)$ for every $b \in (0, \bar{b}]$. ■

Proof of Proposition 4. Since $\tilde{b} > \bar{b}$, it follows $\phi_2(\bar{b}) > \lambda_2(\bar{b})$ and $\phi_1(\bar{b}) > \lambda_1(\bar{b})$. Then, there exists $\epsilon > 0$ such that $\phi_2(b) > \lambda_2(b)$ for every $b \in (\bar{b} - \epsilon, \bar{b})$. Since $\phi_1(b) > \lambda_1(b)$ for every $b \in (0, \bar{b})$, it follows that $p(b) > q(b)$ for every $b \in (\bar{b} - \epsilon, \bar{b})$.

From the system given by (18) and (19), we have

$$\frac{F_2 \circ \phi_2(b)}{\mathrm{D}F_2 \circ \phi_2(b)} < \frac{G_2 \circ \lambda_2(b)}{\mathrm{D}G_2 \circ \lambda_2(b)}$$

Thus,

$$\mathrm{D}\left(\frac{G_2 \circ \lambda_2(b)}{F_2 \circ \phi_2(b)}\right) > 0$$

Since $\tilde{b} > \bar{b}$ and $F_2 \circ \phi_2(\bar{b}) = 1 > G_2 \circ \lambda_2(\bar{b})$, we have $G_2 \circ \lambda_2(b) < F_2 \circ \phi_2(b)$ for some neighborhood around $\bar{b}$. ■

Proof of Proposition 5. Suppose the equilibrium bidding strategy in a second-price auction is denoted by κ_i for every $i \in N$. We show that $\kappa_2(t_2) = t_2$ and $\kappa_1(t_1) = t_1$ is an equilibrium profile. First, consider bidder 2 with valuation t_2. Suppose bidder 1 follows his equilibrium bidding strategy $\kappa_1(t_1) = t_1$. We show that unilateral deviation for bidder 2 is not profitable. If bidder 2 bids according to $\kappa_2(t_2) = t_2$, then he gets a utility of 0. Suppose bidder 2 under bids by bidding b' such that $b' < t_2$. Since we know that $t_2 < t_1$, he loses the auction by bidding b' and cannot buy the object in the resale stage. Therefore, his utility is 0. Thus, under bidding is not profitable. Now, suppose bidder 2 over bids by bidding b'' such that $b'' > t_2$. Since $t_2 < t_1$, then either $t_2 < b'' < t_1$ or $t_2 < t_1 < b''$. Whenever $t_2 < b'' < t_1$, bidder 2 loses the auction and cannot buy the object in the resale stage thereby getting a utility of 0. Whenever, $t_2 < t_1 < b''$, bidder 2 wins the auction but cannot resell the object in the resale stage thereby getting a utility of $t_2 - t_1 < 0$. Thus, in both cases, over bidding is not profitable.

Now consider bidder 1 with valuation t_1. Suppose bidder 2 follows his equilibrium bidding strategy $\kappa_2(t_2) = t_2$. We show that unilateral deviation for bidder 1 is not profitable. If bidder 1 bids according to $\kappa_1(t_1) = t_1$, then he gets a utility of $t_1 - t_2$. Suppose bidder 1 under bids by bidding b' such that $b' < t_1$. Since $t_2 < t_1$, then either $t_2 < b' < t_1$ or $b' < t_2 < t_1$. Whenever

$t_2 < b' < t_1$, bidder 1 wins the auction and does not sell the object in the resale stage thereby getting a utility of $t_1 - t_2$. Whenever, $b' < t_2 < t_1$, bidder 1 loses the auction and may be able to buy the object in the resale stage. If he is able to buy the object in the resale stage, he ends up paying weakly more than t_2, and, on the other hand, if he is not able to buy the object in the resale stage, he ends up getting a utility of 0. Thus, in both the cases, under bidding is not profitable. Now, suppose bidder 1 over bids by bidding b'' such that $b'' > t_1$. Since, $t_1 > t_2$, he wins the auction and does not make any resale offer thereby getting a utility of $t_1 - t_2$. Thus, over bidding is not profitable. Therefore, truth-telling strategy is an equilibrium strategy. ■

Proof of Theorem 6. First, consider a first-price auction. The seller's *ex-ante* expected revenue generated from bidder 2 is

$$R_2^I = \int_0^{\bar{b}} \mathrm{d}bF \circ \phi_2(\mathrm{d}b)bF \circ \phi_1(b)$$

Using integration-by-parts, we get

$$R_2^I = \int_0^{\bar{b}} \mathrm{d}b(1 - F \circ \phi_2(b))\mathrm{D}(bF \circ \phi_1(b))$$

Now, using the first-order differential equations given by (5), we have

$$\begin{aligned} \mathrm{D}(bF \circ \phi_1(b)) &= bf \circ \phi_1(b)\mathrm{D}\phi_1(b) + F \circ \phi_1(b) \\ &= bf \circ \phi_1(b)\mathrm{D}\phi_1(b) + (p(b) - b)f \circ \phi_1(b)\mathrm{D}\phi_1(b) + F \circ \phi_2(b) \\ &= p(b)f \circ \phi_1(b)\mathrm{D}\phi_1(b) + F \circ \phi_2(b) \end{aligned}$$

Using this expression in the previous equation, we have

$$R_2^I = \int_0^{\bar{b}} \mathrm{d}b(1 - F \circ \phi_2(b))(p(b)f \circ \phi_1(b)\mathrm{D}\phi_1(b) + F \circ \phi_2(b))$$

The idea is to define the revenue expression in terms of the valuations instead of bids. From the definition of connecting function, we know $\Omega_1 \circ \phi_2(b) = \phi_1 \circ \phi_2^{-1} \circ \phi_2(b) = \phi_1(b)$ where $t = \phi_2(b)$. Using this in the above equation, we get

$$R_2^I = \int_0^{\bar{a}} \mathrm{d}t(1 - F(t))(f \circ \Omega_1(t)p \circ \beta_2(t)\mathrm{D}\Omega_1(t) + F(t))$$

Integrating-by-parts, we have

$$\begin{aligned} R_2^I = &\int_0^{\bar{a}} \mathrm{d}t(1 - F \circ \Omega_1(t))\mathrm{D}((1 - F(t))p \circ \beta_2(t)) + \\ &\int_0^{\bar{a}} \mathrm{d}t(1 - F(t))F(t) \end{aligned} \tag{20}$$

Similarly, the seller's *ex-ante* expected revenue generated from bidder 1 is

$$\begin{aligned}R_1^I &= \int_0^{\bar{b}} \mathrm{d}bF \circ \phi_1(\mathrm{d}b)bF \circ \phi_2(b) \\ &= \int_0^{\bar{b}} \mathrm{d}b(1 - F \circ \phi_1(b))\mathrm{D}(bF \circ \phi_2(b))\end{aligned}$$

Now, using the first-order differential equations given by (5), we have

$$\begin{aligned}\mathrm{D}(bF \circ \phi_2(b)) &= bf \circ \phi_2(b)\mathrm{D}\phi_2(b) + F \circ \phi_2(b) \\ &= bf \circ \phi_2(b)\mathrm{D}\phi_2(b) + (p(b) - b)f \circ \phi_2(b)\mathrm{D}\phi_2(b) \\ &= p(b)f \circ \phi_2(b)\mathrm{D}\phi_2(b)\end{aligned}$$

Using this expression in the previous equation, we have

$$R_1^I = \int_0^{\bar{b}} \mathrm{d}b(1 - F \circ \phi_1(b))p(b)f \circ \phi_2(b)\mathrm{D}\phi_2(b)$$

Writing the above expression in terms of valuations, we get the following expression for seller's *ex-ante* expected revenue generated from bidder 1

$$R_1^I = \int_0^{\bar{a}} \mathrm{d}t(1 - F \circ \Omega_1(t))f(t)p \circ \beta_2(t) \tag{21}$$

Therefore, the seller's *ex-ante* expected revenue in a first-price auction is

$$\begin{aligned}R^I =& R_2^I + R_1^I \\ =& \int_0^{\bar{a}} \mathrm{d}t(1 - F \circ \Omega_1(t))\{\mathrm{D}((1 - F(t))p \circ \beta_2(t)) + f(t)p \circ \beta_2(t))\} \\ &+ \int_0^{\bar{a}} \mathrm{d}t(1 - F(t))F(t) \\ =& \int_0^{\bar{a}} \mathrm{d}t(1 - F \circ \Omega_1(t))((1 - F(t))\mathrm{D}(p \circ \beta_2(t)) \\ &+ \int_0^{\bar{a}} \mathrm{d}t(1 - F(t))F(t)\end{aligned} \tag{22}$$

Now, consider a second-price auction. The seller's *ex-ante* expected revenue is

$$R^{II} = \int_0^{\bar{a}} \mathrm{d}t(1 - F(t))^2 \tag{23}$$

The difference between the seller's revenue from a first-price and a second-price auction is

$$\begin{aligned}R^I - R^{II} &= \int_0^{\bar{a}} \mathrm{d}t(1 - F \circ \Omega_1(t))((1 - F(t))\mathrm{D}(p \circ \beta_2(t)) \\ &\quad + \int_0^{\bar{a}} \mathrm{d}t(1 - F(t))(2F(t) - 1) \\ &> \int_0^{\bar{a}} \mathrm{d}t(1 - F(t))(2F(t) - 1) \\ &> 0\end{aligned}$$

■

Proof of Theorem 7. Consider bidder 1 with valuation t_1. Suppose b^* is the optimal bid made by bidder 1 and p^* is the optimal resale offer made by bidder 2. Then, the value function of bidder 1 is

$$V_1^I(t_1) = \frac{F_2 \circ \phi_2(b^*)}{F_2(t_1)}(t_1 - b^*) + \frac{F_2(t_1) - F_2 \circ \phi_2(b^*)}{F_2(t_1)} \max\{t_1 - p^*, 0\}$$

Notice that $V_1^I(0) = 0$. Using Envelope Theorem, we have

$$\mathrm{D}V_1^I(t_1) = 1 - \frac{F_2 \circ \phi_2(b^*) f_2(t_1)}{(F_2(t_1))^2}(t_1 - b^* - \max\{t_1 - p^*, 0\})$$

Notice that $\mathrm{D}V_1^I(0) = 1$. Now, consider the second-price auction. Since truth-telling strategy is an equilibrium strategy and $t_2 < t_1$, bidder 2 always loses the auction and cannot buy the object in the resale stage. Therefore, the value function of bidder 1 with type t_1 is

$$V_1^{II}(t_1) = \int_0^{t_1} \mathrm{d}z f_2(z)(t_1 - z)$$

Notice that $V_1^{II}(0) = 0$. The derivative of the value function is

$$\mathrm{D}V_1^{II}(t_1) = F_2(t_1) - F_2(0)$$

Notice that $\mathrm{D}V_1^{II}(0) = 0$. Since $V_1^I(0) = V_1^{II}(0) = 0$ and $\mathrm{D}V_1^I(0) > \mathrm{D}V_1^{II}(0)$, it follows that $V_1^I > V_1^{II}$ around some neighborhood of 0. Since $\mathrm{D}V_1^I(0) > \mathrm{D}V_1^{II}(0)$, it follows that there exists $\epsilon > 0$ such that $V_1^I(t_1) > V_1^{II}(t_1)$ for every $t_1 \in (0, \epsilon)$. Suppose there exists $t_1^* > 0$ such that $V_1^I(t_1^*) = V_1^{II}(t_1^*)$ and $V_1^I(t_1) > V_1^{II}(t_1)$ for every $t_1 \in (0, t_1^*)$. Then, from the value functions, we have

$$t_1^* - b^* - \max\{t_1^* - p^*, 0\} = \frac{F_2(t_1^*)}{F_2 \circ \phi_2(b^*)}\left(\int_0^{t_1^*} \mathrm{d}z f_2(z)(t_1^* - z) - \max\{t_1^* - p^*, 0\}\right)$$

Using the above equation in the derivative equation of the value functions, we have

$$\mathrm{D}V_1^I(t_1^*) = 1 - \frac{f_2(t_1^*)}{F_2(t_1^*)}\int_0^{t_1^*} \mathrm{d}z f_2(z)(t_1^* - z) + \frac{f_2(t_1^*)}{F_2(t_1^*)} \max\{t_1^* - p^*, 0\}$$

and

$$\mathrm{D}V_1^{II}(t_1^*) = F_2(t_1^*) - F_2(0)$$

From Assumption 1, it follows $\mathrm{D}V_1^I(t_1^*) > \mathrm{D}V_1^{II}(t_1^*)$. Therefore, $V_1^I(t_1) > V_1^{II}(t_1)$ for every $t_1 \in (0, \bar{a})$. ■

References

[1] Harrison Cheng. Ranking sealed high-bid and open asymmetric auction. *Journal of Mathematical Economics*, 42:471–98, 2006.

[2] Harrison H Cheng and Guofu Tan. Asymmetric common-value auctions with applications to private-value auctions with resale. *Economic Theory*, 45(1-2):253–290, 2010.

[3] Isa Hafalir and Vijay Krishna. Asymmetric auctions with resale. *American Economic Review*, 98(1):87–112, 2008.

[4] Isa Hafalir and Vijay Krishna. Revenue and efficiency effects of resale in first-price auctions. *Journal of Mathematical Economics*, 45(9-10): 589–602, 2009.

[5] René Kirkegaard. Asymmetric first price auctions. *Journal of Economic Theory*, 144(4):1617–1635, 2009.

[6] Vijay Krishna. *Auction theory.* Academic press, 2009.

[7] Michael Landsberger, Jacob Rubinstein, Elmar Wolfstetter, and Shmuel Zamir. First–price auctions when the ranking of valuations is common knowledge. *Review of Economic Design*, 6(3-4):461–480, 2001.

[8] Bernard Lebrun. Comparative statics in first price auctions. *Games and Economic Behavior*, 25(1):97–110, 1998.

[9] Bernard Lebrun. First price auctions in the asymmetric n bidder case. *International Economic Review*, 40(1):125–142, 1999.

[10] Bernard Lebrun. First-price auctions with resale and with outcomes robust to bid disclosure. *The RAND Journal of Economics*, 41(1):165–178, 2010.

[11] Eric Maskin and John Riley. Asymmetric auctions. *The Review of Economic Studies*, 67(3):413–438, 2000.

[12] Gábor Virág. First-price auctions with resale: the case of many bidders. *Economic Theory*, 52(1):129–163, 2013.

[13] Gábor Virág. Auctions with resale: Reserve prices and revenues. *Games and Economic Behavior*, 99:239–249, 2016.